Outline

1. TQFTs via fields & local relations

2. Blob homology joint work w/ Scott Morrison

3. ~~Derived categories of partial orders and contact categories for the disk~~ joint work w/ Alex Dugas

4. Wild speculation prompted by Dennis Sullivan

Before definitions of fields and local relations, some examples.

$$\mathcal{C}(X^n) \stackrel{\text{def}}{=} \{\text{maps } X \to BG\}$$

BG is classifying space of finite group, or maybe any topological space.

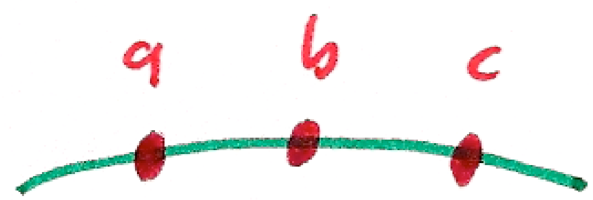
$$A(X) \stackrel{\text{def}}{=} \mathbb{C}[\mathcal{C}(X)]/\sim$$

local relation

Equivalence relation generated by $f \sim g$ if f is homotopic to g via a homotopy supported on a ball $B \subset X$.

Let C be a *-algebra (more generally, *-1-category).

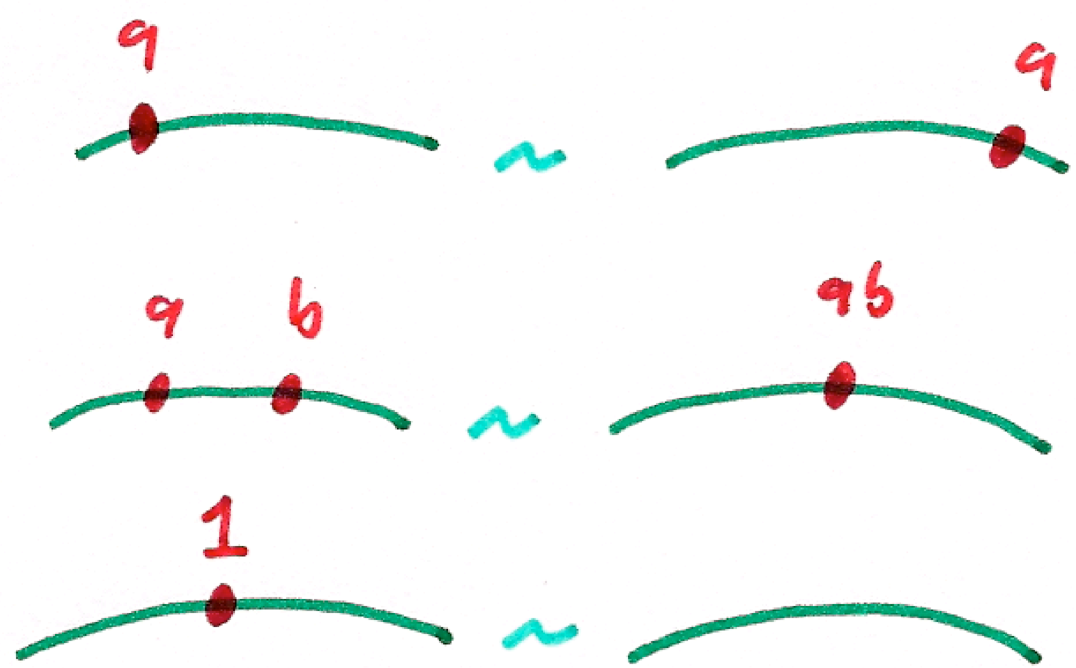
$\mathcal{C}(X^1) \stackrel{\text{def}}{=} \{\text{all } C\text{-pictures on } X\}$



$a, b, c \in C$

$A(X) \stackrel{\text{def}}{=} \mathbb{C}[\mathcal{C}(X)]/\sim$

local relation

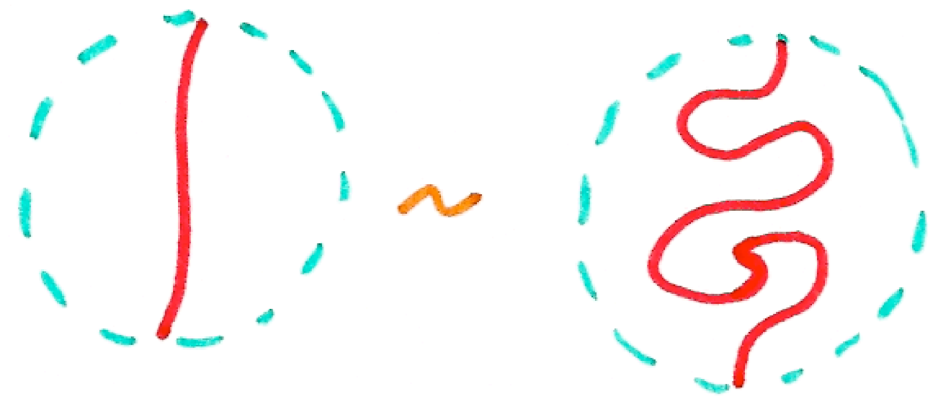


(isotopy)

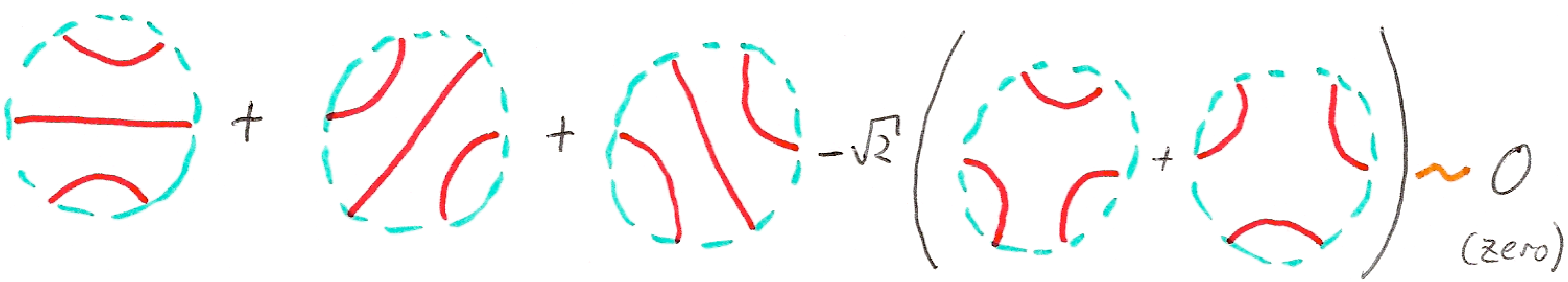
Let C be a pivotal tensor category (more generally, pivotal 2-category).

$\mathcal{C}(X^2) \stackrel{\text{def}}{=} \{\text{all } C\text{-pictures on } X\} = \{\text{all } C\text{-labeled 1-complexes in } X\}$

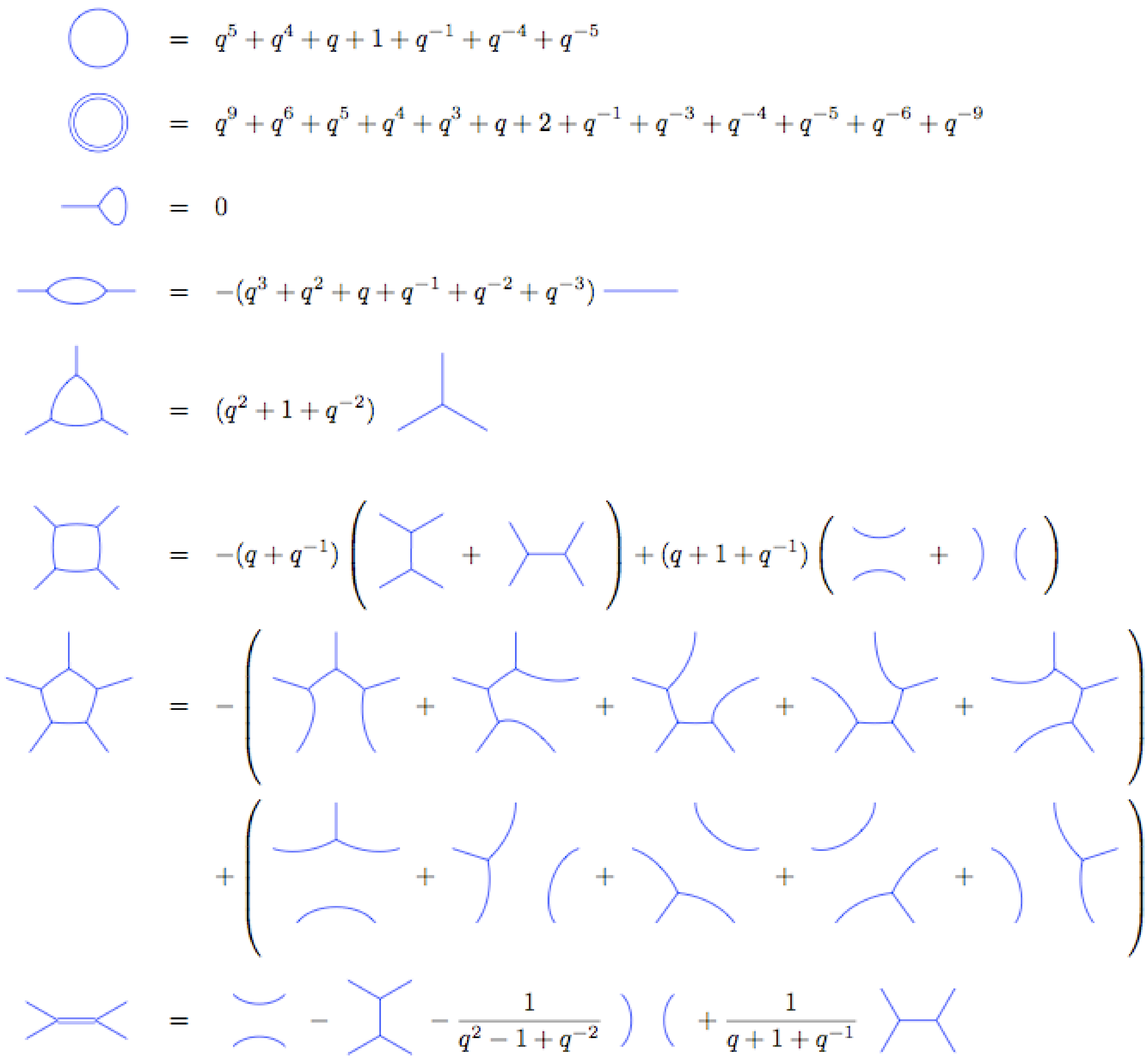
$A(X) \stackrel{\text{def}}{=} \mathbb{C}[\mathcal{C}(X)]/\sim$



