Blob homology, part I

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http://tqft.net/UCR-blobs1

Blob homology

... homotopical topology and TQFT have grown so close that I have started thinking that they are turning into the language of new foundations.

— Yuri Manin, September 2008

What is *blob homology*?

The blob complex takes an n-manifold $\mathcal{M}$ and an 'n-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$.

$HH_*(\mathcal{C})$ (the Hochschild homology)

$\mathcal{M} = S^1$

$$H(\mathcal{B}_*(\mathcal{M};\mathcal{C})) \xrightarrow{\;* = 0\;} A(\mathcal{M};\mathcal{C})$$

(the usual TQFT Hilbert space)

$\mathcal{C} = k[t]$

$H_*(\Delta^\infty(\mathcal{M}), k)$ (singular homology of the infinite configuration space)

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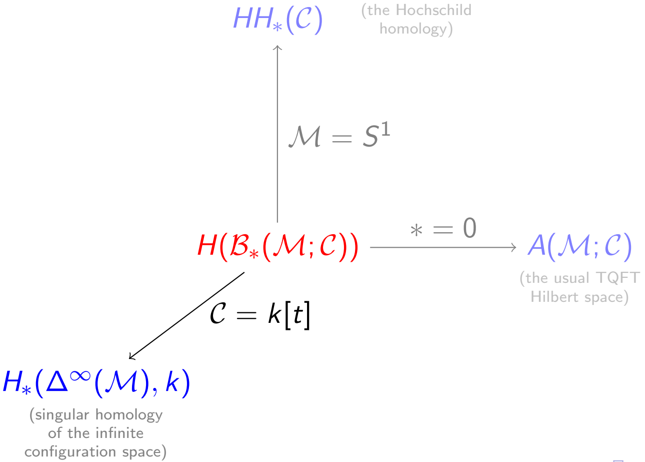


n-categories

Defining n-categories is fraught with difficulties

I'm not going to go into details; I'll draw 2-dimensional pictures, and rely on your intuition for pivotal 2-categories.

Kevin's talk (part $\mathbb{III}$) will explain the notions of 'topological n-categories' and 'A_∞ n-categories'.

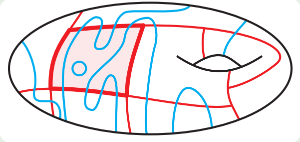
- Defining n-categories: a choice of 'shape' for morphisms.
- We allow all shapes! A vector space for every ball.
- 'Strong duality' is integral in our definition.

Fields and pasting diagrams

Pasting diagrams

Fix an n-category with strong duality $\mathcal{C}$. A *field* on $\mathcal{M}$ is a pasting diagram drawn on $\mathcal{M}$, with cells labelled by morphisms from $\mathcal{C}$.

Example ($\mathcal{C} = \mathrm{TL}_d$ the Temperley-Lieb category)



$\in \mathcal{F}^{\mathrm{TL}_d}\left(T^2\right)$

Given a pasting diagram on a ball, we can evaluate it to a morphism. We call the kernel the *null fields*.

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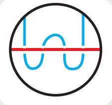
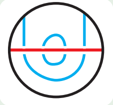


$- \frac{1}{d}$



) $= 0$

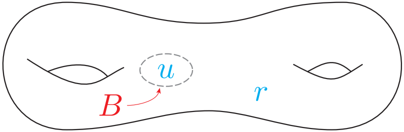
Definition of the blob complex, $k = 0, 1$

Motivation

A *local* construction, such that when $\mathcal{M}$ is a ball, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$ is a resolution of $A(\mathcal{M},; \mathcal{C})$.

$$\mathcal{B}_0(\mathcal{M}; \mathcal{C}) = \mathcal{F}(\mathcal{M}), \text{ arbitrary pasting diagrams on } \mathcal{M}.$$

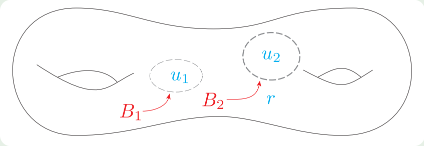
$$\mathcal{B}_1(\mathcal{M}; \mathcal{C}) = \left\{ (B, u, r) \;\middle|\; \begin{array}{c} B \text{ an embedded ball} \\ u \in \mathcal{F}(B) \text{ in the kernel} \\ r \in \mathcal{F}(\mathcal{M} \setminus B) \end{array} \right\}.$$



$$d_1 : (B, u, r) \mapsto u \circ r \qquad \mathcal{B}_0 / \operatorname{im}(d_1) \cong A(\mathcal{M}; \mathcal{C})$$

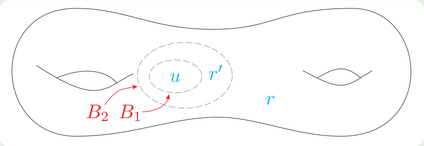
Definition, $k = 2$

$$\mathcal{B}_2 = \mathcal{B}_2^{\text{disjoint}} \oplus \mathcal{B}_2^{\text{nested}}$$

$\mathcal{B}_2^{\text{disjoint}} = \Bigg\{$ 

$\Bigg| \ \mathrm{ev}_{B_i}(u_i) = 0 \Bigg\}$

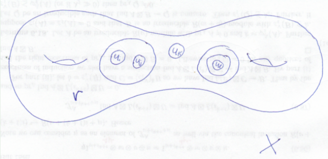
$$d_2 : (B_1, B_2, u_1, u_2, r) \mapsto (B_2, u_2, r \circ u_1) - (B_1, u_1, r \circ u_2)$$

$\mathcal{B}_2^{\text{nested}} = \Bigg\{$ 

$\Bigg| \ \mathrm{ev}_{B_1}(u) = 0 \Bigg\}$

$$d_2 : (B_1, B_2, u, r', r) \mapsto (B_2, u \circ r', r) - (B_1, u, r \circ r')$$

Definition, general case

$\mathcal{B}_k = \Bigg\{$  $\Bigg\}$

k blobs, properly nested or disjoint, with "innermost" blobs labelled by pasting diagrams that evaluate to zero.

$$d_k : \mathcal{B}_k \to \mathcal{B}_{k-1} = \sum_i (-1)^i (\text{erase blob } i)$$

An action of $C_*(\mathrm{Homeo}(\mathcal{M}))$

Theorem

There's a chain map

$$C_*(\mathrm{Homeo}(\mathcal{M})) \otimes \mathcal{B}_*(\mathcal{M}) \to \mathcal{B}_*(\mathcal{M}).$$

which is associative up to homotopy, and compatible with gluing.

Taking H_0, this is the mapping class group acting on a TQFT skein module.

Gluing

$\mathcal{B}_*(Y \times [0,1])$ is naturally an A_∞ category

m_2: gluing $[0,1] \simeq [0,1] \cup [0,1]$

m_k: reparametrising $[0,1]$

If $Y \subset \partial X$ then $\mathcal{B}_*(X)$ is an A_∞ module over $\mathcal{B}_*(Y)$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{B}_*(X \bigcup_Y \circlearrowleft) \cong \mathcal{B}_*(X) \bigotimes_{\mathcal{B}_*(Y)}^{A_\infty} \circlearrowleft.$$

In principle, we can compute blob homology from a handle decomposition, by iterated Hochschild homology.