Product and gluing formulas for the blob complex

Today we'll aim for

Product formula

If F is an n-k manifold, $B_*(F\times -)$ is an A_∞ k-category.

Call this $\mathcal{C}_F$.

If Y is a k manifold.

$$B_*(Y\times F) \cong \underrightarrow{\mathcal{C}_{F}}_h(Y)$$

This is actually a special case ~~of~~ a more general formula.

Given any map $\pi: E \to X$,

we can build a collection Π of modules out of π,

and then $B_*(E) \cong \underrightarrow{\Pi}_h(X)$.

We'll also want

Gluing formula

If Y is an n-1 manifold, $B_*(Y\times -)$ is an A_∞ 1-category

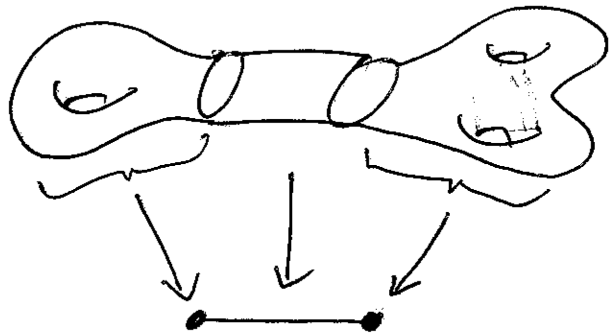
If $Y \subset \partial M_1$ and $Y \subset \partial M_2$, then $B_*(M_i)$ is an

$\cancel{B_*(M_1 \cup_Y M_2)}$ A_∞ module over $B_*(Y)$, and

$$B_*(M_1 \cup_Y M_2) \simeq B_*(M_1) \underset{B_*(Y)}{\overset{A_\infty}{\otimes}} B_*(M_2)$$

We can actually see this as a special case of the formula for a map.

Consider $\pi : M_1 \cup Y \times I \cup M_2 \longrightarrow I$



The collection of modules Π that we build out of the map is exactly $B_*(M_1)$ and $B_*(M_2)$ as $B_*(Y)$ modules, so

$$\underrightarrow{\Pi_h}(X) = B_*(M_1) \overset{A_\infty}{\underset{B_*(Y)}{\otimes}} B_*(M_2).$$

Today we'll just aim to prove this directly.

Notice that I've only stated this for codimension 1 gluing. (So in particular, we won't have shown that B_* really is a TQFT!)

It ought to be true in arbitrary codimension, but to state the result we'd need a notion of equivalence between disklike k-categories.

Given such a notion of functor, we know what the functor ought to be (follow the same proof for codimension 1), but this is still for the future

Let's prove the product formula.

First, let's do a little digression into acyclic models; we'll be using these to construct chain maps.

Suppose we want to construct a chain map $C_* \to D_*$, but we're not quite sure how to do it: there are lots of choices to be made, and it's hard to see if they can be made consistently.

- Say we have a basis for C_k, $\{x_{kj}\}_j$, and for each x_{kj} we have a subcomplex $Z_*^{kj} \subset D_*$ of "possible choices" we might make.

Consider $\mathrm{Maps}(C_* \to D_*)$, the complex of (not necessarily degree preserving) chain maps.
(think of Maps_0 as being honest chain maps, Maps_1 being homotopies, etc)

Define $\text{Compat-}Z(C_* \to D_*)$ to be the subcomplex of maps so $f(x_{kj}) \in Z_*^{kj}$
(if f is degree l we'll in fact have $f(x_{kj}) \in Z_{k+l}^{kj}$)

<u>Theorem</u> (Acyclic models) (Spanier, chapter 4).

Suppose • $D_*^{k-1,l} \subset D_*^{kj}$ whenever $x_{k-1,l}$ occurs in ∂x_{kj}.

• D_0^{0j} is nonempty for all j

• $H_m(D_*^{kj}) = 0 \quad \forall k, j$ and $m \geq k-1$.