(isotopy)



Kuperberg G2 “spider”:



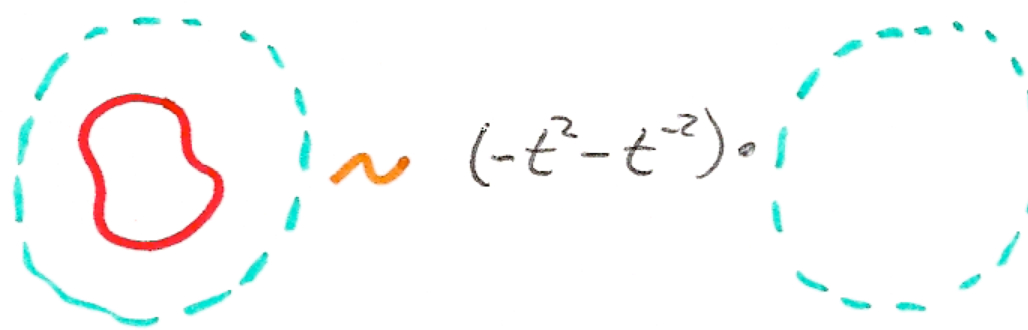
Exercise: Find the sign error in these relations

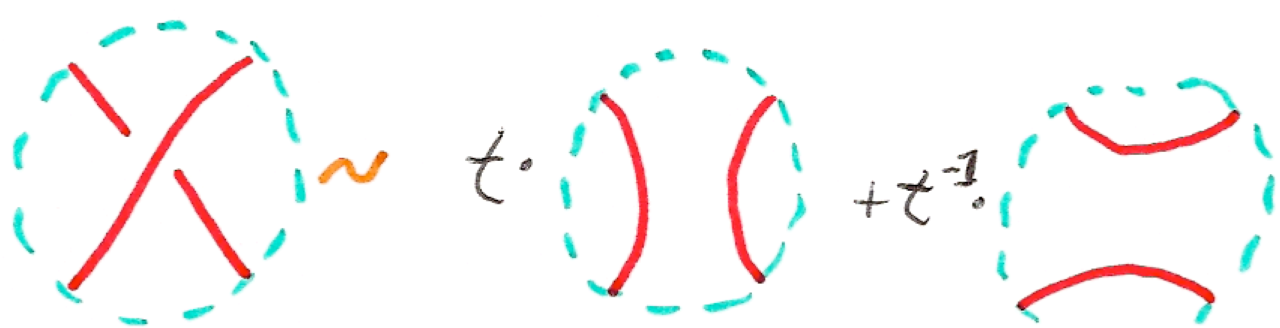
Let C be a braided ribbon category (more generally, 3-category with strong duality).

$$\mathcal{C}(X^3) \stackrel{\text{def}}{=} \{\text{all } C\text{-pictures on } X\} = \{\text{all } C\text{-labeled framed 1-complexes in } X\}$$

$$A(X) \stackrel{\text{def}}{=} \mathbb{C}[\mathcal{C}(X)]/\sim$$

(isotopy)





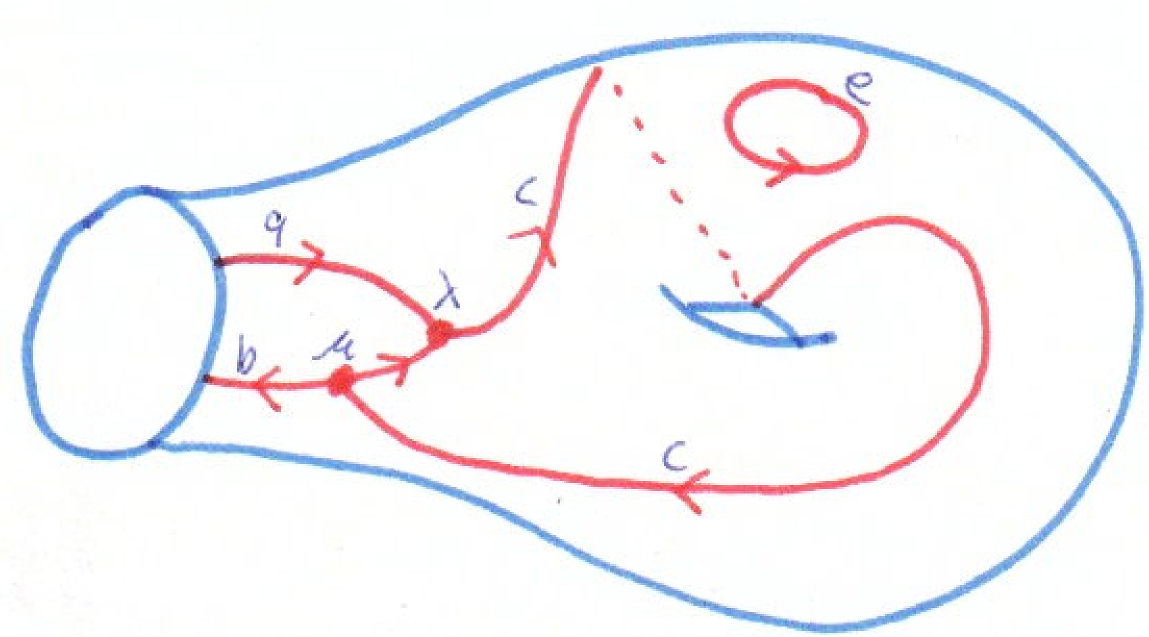
Now for the rigorous definitions:

A system of fields for manifolds of dimension $\leq n$ consists of:

- a collection of functors $\mathcal{C}_k : \mathcal{M}_k \to \mathbf{Set}$, $k \leq n$, where $\mathcal{M}_k$ denotes the category of k-manifolds and homeomorphisms (PL, say)
- additional data [...] (see below)
- satisfying conditions [...] (see below)

Main examples:

- $\mathcal{C}(X) = \{\text{maps } X \to C\}$, e.g. $C = B\Gamma$, where Γ is a finite group
- $\mathcal{C}(X) = \{\text{decorated cell complexes } \subset X\}$



The rest of the "fields" definition:

- restriction maps $\mathcal{C}(X) \to \mathcal{C}(\partial X)$ (natural transformation of functors)
- orientation reversal maps $\mathcal{C}(X) \to \mathcal{C}(-X)$ (natural transformation of functors)
- compatibility with monoidal structure $\mathcal{C}(X \sqcup W) \cong \mathcal{C}(X) \times \mathcal{C}(W)$
- gluing along $Y \subset \partial X$, $-Y \subset \partial W$ corresponds to a fibered product

$$\begin{tikzcd}
 & & \mathcal{C}(X) \arrow[dr] & \\
\mathcal{C}(X \cup_Y W) & \mathcal{C}(X) \times_{\mathcal{C}(Y)} \mathcal{C}(Y) \arrow[l, hook'] \arrow[ur] \arrow[dr] & & \mathcal{C}(Y) \\
 & & \mathcal{C}(W) \arrow[ur] &
\end{tikzcd}$$

  (and similarly for self-gluing, gluing with corners). (Map to $\mathcal{C}(X \cup_Y W)$ is surjective up to appropriate equivalence relation.)

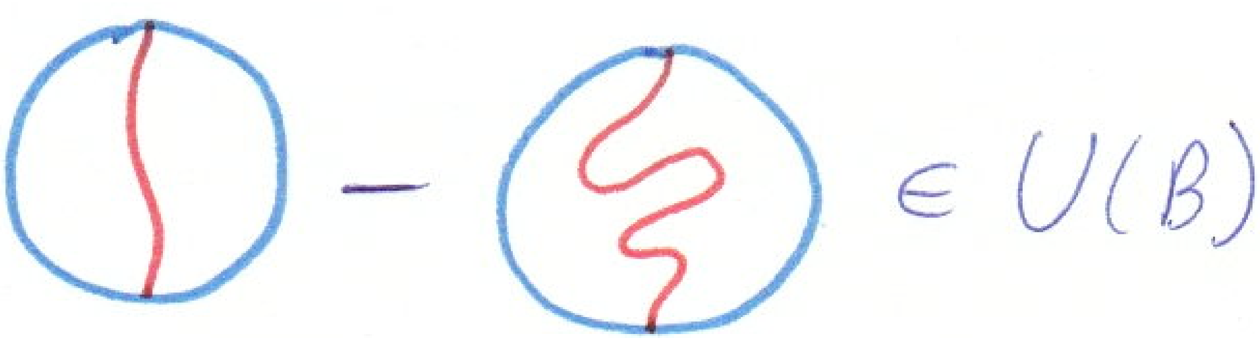
- "product with I" maps $\mathcal{C}(Y) \to \mathcal{C}(Y \times I)$; fiber-preserving homeos of $Y \times I$ act compatibly on image

Definition of "local relations":

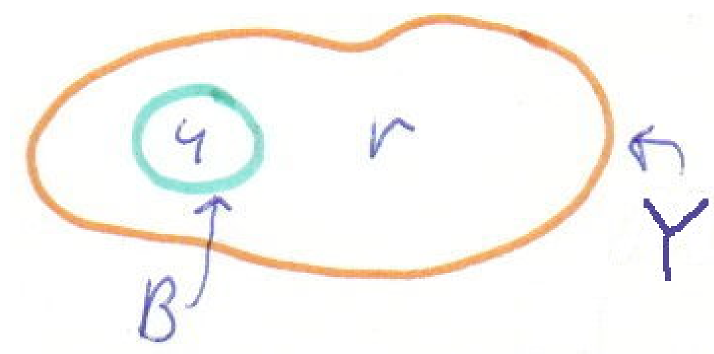
For each n-manifold $B \cong B^n$ and $c \in \mathcal{C}(\partial B)$, a subspace $U(B;c) \subset \mathbb{C}[\mathcal{C}(B;c)]$, preserved under homeomorphisms, such that

- local relations are at least as strong as isotopy: for all $a, b \in \mathcal{C}(B)$ with a isotopic to b (pseudo-isotopic or extended isotopic), we have $a - b \in U(B)$.

