The Blob Complex

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Abstract

Given an n-manifold M and an n-category $\mathcal{C}$, we define a chain complex (the "blob complex") $\mathcal{B}_*(M;\mathcal{C})$. The blob complex can be thought of as a derived category version of the Hilbert space of a TQFT, or as a generalization of Hochschild homology to n-categories and n-manifolds. It enjoys a number of nice formal properties, including a higher dimensional generalization of Deligne's conjecture about the action of the little disks operad on Hochschild cochains. Along the way, we give a definition of a weak n-category with strong duality which is particularly well suited for work with TQFTs.

Contents

1 Introduction

We construct a chain complex $\mathcal{B}_*(M;\mathcal{C})$ — the "blob complex" — associated to an n-manifold M and a linear n-category $\mathcal{C}$ with strong duality. This blob complex provides a simultaneous generalization of several well known constructions:

- The 0-th homology $H_0(\mathcal{B}_*(M;\mathcal{C}))$ is isomorphic to the usual topological quantum field theory invariant of M associated to $\mathcal{C}$. (See Proposition 3.1.1 later in the introduction and §2.4.)
- When $n = 1$ and $\mathcal{C}$ is just a 1-category (e.g. an associative algebra), the blob complex $\mathcal{B}_*(S^1;\mathcal{C})$ is quasi-isomorphic to the Hochschild complex $\mathrm{Hoch}_*(\mathcal{C})$. (See Theorem 4.1.1 and §4.)
- When $\mathcal{C}$ is $\pi^\infty_{\le n}(T)$, the A_∞ version of the fundamental n-groupoid of the space T (Example 6.2.7), $\mathcal{B}_*(M;\mathcal{C})$ is homotopy equivalent to $C_*(\mathrm{Maps}(M \to T))$, the singular chains on the space of maps from M to T. (See Theorem 7.3.1.)

The blob complex definition is motivated by the desire for a derived analogue of the usual TQFT Hilbert space (replacing the quotient of fields by local relations with some sort of resolution), and for a generalization of Hochschild homology to higher n-categories. One can think of it as the push-out of these two familiar constructions. More detailed motivations are described in §1.2.

The blob complex has good formal properties, summarized in §1.3. These include an action of $C_*(\mathrm{Homeo}(M))$, extending the usual $\mathrm{Homeo}(M)$ action on the TQFT space H_0 (Theorem 5.2.1) and a gluing formula allowing calculations by cutting manifolds into smaller parts (Theorem 7.2.1).

We expect applications of the blob complex to contact topology and Khovanov homology but do not address these in this paper.

Throughout, we have resisted the temptation to work in the greatest possible generality. (Don't worry, it wasn't that hard.) In most of the places where we say "set" or "vector space", any symmetric monoidal category with sufficient limits and colimits would do. We could also replace many of our chain complexes with topological spaces (or indeed, work at the generality of model categories).

1.1 Structure of the paper

The subsections of the introduction explain our motivations in defining the blob complex (see §1.2), summarize the formal properties of the blob complex (see §1.3), describe known specializations (see §1.4), and outline the major results of the paper (see §1.5 and §1.6).

The first part of the paper (sections §2–§5) gives the definition of the blob complex, and establishes some of its properties. There are many alternative definitions of n-categories, and part of the challenge of defining the blob complex is simply explaining what we mean by an "n-category with strong duality" as one of the inputs. At first we entirely avoid this problem by introducing the notion of a "system of fields", and define the blob complex associated to an n-manifold and an n-dimensional system of fields. We sketch the construction of a system of fields from a *-1-category and from a pivotal 2-category.

Nevertheless, when we attempt to establish all of the observed properties of the blob complex, we find this situation unsatisfactory. Thus, in the second part of the paper (§§6-7) we give yet another definition of an n-category, or rather a definition of an n-category with strong duality. (Removing the duality conditions from our definition would make it more complicated rather than less.) We call these "topological n-categories", to differentiate them from previous versions. Moreover, we find that we need analogous A_∞ n-categories, and we define these as well following very similar axioms.

