Blob homology, part $\mathbb{I}$

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Outline

What is *blob homology*?

The blob complex takes an n-manifold $\mathcal{M}$ and an 'n-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$.

$HH_*(\mathcal{C})$ (the Hochschild homology)

$\mathcal{M} = S^1$

$$H(\mathcal{B}_*(\mathcal{M};\mathcal{C})) \xrightarrow{* = 0} A(\mathcal{M};\mathcal{C})$$

(the usual TQFT Hilbert space)

$\mathcal{C} = k[t]$

$H_*(\Delta^\infty(\mathcal{M}), k)$

(singular homology of the infinite configuration space)

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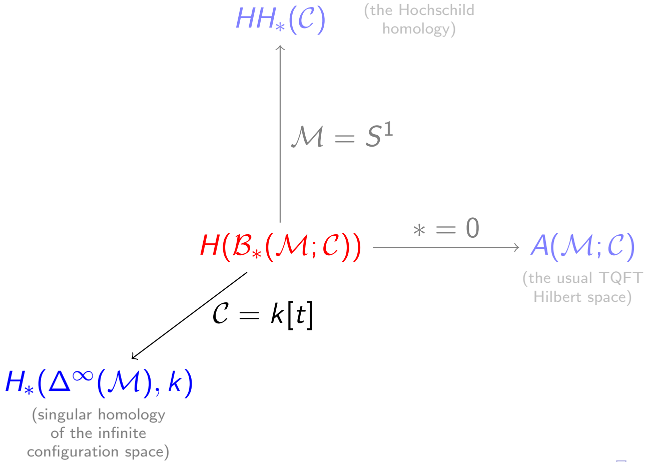


n-categories

Defining n-categories is fraught with difficulties

I'm not going to go into details; I'll draw 2-dimensional pictures, and rely on your intuition for pivotal 2-categories.

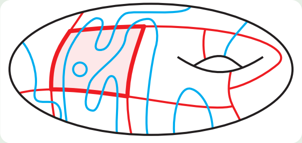
- Kevin's talk (part $\mathbb{III}$) will explain the notions of 'topological n-categories' and 'A_∞ n-categories'.
- Defining n-categories: a choice of 'shape' for morphisms.
- We allow all shapes! A vector space for every ball.
- 'Strong duality' is integral in our definition.

Fields and pasting diagrams

Pasting diagrams

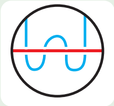
Fix an n-category with strong duality $\mathcal{C}$. A *field* on $\mathcal{M}$ is a pasting diagram drawn on $\mathcal{M}$, with cells labelled by morphisms from $\mathcal{C}$.

Example ($\mathcal{C} = \mathrm{TL}_d$ the Temperley-Lieb category)



$\in \mathcal{F}^{\mathrm{TL}_d}(T^2)$

Given a field on a ball, we can evaluate it to a morphism. We call the kernel the *null fields*.

ev ( $- \frac{1}{d}$ ) $= 0$

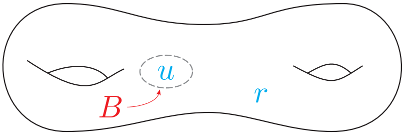
Definition of the blob complex, $k = 0, 1$

Motivation

A *local* construction, such that when $\mathcal{M}$ is a ball, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$ is a resolution of $A(\mathcal{M},;\mathcal{C})$.

$$\mathcal{B}_0(\mathcal{M};\mathcal{C}) = \mathcal{F}(\mathcal{M}), \text{ arbitrary fields on } \mathcal{M}.$$

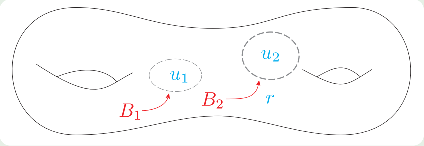
$$\mathcal{B}_1(\mathcal{M};\mathcal{C}) = \left\{ (B, u, r) \;\middle|\; \begin{array}{c} B \text{ an embedded ball} \\ u \in \mathcal{F}(B) \text{ in the kernel} \\ r \in \mathcal{F}(\mathcal{M} \setminus B) \end{array} \right\}.$$



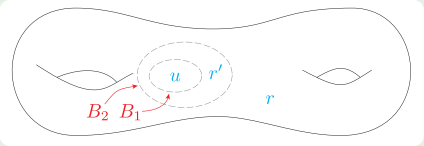
$$d_1 : (B, u, r) \mapsto u \circ r \qquad \mathcal{B}_0 / \operatorname{im}(d_1) \cong A(\mathcal{M};\mathcal{C})$$

Definition, $k = 2$

$$\mathcal{B}_2 = \mathcal{B}_2^{\text{disjoint}} \oplus \mathcal{B}_2^{\text{nested}}$$

$\mathcal{B}_2^{\text{disjoint}} =$  $u_i \in \ker \mathrm{ev}_{B_i}$

$$d_2 : (B_1, B_2, u_1, u_2, r) \mapsto (B_2, u_2, r \circ u_1) - (B_1, u_1, r \circ u_2)$$

$\mathcal{B}_2^{\text{nested}} =$  $u \in \ker \mathrm{ev}_{B_1}$

$$d_2 : (B_1, B_2, u, r', r) \mapsto (B_2, u \circ r', r) - (B_1, u, r \circ r')$$