- local relations are an ideal with respect to gluing: for all $B = B_1 \cup B_2$, $u \in U(B_1)$, $r \in \mathcal{C}(B_2)$, we have $u \bullet r \in U(B)$

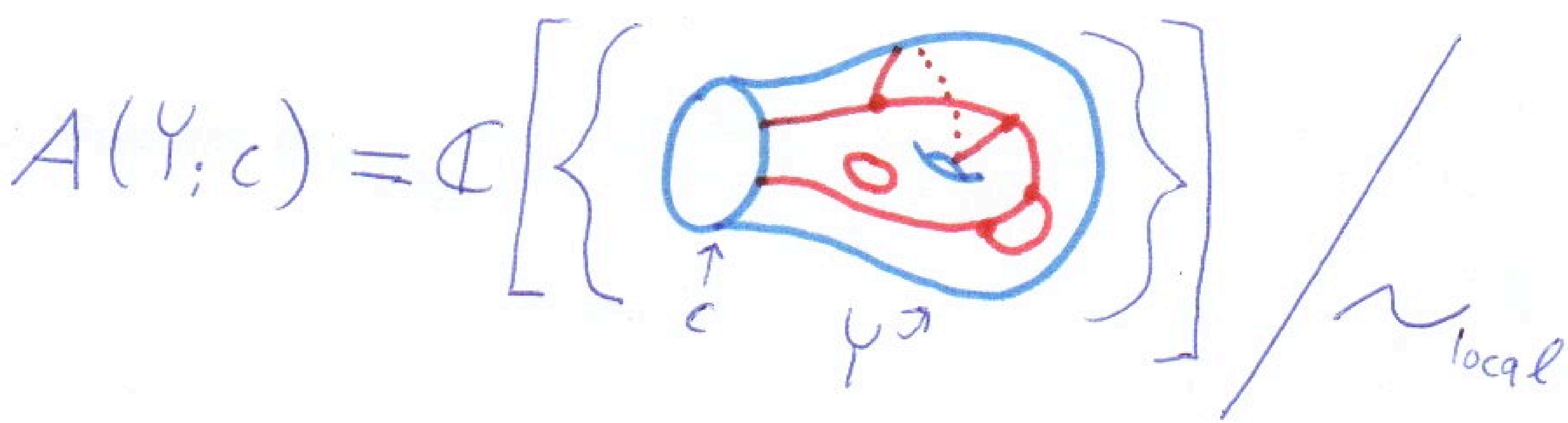


Basic constructions, dimension n



$$A(Y^n; c) \stackrel{\text{def}}{=} \mathbb{C}[\mathcal{C}(Y)]/LR(Y)$$

where $c \in \mathcal{C}(\partial Y)$ is a boundary condition and $LR(Y)$ is the span of all $u \bullet r$, $B \subset Y$, $u \in U(B)$, $r \in \mathcal{C}(Y \setminus B)$



$$Z(M^{n+1}; c) \stackrel{\text{def}}{=} \int_{x \in \mathcal{C}(M;c)} T(x)$$

There is a an argument "at the physical level of rigor" that the Hilbert space for *any* topologically invariant path integral can be described as above via local relations.

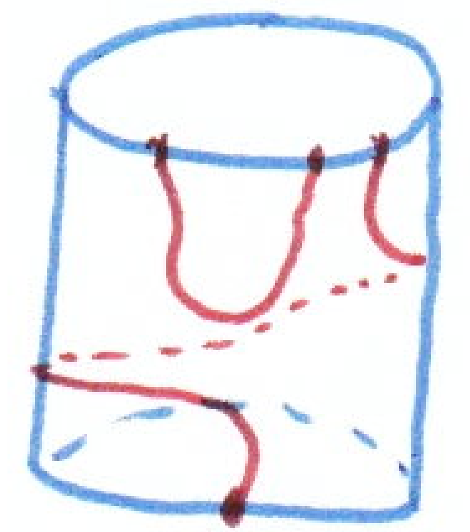
The path integral $Z(B^{n+1})$ can be thought of as a local projection from the southern hemisphere to the northern hemisphere, and this is dual to a local relation.

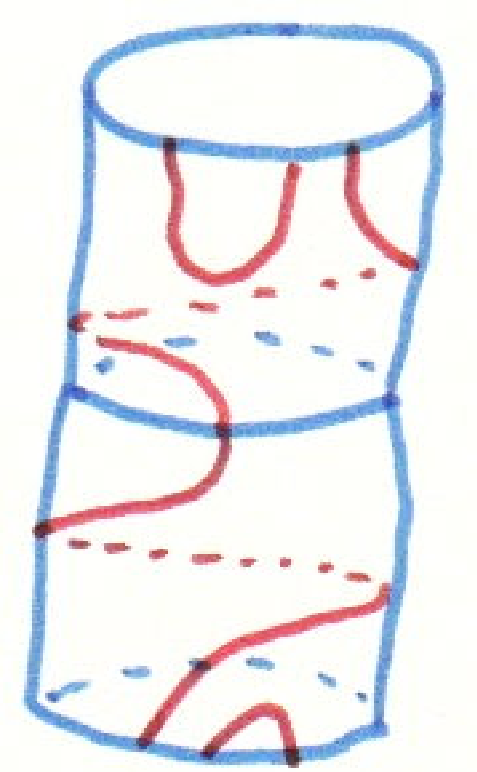
So we *define* a TQFT to be anything that arises from the above construction, starting with fields and local relations. Atiyah-Segal style statements about functors from cobordism categories become propositions rather than axioms.

Basic constructions, dimension n-1

$A(W^{n-1}; c)$ is a *-1-category:

- objects: $\mathcal{C}(W; c)$
- morphisms from a to b: $A(W \times I; a, b)$
- composition: gluing





Basic constructions, dimension n-2

$A(Q^{n-2}; c)$ is a pivotal 2-category:

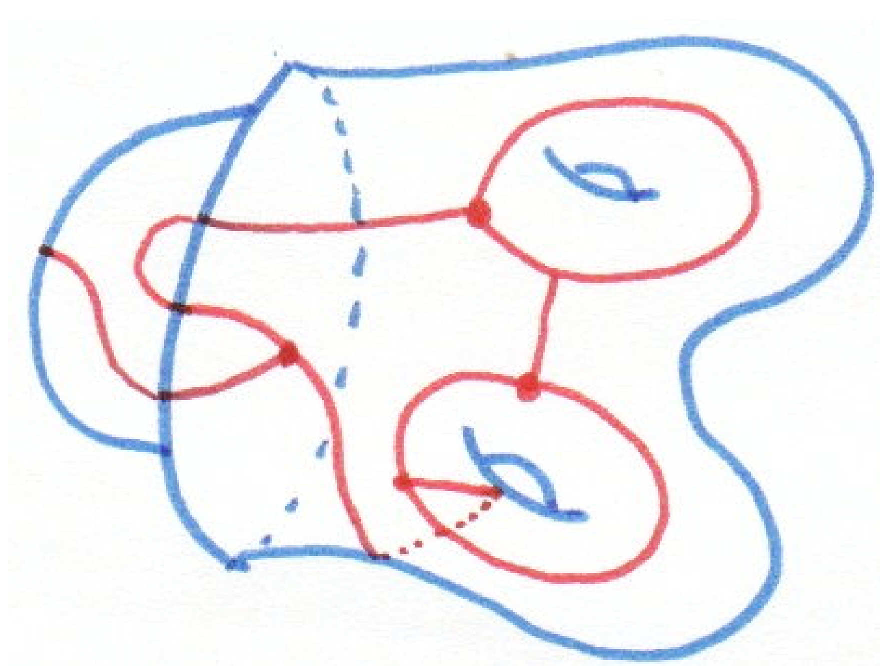
- 0-morphisms: $\mathcal{C}(Q; c)$
- 1-morphisms from a to b: $\mathcal{C}(Q \times I; a, b)$
- 2-morphisms from e to f: $A((Q \times I) \times I; e, f)$
- composition: gluing

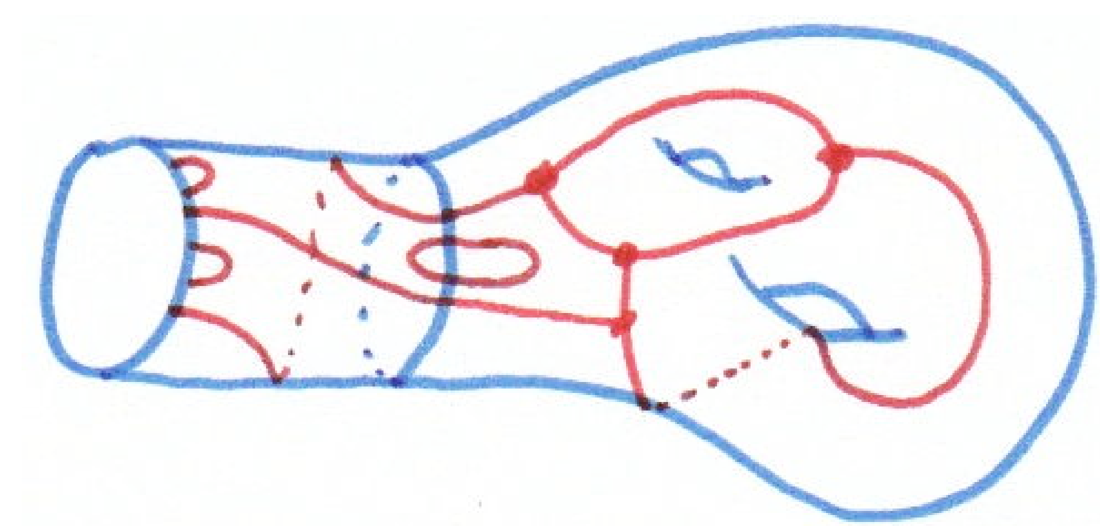
And so on: $A(X^{n-k})$ is a linear k-category with strong duality. For $j < k$, the j-morphisms are $\mathcal{C}(X \times I^j; \cdot)$. The k-morphisms are $A(X \times I^k; \cdot)$.

Manifolds afford representations of their boundary categories

$\{A(W^{n-k}; c)\}$, where c runs through all of $\mathcal{C}(\partial W)$, affords a representation of the $k+1$-category $A(\partial W)$, via gluing of collars.

More generally, let $Z \subset \partial W$ be a codimension-0 submanifold, and $b \in \mathcal{C}(\partial W \setminus Z)$. Then $\{A(W^{n-k}; b, c)\}$, where c runs through all of $\mathcal{C}(Z, \partial b)$, affords a representation of the $k+1$-category $A(Z; \partial b)$, via gluing of partial collars.