The basic idea is that each potential definition of an n-category makes a choice about the "shape" of morphisms. We try to be as lax as possible: a topological n-category associates a vector space to every B homeomorphic to the n-ball. These vector spaces glue together associatively, and we require that there is an action of the homeomorphism groupoid. For an A_∞ n-category, we associate a chain complex instead of a vector space to each such B and ask that the action of homeomorphisms extends to a suitably defined action of the complex of singular chains of homeomorphisms. The axioms for an A_∞ n-category are designed to capture two main examples: the blob complexes of n-balls labelled by a topological n-category, and the complex $C_*(\mathrm{Maps}(- \to T))$ of maps to a fixed target space T.

In §6.3 we explain how to construct a system of fields from a topological n-category (using a colimit along certain decompositions of a manifold into balls). With this in hand, we write $\mathcal{B}_*(M; \mathcal{C})$ to indicate the blob complex of a manifold M with the system of fields constructed from the n-category $\mathcal{C}$. In §7 we give an alternative definition of the blob complex for an A_∞ n-category on an n-manifold (analogously, using a homotopy colimit). Using these definitions, we show how to use the blob complex to "resolve" any topological n-category as an A_∞ n-category, and relate the first and second definitions of the blob complex. We use the blob complex for A_∞ n-categories to establish important properties of the blob complex (in both variants), in particular the "gluing formula" of Theorem 7.2.1 below.

The relationship between all these ideas is sketched in Figure 1.