Then $\text{Compat-}\mathbb{Z}(C_* \to D_*)$ is non-empty and contractible.

Proof of the product theorem

First we'll define $\varphi: \mathcal{C}_{\underrightarrow{F,h}}(Y) \to B_*(Y \times F; \mathcal{C})$

Recall elements of $\mathcal{C}_{\underrightarrow{F,h}}(Y)$ are simplices for the functor

$$\mathcal{U}_{\mathcal{C}_{F}, Y} : \mathcal{D}(Y) \to \mathrm{Chain}$$

$$Y = \cup B_\alpha \longrightarrow \underset{\substack{\alpha, \\ \text{fibered} \\ \text{over} \\ \text{boundary} \\ \text{conditions}}}{\bigotimes} B_*(B_\alpha \times F)$$

i.e an m-simplex is a sequence

$x_0 \leq x_1 \leq \cdots \leq x_m$ of permissible decompositions, along with

$$a \in \mathcal{U}_{\mathcal{C}_F, Y}(x_0).$$

We define φ on 0-simplices by

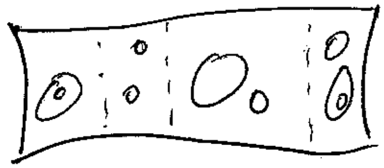
$$\varphi(a, (x_0)) = gl(a) \in B_*(Y \times F).$$

Define φ on all higher simplices to be zero!

We can readily see this is a chain map.

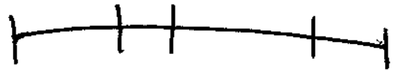
We need to define a map back. In fact, we won't do this on all of $B_*(Y \times F; \mathcal{C})$.

First define $G_* \subset B_*(Y \times F; \mathcal{C})$ to be the image of φ, i.e. all blob diagrams splittable along <u>some</u> decomposition of Y.



$Y \times F$

In fact, $B_*(Y\times F; \mathcal{C})$ is homotopy equivalent to ~~some subcomplex of~~ G_*, by the small blobs lemma.



Y

(Roughly: Choose an open cover of Y [redacted] such that any k open sets is contained in some disjoint union of balls.

This doesn't quite work, since it has to work for all k at once.

~~Since~~ G_* ~~and $B_*(Y\times F;\mathcal{C})$ are free it suffices to show~~ that the inclusion induces an isomorphism of homotopy groups, and in turn that for any

$$\begin{array}{ccc} C_* & \subset & B_*(Y\times F;\mathcal{C}) \\ \cup & & \cup \\ D_* & \subset & G_* \end{array}$$

C_*, D_* finitely generated, we can find a homotopy

$$h: C_* \to B_{*+1}(Y\times F; \mathcal{C})$$
$$h(D_*) \subset G_{*+1}$$

and $h\partial x + \partial h x + x \in G_*$ for all $x \in C_*$.

Now we can use small blobs to move C_* into G_*,

~~since there is a maximal k.~~

(There's an easier argument: roughly, use small blobs, then truncate, and show the inclusion is an isomorphism on homology arbitrarily high.)

We're just going to construct a map back

$$\xi : G_* \to \underrightarrow{\mathcal{C}}_{F_h}(Y)$$

and we'll do this by acyclic models.

Associated to a blob diagram α, we have

$d^\bullet \alpha$, the set of all iterated boundaries,

i.e. all ways of forgetting some subset of the blobs.

Define $Z(\alpha) = \{ (\beta, \underline{x}) \mid \beta \in d^\bullet \alpha,\ \alpha \text{ splits along } x_0 \}$.

<u>Lemma</u> $Z(\alpha)$ is acyclic.

Thus we can choose $\xi : G_* \to \underrightarrow{\mathcal{C}}_{F_h}(Y)$

so $\xi(\alpha) \in Z(\alpha)$, and in fact

$$\xi(\alpha) = (\alpha, (x_0)) + r$$

where r is a sum of higher simplices.