Gluing

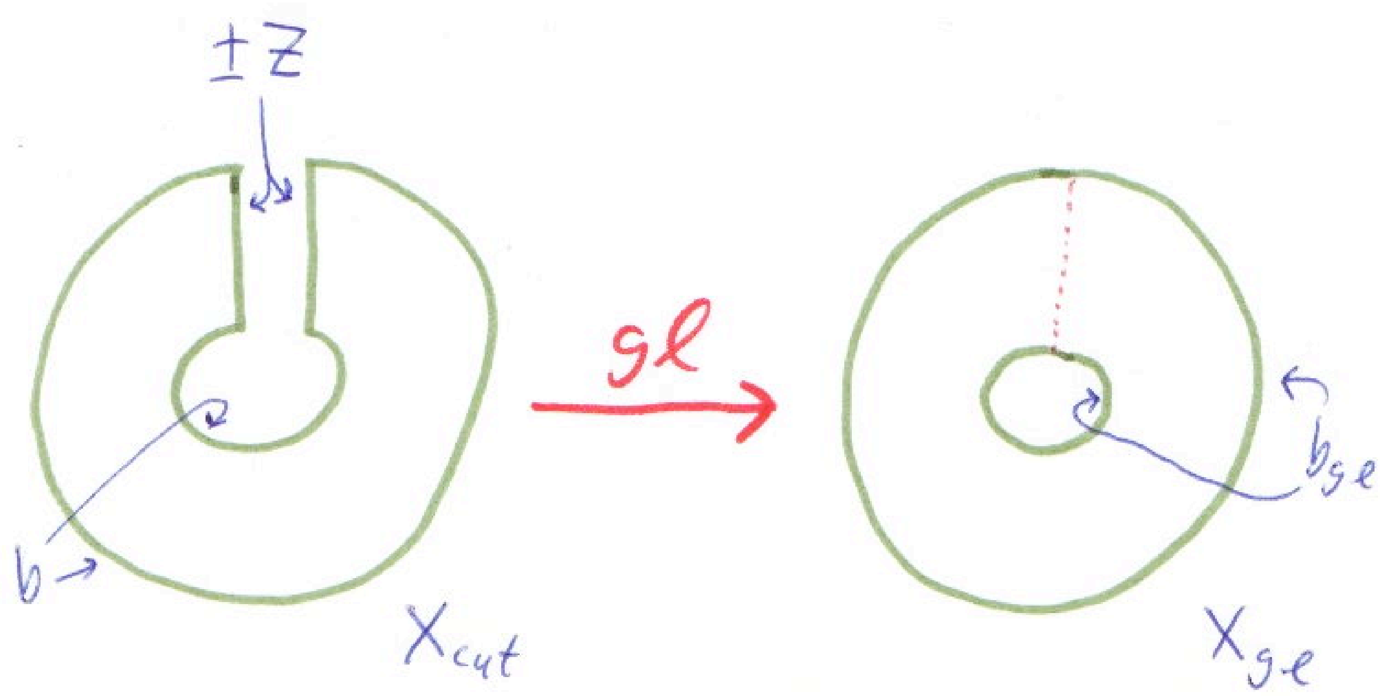
Let X_{cut} be an $n-k$-manifold, with $Z \sqcup -Z$ embedded as a codim-0 submanifold of ∂X_{cut}. Identifying the copies of Z yields a manifold X_{gl}.

Theorem:

$$A(X_{\text{gl}}; b_{\text{gl}}) \cong \bigotimes_{A(Z;\partial b)} \{A(X_{\text{cut}}; b, \cdot)\}$$

(k times categorified coend)

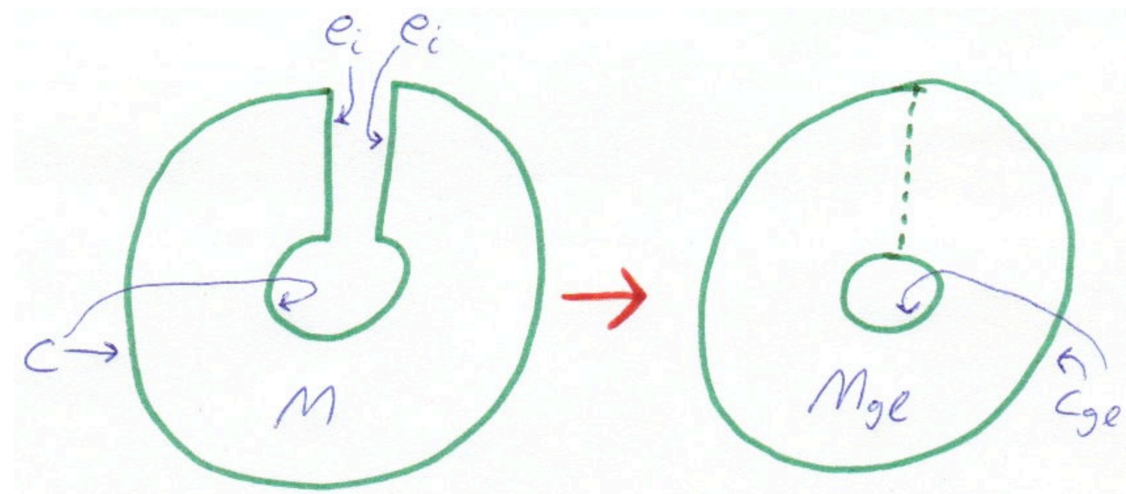
(Drinfeld double is a special case of the once categorified coend. Drinfeld center is a special case of the once categorified end.)



What about Z?

- $Z(Y^n) \stackrel{\text{def}}{=} A(Y)^* = \{f : \mathbb{C}[\mathcal{C}(Y)] \to \mathbb{C} \mid f(u \bullet r) = 0 \text{ for all } B, u, r \text{ as above}\}$
- $Z(W^{n-1}) \stackrel{\text{def}}{=} \mathrm{Rep}(A(W))$ (i.e. functors from $A(W)$ to **Vect**)
- and in general, $Z(X^{n-k}) \stackrel{\text{def}}{=} \mathrm{Rep}(A(X))$
- for $\dim X \leq n$, we have $Z(X) \in Z(\partial X)$

What about dimension n+1?



What we want from a path integral:

- For all M^{n+1}, $Z(M) \in Z(\partial M)$, i.e.

$$Z(M) : A(\partial M) \to \mathbb{C}$$

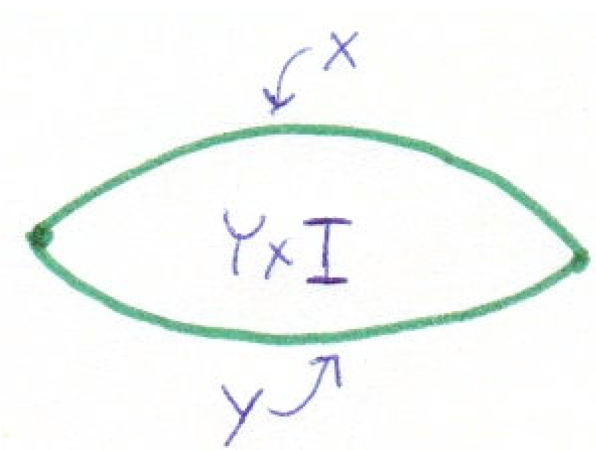
- satisfying the gluing formula

$$Z(M_{\mathrm{gl}})(c_{\mathrm{gl}}) = \sum_i Z(M)(e_i \bullet e_i \bullet c) \frac{1}{\langle e_i, e_i \rangle},$$

  where e_i runs through an *orthogonal* basis of $A(Y; \partial c)$

- and where the (non-degenerate) inner products of $A(Y^n; b)$ are related to the path integral via

$$\langle x, y \rangle = Z(Y \times I)(x \bullet y)$$



Theorem. Suppose

1. there exists $z \in Z(S^n)$ such that the induced inner product $A(B^n; c) \otimes A(B^n; c) \to \mathbb{C}$ given by $a \otimes b \mapsto z(a \bullet b)$ is positive definite for all $c \in \mathcal{C}(S^{n-1})$; and

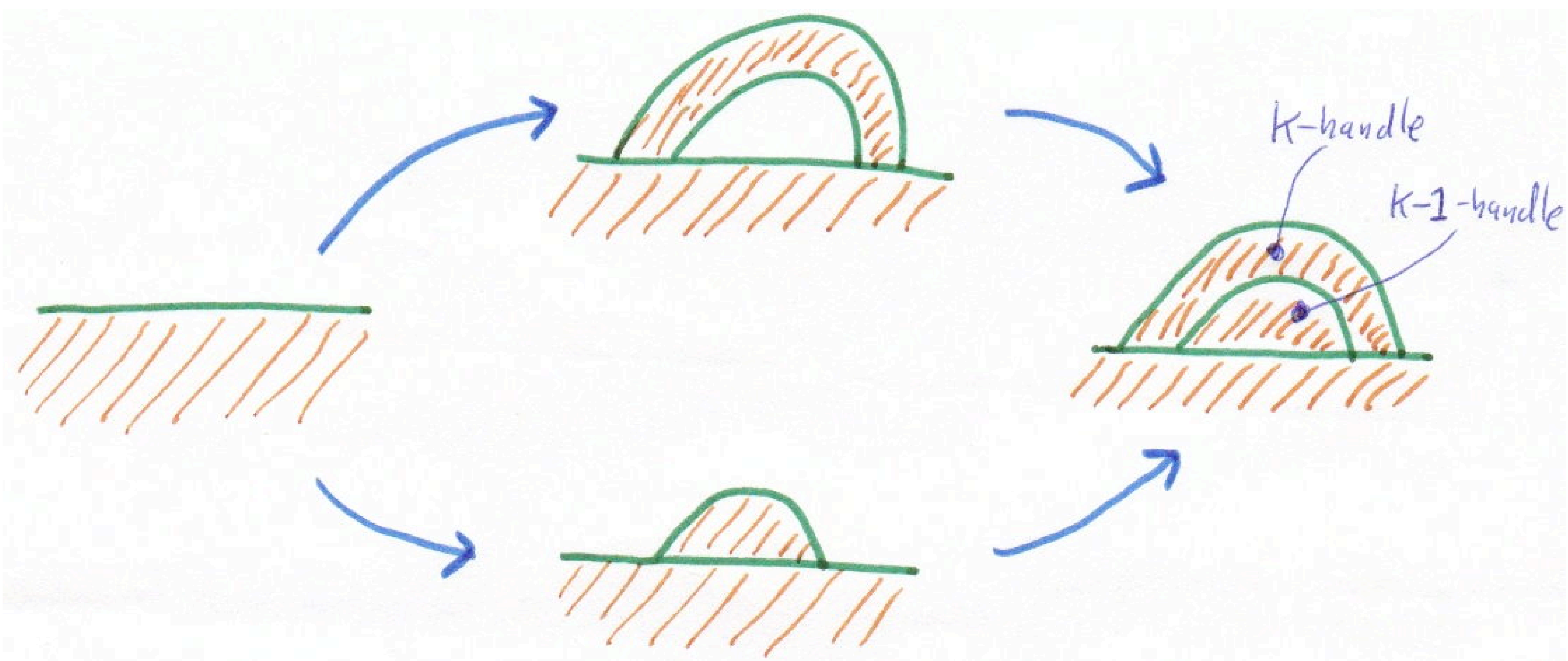
2. $\dim A(Y^n; c) < \infty$ for all n-manifolds Y and all $c \in \mathcal{C}(\partial Y)$.

Then there exists a unique path integral $Z(M^{n+1}) \in Z(\partial M)$ (for all $n+1$-manifolds M) satisfying the the above conditions and such that $Z(B^{n+1}) = z$.

Sketch of proof:

- Choose a handle decomposition of M. Adding the handles one at a time (lowest index first) determines $Z(M)$ via the gluing formula. This proves uniqueness.

- To prove existence, must show that the computation of the previous step does not change if we cancel a pair of handles. This follows from the more general fact (lemma) that the gluing formula is associative. So we can add the canceling pair of handles in reverse order, but this is equivalent to adding partial collars, and hence has no effect on the computation.



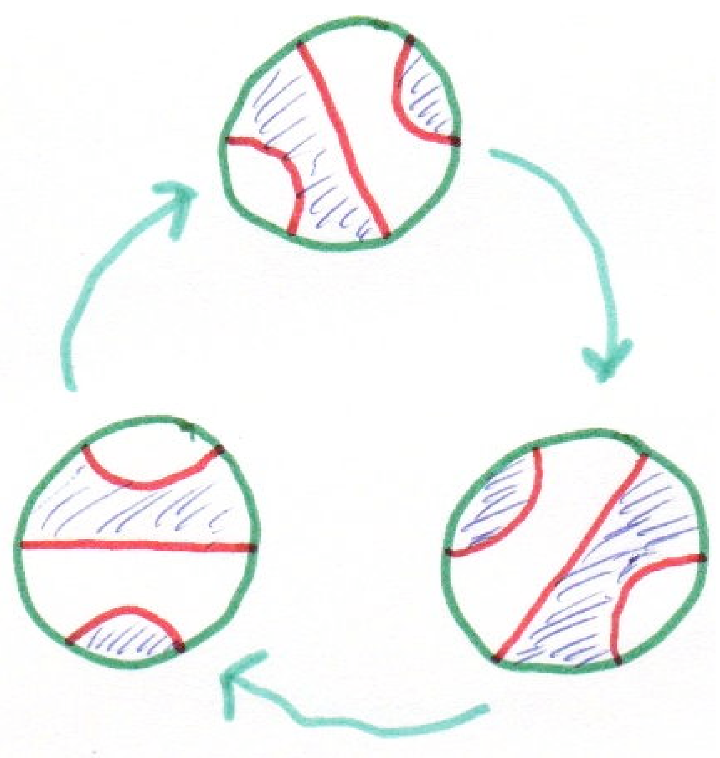
Fields	Local Relation	State Sum
Maps into BG (G a finite group) n = arbitrary	Homotopy of maps	Dijkgraaf-Witten sum on a triangulation
Pictures based on a *-1-category n=1	Isotopy plus relations coming from the category	Euler characteristic of a surface
Pictures based on a pivotal 2-category n = 2	Isotopy plus relations coming from the category	Turaev-Viro sum
Pictures based on a modular ribbon category (a disklike 3-category) n=3	Isotopy plus relations coming from the category	For a generic cell handle decomposition of a 4-manifold, the Crane-Yetter state sum For 2-handles attached to the 4-ball, the Witten-Reshetikhin-Turaev surgery formula For a “special spine” of a 4-manifold, the Turaev shadow state sum

Part 2

Goal: Apply the machinery from part 1 to interesting new examples

Contact structures as a TQFT (n=3)

- $\mathcal{C}(M^3) = \{\text{contact structures on } M\}$
- $\mathcal{C}(Y^2) = \{\text{germs of contact structures on } Y \times [-\varepsilon, \varepsilon]\}$
- local relations: (1) isotopy, (2) overtwisted disk ~ 0.
- A basis for $A(M^3, c)$ is the set of tight contact structures on M restricting to c on ∂M, modulo isotopy.

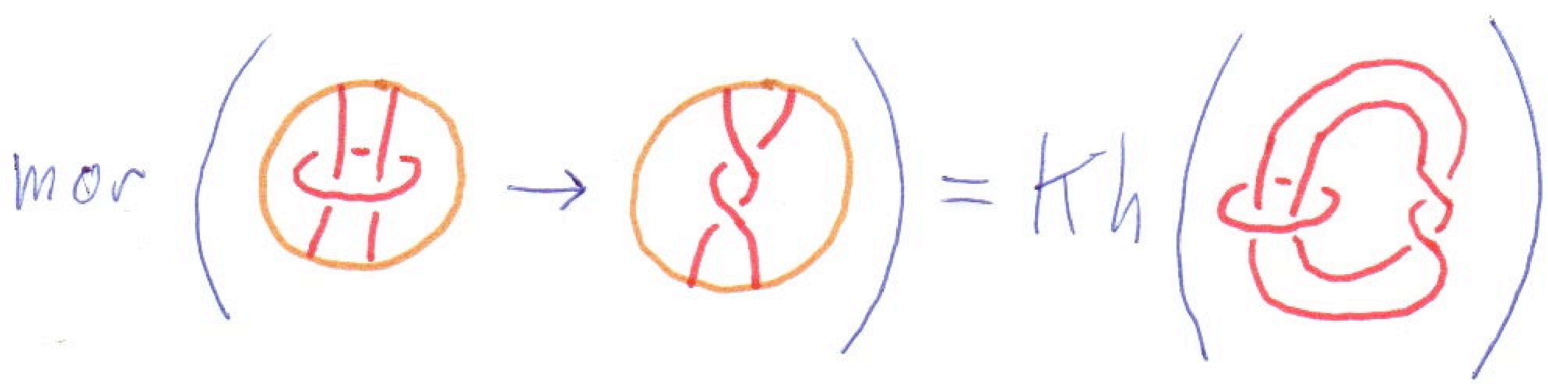


Khovanov homology as a TQFT (n=4)

Khovanov homology has the structure of a disk-like 4-category:

- 0-morphisms: nothing in B^0
- 1-morphisms: nothing in B^1
- 2-morphisms: points in B^2
- 3-morphisms: tangles in B^3
- 4-morphisms $T_1 \to T_2$: $\mathrm{Kh}(\overline{T_1} \bullet T_2)$ (a bigraded module)

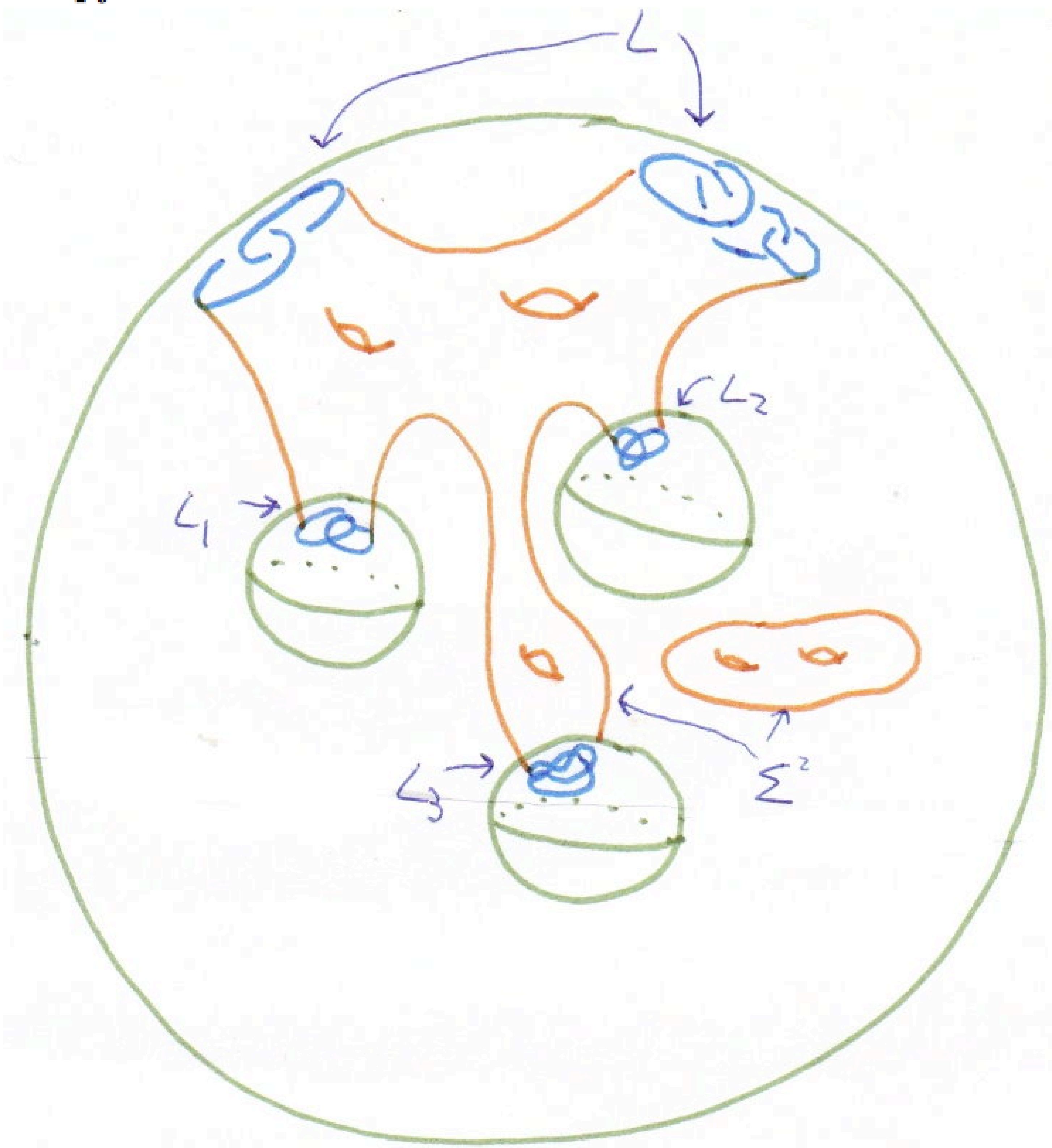
Composition of 0, 1, 2, 3-morphisms is obvious, as is duality.
Composition and duality of 4-morphisms...