Now $\phi(\xi(\alpha)) = \phi(\alpha, (x_0)) + \phi(r)$

$$= \alpha$$

Why is $\xi \circ \phi$ homotopic to the identity on $\underrightarrow{\mathcal{C}}_{F_h}(Y)$?

Consider the acyclic model for a chain map

$$\underrightarrow{\mathcal{C}}_{F_h}(Y) \longrightarrow \underrightarrow{\mathcal{C}}_{F_h}(Y)$$

$$\alpha \longmapsto Z(\phi(\alpha)).$$

Both $\xi \circ \phi$ and id are compatible with this model, so must be homotopic. □

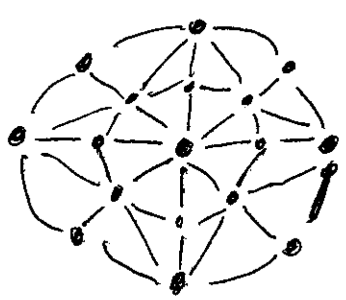
Proof of the Lemma

$Z(\alpha)$ is a tensor product; the first factor $d^{\bullet}\alpha$ is acyclic, so we really just need to worry about al simplices of decompositions on which α is splittable.

Suppose we have 0-simplices K and K'.
There isn't necessarily a ~~decompos~~ common refinement, but we can pick a generic decomposition L ~~so~~ with has common refinements KL and $K'L$ with K and L.
This provides a 1-cell connecting K and K'

$$\begin{tikzcd} K & & L & & K' \\ & KL \arrow[ul] \arrow[ur] & & K'L \arrow[ul] \arrow[ur] \end{tikzcd}$$

Generalizing, given a k-cycle, pick a generic decomposition L and ~~the~~ a cone ~~of~~ spans between the k-cycle and L.



□

Proof of the gluing formula

Recall first what the gluing formula says:

If M_1 and M_2 are modules over an A_∞ category C,

$$M_1 \overset{A_\infty}{\underset{C}{\otimes}} M_2 := \operatorname{hocolim}(\mathcal{V} : \mathcal{D}(I) \to \mathrm{Chain})$$

Define $\varphi : B_*(M_1) \overset{A_\infty}{\underset{B_*(Y)}{\otimes}} B_*(M_2) \to B_*(M_1 \cup Y \times I \cup M_2)$

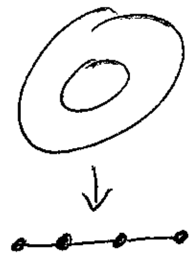
to be the gluing map on 0-simplices and zero on higher simplices.

Again, we'll define the map back on the image $C_* := \operatorname{im}\varphi$, which is homotopy equivalent to the ~~big~~ full blob complex.

$Z(\alpha)$ is defined pretty much the same way, although here acyclicity is actually easier, as any two decomposition of I have a common refinement.

Final bonus section on general maps.

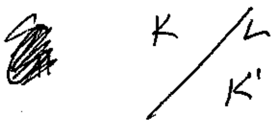
Given $\pi : E^n \to X^k_1$ pick a cell decomposition of X so π is trivial over each cell.



For each codimension 0 cell, K, we have a k-category $\boxed{\,}_K$ based on balls in K.

$$\boxed{\,}_K(D) = \mathcal{B}_*(\pi^{-1}(D))$$

For each codimension 1 cell L, we have a bimodule ($= S^0$ module) between the k-categories on either side:



Given a disc D which is a neighbourhood of a disc in L,

$$\boxed{\,}_L(D) = \mathcal{B}_*(\pi^{-1}(D)).$$ The bimodule action is obvious here!

Going up, for a codimension j cell P, we have an S^{j-1}-module for the annular category corresponding to the link of P.

~~Pathologically, $\mathcal{B}_*($~~

By the same arguments as before

$$\mathcal{B}_*(E) = \underrightarrow{\boxed{\,}}_h(X)$$

hocolim along decompositions of X compatible with the cell structure, using the appropriate sphere module labels.