Operadish product on Kh:

$$\mathrm{Kh}(L_1) \otimes \cdots \otimes \mathrm{Kh}(L_k) \to \mathrm{Kh}(L)$$

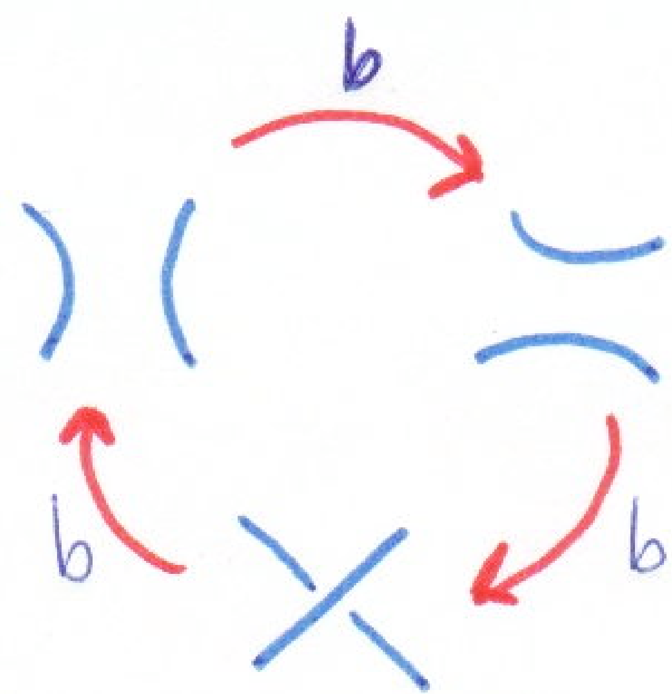
Invariant under isotopy.



Applying the above constructions, we get a 4+1-dimensional TQFT (minus the 5-dimension~~s~~al part). It assigns a bigraded $\mathbb{C}[\alpha]$ module $A_{\mathrm{Kh}}(W^4; L)$ to each 4-manifold W. $A_{\mathrm{Kh}}(B^4; L) \cong \mathrm{Kh}(L)$.

How to calculate?

For $\mathrm{Kh}(L)$ (a.k.a. $A_{\mathrm{Kh}}(B^4; L)$), one makes extensive use of the exact triangle (long exact sequence)

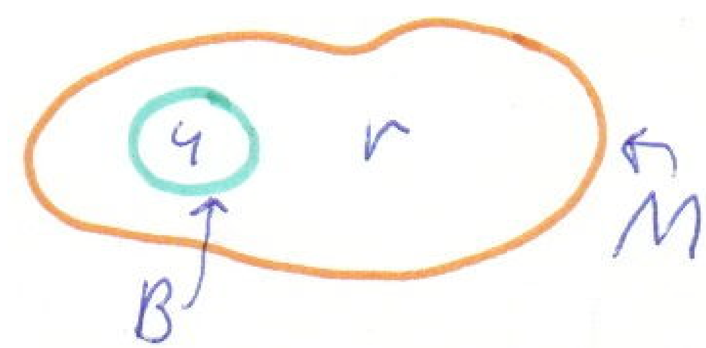


Blob homology

$$\left.\begin{array}{r} n\text{-manifold } M \\ n\text{-category } C \end{array}\right\} \longrightarrow \text{chain complex } \mathcal{B}_*(M,C)$$

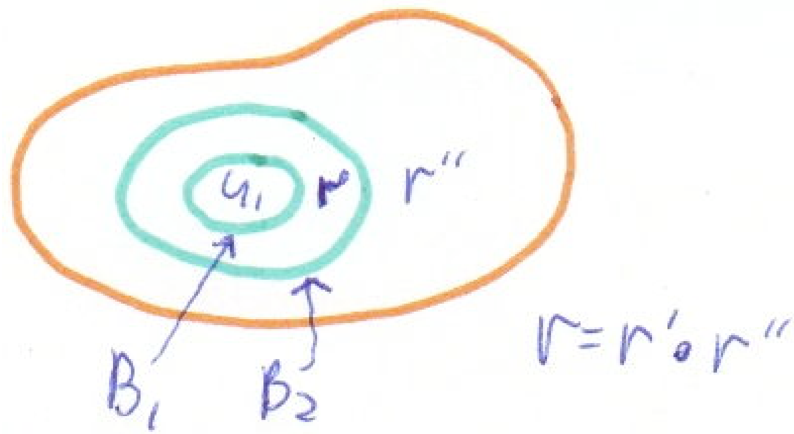
$$A_C(M) \overset{\text{def}}{=} \mathbb{C}[\mathcal{C}_C(M)]/\mathbb{C}[\{(B,u,r)\}]$$

where $B \subset M$, $u \in U(B)$, and $r \in \mathcal{C}(M \setminus B)$

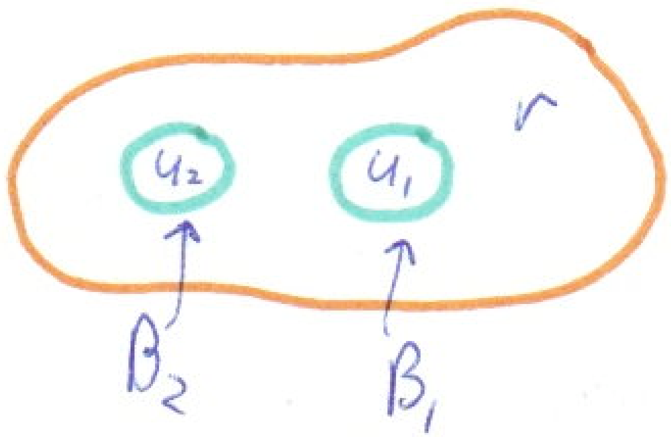


Replace quotient with resolution:

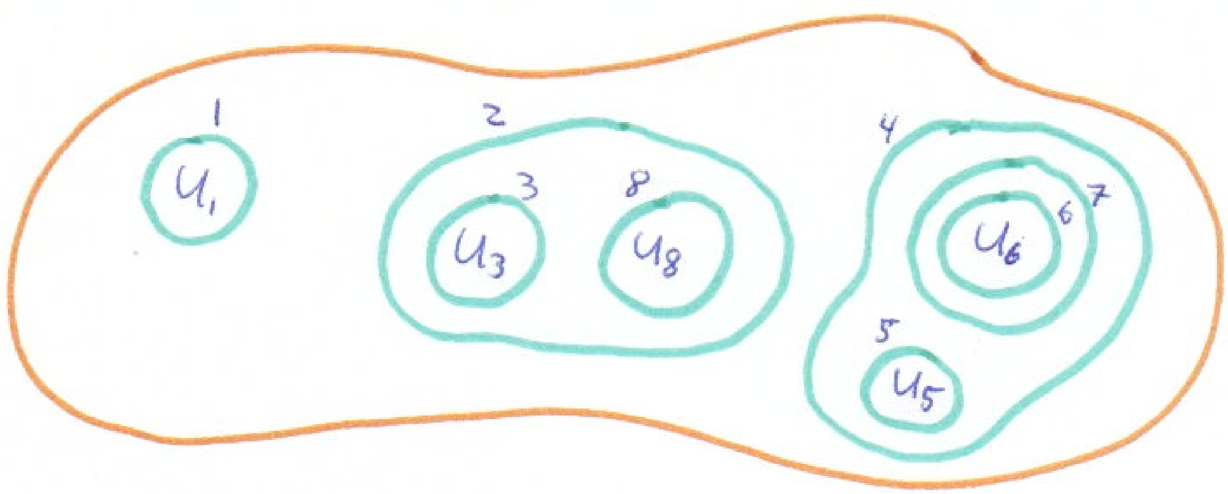
$$\begin{tikzcd} & \mathbb{C}[\{(B,u,r)\}] \arrow[d, equal] & \mathbb{C}[\mathcal{C}(M)] \arrow[d, equal] \\ {} \arrow[r, "\partial"] & \mathcal{B}_1(M,C) \arrow[r, "\partial"] & \mathcal{B}_0(M,C) \\ & (B,u,r) \arrow[r, mapsto, "\partial"] & u \circ r \end{tikzcd}$$



$$(B_1, B_2, u_1, r) \stackrel{\partial}{\longmapsto} (B_2, r' \bullet u_1, r'') - (B_1, u_1, r)$$



$$(B_1, B_2, u_1, u_2, r) \stackrel{\partial}{\longmapsto} (B_2, u_2, u_1 \bullet r) - (B_1, u_1, u_2 \bullet r)$$



$\mathcal{B}_k(M, C)$ is defined to be finite linear combinations of k-blob diagrams. A k-blob diagram consists of k blobs (balls) $B_0, \ldots, B_{k-1}$ in M. Each pair B_i and B_j is required to be either disjoint or nested. Each innermost blob B_i is equipped with a null field $u_i \in U$. There is also a C-picture r on the complement of the innermost blobs. The boundary map $\partial : \mathcal{B}_k(M, C) \to \mathcal{B}_{k-1}(M, C)$ is defined to be the alternating sum of forgetting the i-th blob.

Easy consequences of the definition

- **Functoriality.** The blob complex is functorial with respect to diffeomorphisms. That is, fixing C, the association

$$M \mapsto \mathcal{B}_*(M, C)$$

  is a functor from n-manifolds and diffeomorphisms between them to chain complexes and isomorphisms between them.

- **Contractibility for B^n.** The blob complex of the n-ball, $\mathcal{B}_*(B^n, C)$, is quasi-isomorphic to the 1-step complex consisting of n-morphisms of C. (The domain and range of the n-morphisms correspond to the boundary conditions on B^n. Both are suppressed from the notation.) Thus $\mathcal{B}_*(B^n, C)$ can be thought of as a free resolution of C.

- **Disjoint union.** There is a natural isomorphism

$$\mathcal{B}_*(M_1 \sqcup M_2, C) \cong \mathcal{B}_*(M_1, C) \otimes \mathcal{B}_*(M_2, C).$$

- **Gluing.** Let M_1 and M_2 be n-manifolds, with Y a codimension-0 submanifold of ∂M_1 and $-Y$ a codimension-0 submanifold of ∂M_2. Then there is a chain map

$$\mathrm{gl}_Y : \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \to \mathcal{B}_*(M_1 \cup_Y M_2).$$

Nontriviality

$$\mathcal{B}_k(M, C)$$

- $\boldsymbol{k = 0}$**.** $H_0(\mathcal{B}_*(M, C))$ is isomorphic to $A_C(M)$, the dual Hilbert space of the n+1-dimensional TQFT based on C.
- $\boldsymbol{M = S^1}$**.** When C is a 1-category, $\mathcal{B}_*(S^1, C)$ is homotopy equivalent to the Hochschild complex $\mathrm{Hoch}_*(C)$.
- **Very simple $\boldsymbol{n}$-categories.** If C is a polynomial algebra viewed as an n-category, then $\mathcal{B}_*(M^n, C)$ is homotopy equivalent to singular chains on a configuration space of M (possibly mod a generalized diagonal).

- **Evaluation map.** There is an 'evaluation' chain map

$$\mathrm{ev}_M : C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \to \mathcal{B}_*(M).$$

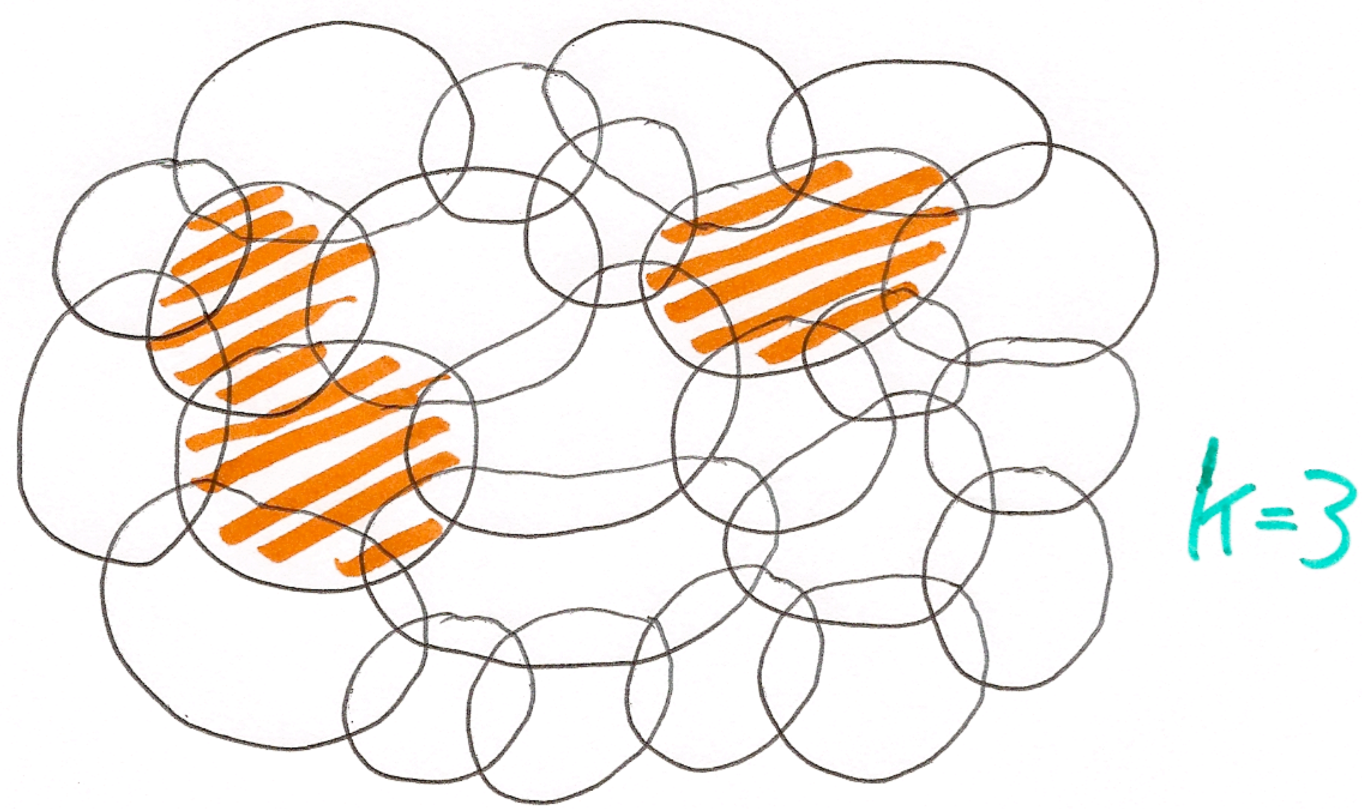
(Here $C_*(\mathrm{Diff}(M))$ is the singular chain complex of the space of diffeomorphisms of M, fixed on ∂M.)

Restricted to $C_0(\mathrm{Diff}(M))$ this is just the action of diffeomorphisms described above. Further, for any codimension-1 submanifold $Y \subset M$ dividing M into $M_1 \cup_Y M_2$, the following diagram (using the gluing maps described above) commutes.

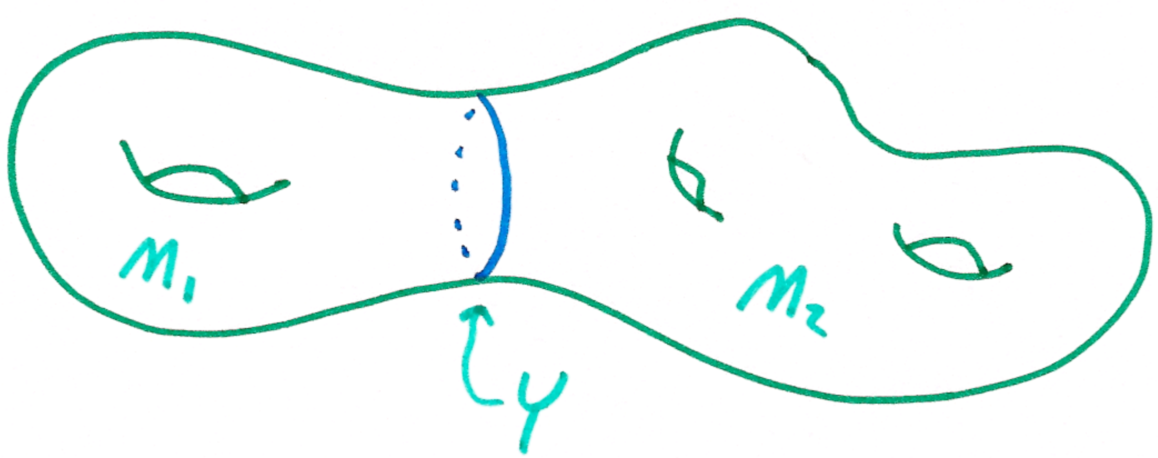
$$\begin{tikzcd}
C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \arrow[r, "\mathrm{ev}_M"] & \mathcal{B}_*(M) \\
C_*(\mathrm{Diff}(M)) \otimes C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y^{\mathrm{Diff}} \otimes \mathrm{gl}_Y"] \arrow[r, bend right, "\mathrm{ev}_{M_1} \otimes \mathrm{ev}_{M_2}"'] & \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y"']
\end{tikzcd}$$

In fact, up to homotopy the evaluation maps are uniquely characterized by these two properties.

Lemma. Let $f : P^k \times M \to M$ be a k-parameter family of diffeomorphisms and $\{U_i\}$ be an open cover of M. Then f is homotopic in $C_k(\mathrm{Diff}(M))$ to $\sum f_j$, where is each f_j is supported on a union of at most k of the U_i's. (That is, if $f_j : Q^k \times M \to M$, then $f(q, x) = f(q', x)$ for all q, q' unless x is in the aforementioned union of U_i's.)



A_∞ categories for $n{-}1$-manifolds. For Y an $n{-}1$-manifold, the blob complex $\mathcal{B}_*(Y \times I, C)$ has the structure of an A_∞ category. The multiplication (m_2) is given by stacking copies of the cylinder $Y \times I$ together. The higher m_i's are obtained by applying the evaluation map to $i{-}2$-dimensional families of diffeomorphisms in $\mathrm{Diff}(I) \subset \mathrm{Diff}(Y \times I)$. Furthermore, $\mathcal{B}_*(M, C)$ affords a representation of the A_∞ category $\mathcal{B}_*(\partial M \times I, C)$.



Gluing formula. Let $Y \subset M$ divide M into manifolds M_1 and M_2. Let $A(Y)$ be the A_∞ category $\mathcal{B}_*(Y \times I, C)$. Then $\mathcal{B}_*(M_1, C)$ affords a right representation of $A(Y)$, $\mathcal{B}_*(M_2, C)$ affords a left representation of $A(Y)$, and $\mathcal{B}_*(M, C)$ is homotopy equivalent to $\mathcal{B}_*(M_1, C) \otimes_{A(Y)} \mathcal{B}_*(M_2, C)$.

(More generally, can define an A_∞ k-category for $n{-}k$-manifolds, and prove a similar gluing theorem.)

There is a version of the blob complex for C an A_∞ n-category.
Let $\pi^\infty_{\leq n}(W)$ denote the A_∞ n-category based on maps $B^n \to W$. (The case $n = 1$ is the usual A_∞ category of paths in W.)

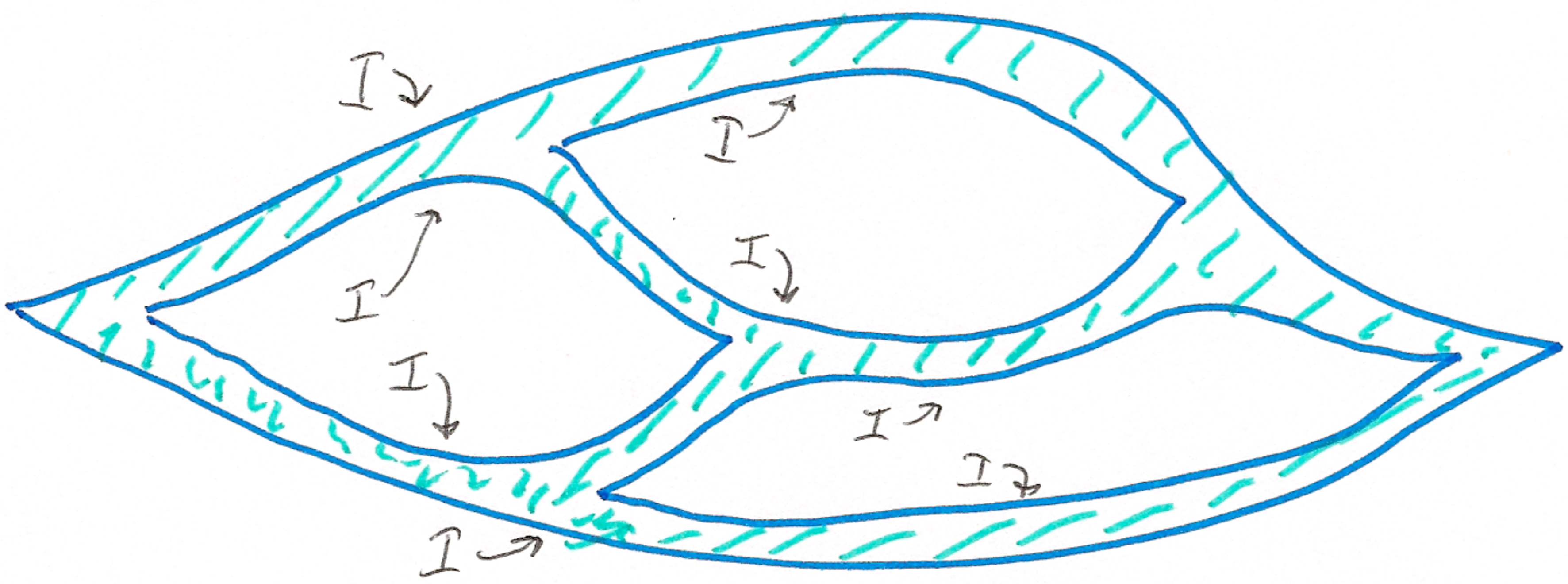
$\mathcal{B}_*(M, \pi^\infty_{\leq n}(W))$ is homotopy equivalent to $C_*(\{\text{maps } M \to W\})$.

Let $M^n = Y^{n-k} \times W^k$ and let C be an n-category. Let $A_*(Y)$ be the A_∞ k-category associated to Y via blob homology. Then

$$\mathcal{B}_*(Y^{n-k} \times W^k, C) \simeq \mathcal{B}_*(W, A_*(Y)).$$

There is a similar result for general fiber bundles.

Deligne conjecture



$$C_*(LD_k) \otimes \overbrace{Hoch^*(C,C) \otimes \cdots \otimes Hoch^*(C,C)}^{k \text{ copies}} \to Hoch^*(C,C)$$

$$C_*(FG_k) \otimes \mathrm{Map}_\infty(\mathcal{B}_*(I,C), \mathcal{B}_*(I,C)) \otimes \cdots$$
$$\otimes \mathrm{Map}_\infty(\mathcal{B}_*(I,C), \mathcal{B}_*(I,C)) \to \mathrm{Map}_\infty(\mathcal{B}_*(I,C), \mathcal{B}_*(I,C))$$

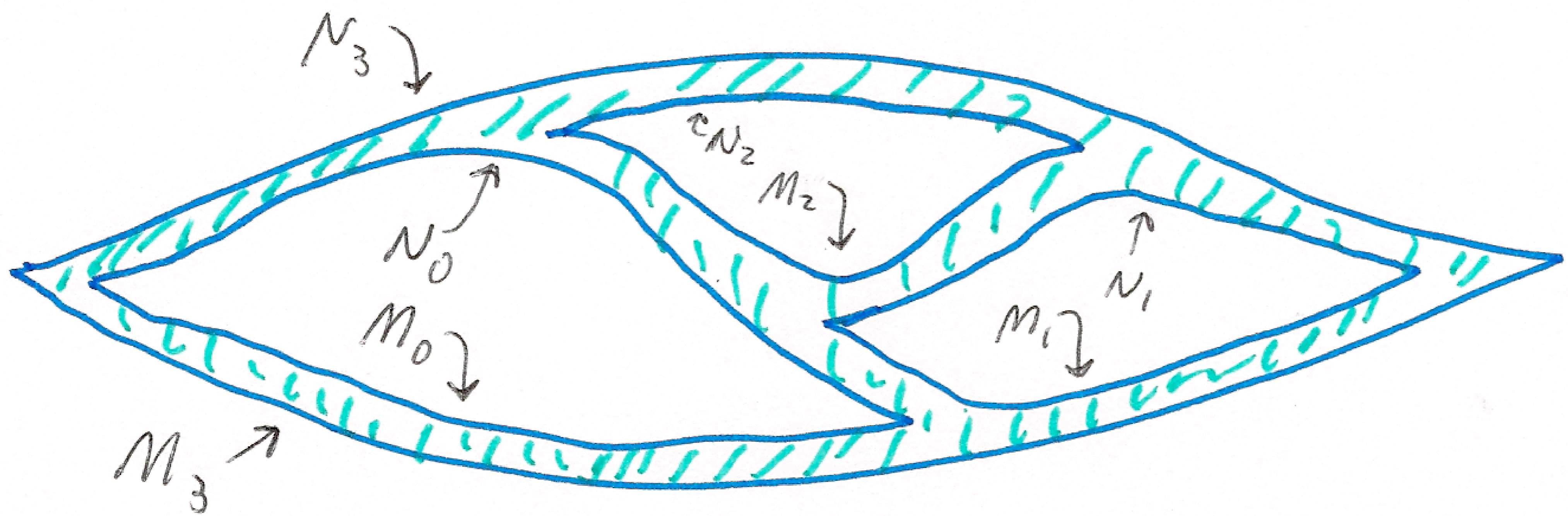
$LD_\bullet$ is little disks operad

$FG_\bullet$ is fat graph operad

$C_*(\cdot)$ denotes singular chains

$\mathrm{Map}_\infty(\cdot,\cdot)$ denotes A_∞ maps between A_∞ modules

Higher dimensional Deligne conjecture



$$C_*(FG^n_{\overline{M},\overline{N}}) \otimes \mathrm{Map}_\infty(\mathcal{B}_*(M_0,C),\mathcal{B}_*(N_0,C)) \otimes \cdots$$
$$\otimes \mathrm{Map}_\infty(\mathcal{B}_*(M_{k-1},C),\mathcal{B}_*(N_{k-1},C)) \to \mathrm{Map}_\infty(\mathcal{B}_*(M_k,C),\mathcal{B}_*(N_k,C))$$

$FG^n_\bullet$ is n-dimensional fat graph operad

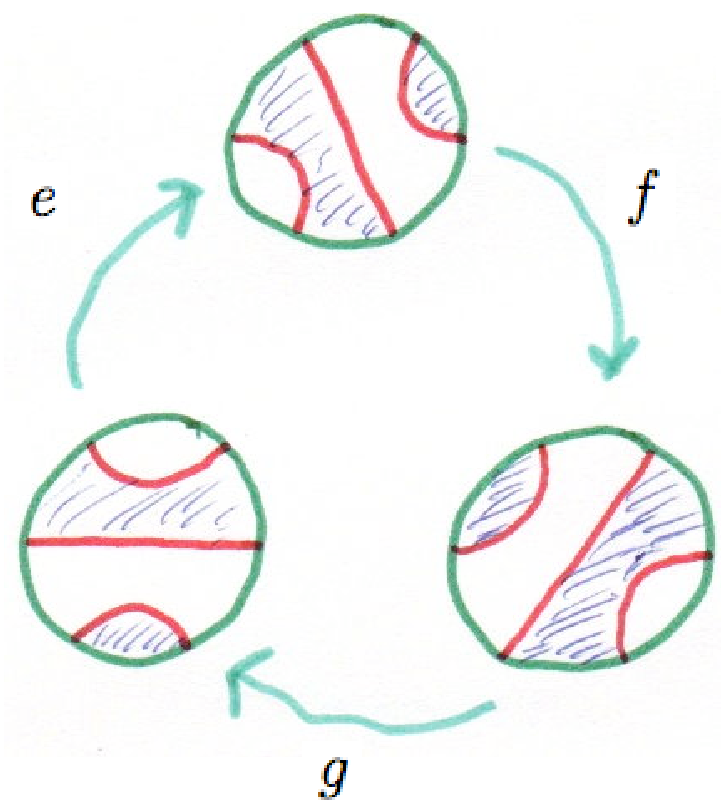
$C_*(\cdot)$ denotes singular chains

$\mathrm{Map}_\infty(\cdot,\cdot)$ denotes A_∞ maps between A_∞ modules

(Note: Evaluation map is case $k=0$.)

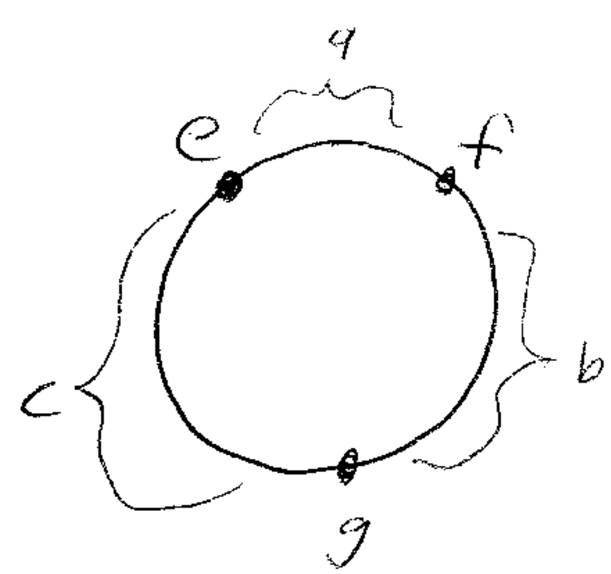
Let C be the contact category, with overtwisted disks set to zero. Let $CS(M)$ denote the space of (all) contact structures on M and let $CS_\epsilon(M)$ denote the space of all contact structures which contain an overtwisted disk of diameter $\leq \epsilon$. Then

Guess: $\mathcal{B}_*(M, C) \simeq C_*(CS(M), CS_\epsilon(M))$ (as $\epsilon \to 0$).



Nontrivial class in $H\mathcal{B}_2(D^2 \times S^1, 6 \text{ longitudes})$:

$$(\Delta^2, \partial) \to (CS, CS_\epsilon)$$



$a + b + c = 2\pi$

4. Wild speculation

Sullivan: "Be bold"

Consider again the A_∞ n-category $\pi^\infty_{\leq n}(W)$. Recall that $H\mathcal{B}_*(M, \pi^\infty_{\leq n}(W))$ computes $H_*(\{\text{maps } M \to W\})$.

In order for the proof to work, we need a well behaved functor $\mathrm{Map}(\cdot, W)$ from manifolds to spaces (and thence to chain complexes). ("Well behaved" means, among other things, that gluing of manifolds corresponds to fibered products of the associated spaces.)

Let's replace this functor with one that associates to a manifold M^n (perhaps with extra structure) the space $S(M)$ of solutions to some PDEs (e.g. Seiberg-Witten equations). To Y^k associate germs of solutions on $S(Y \times B^{n-k}_\epsilon)$.

Under the mild assumption that these solutions glue together via a fibered product rule, then we should be able to construct an A_∞ n-category C such that $H\mathcal{B}_*(M, C)$ is computes $H_*(S(M))$. (In the case of SW theory, we should replace singular chains with fancier $\frac{\infty}{2}$-dimensional chains.)