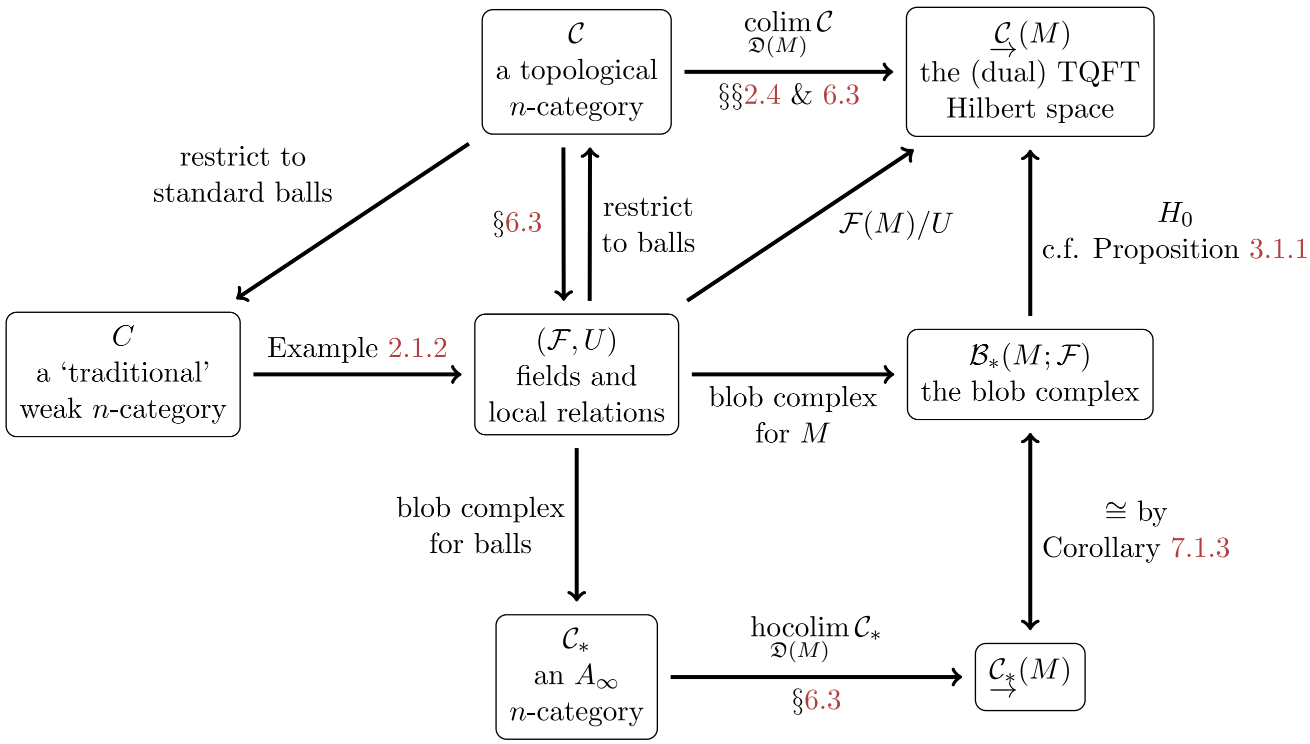

Figure 1: The main gadgets and constructions of the paper.

Later sections address other topics. Section §8 gives a higher dimensional generalization of the Deligne conjecture (that the little discs operad acts on Hochschild cochains) in terms of the blob complex. The appendices prove technical results about $C_*(\mathrm{Homeo}(M))$ and make connections between our definitions of n-categories and familiar definitions for $n = 1$ and $n = 2$, as well as relating the $n = 1$ case of our A_∞ n-categories with usual A_∞ algebras.

1.2 Motivations

We will briefly sketch our original motivation for defining the blob complex.

As a starting point, consider TQFTs constructed via fields and local relations. (See §2 or [14].) This gives a satisfactory treatment for semisimple TQFTs (i.e. TQFTs for which the cylinder 1-category associated to an $n-1$-manifold Y is semisimple for all Y).

For non-semi-simple TQFTs, this approach is less satisfactory. Our main motivating example (though we will not develop it in this paper) is the (decapitated) 4+1-dimensional TQFT associated to Khovanov homology. It associates a bigraded vector space $A_{Kh}(W^4, L)$ to a 4-manifold W together with a link $L \subset \partial W$. The original Khovanov homology of a link in S^3 is recovered as

$A_{Kh}(B^4, L)$.

How would we go about computing $A_{Kh}(W^4, L)$? For the Khovanov homology of a link in S^3 the main tool is the exact triangle (long exact sequence) relating resolutions of a crossing. Unfortunately, the exactness breaks if we glue B^4 to itself and attempt to compute $A_{Kh}(S^1 \times B^3, L)$. According to the gluing theorem for TQFTs, gluing along $B^3 \subset \partial B^4$ corresponds to taking a coend (self tensor product) over the cylinder category associated to B^3 (with appropriate boundary conditions). The coend is not an exact functor, so the exactness of the triangle breaks.

The obvious solution to this problem is to replace the coend with its derived counterpart, Hochschild homology. This presumably works fine for $S^1 \times B^3$ (the answer being the Hochschild homology of an appropriate bimodule), but for more complicated 4-manifolds this leaves much to be desired. If we build our manifold up via a handle decomposition, the computation would be a sequence of derived coends. A different handle decomposition of the same manifold would yield a different sequence of derived coends. To show that our definition in terms of derived coends is well-defined, we would need to show that the above two sequences of derived coends yield isomorphic answers, and that the isomorphism does not depend on any choices we made along the way. This is probably not easy to do.

Instead, we would prefer a definition for a derived version of $A_{Kh}(W^4, L)$ which is manifestly invariant. (That is, a definition that does not involve choosing a decomposition of W. After all, one of the virtues of our starting point — TQFTs via field and local relations — is that it has just this sort of manifest invariance.)

The solution is to replace $A_{Kh}(W^4, L)$, which is a quotient

$$\text{linear combinations of fields} \Big/ \text{local relations},$$

with an appropriately free resolution (the blob complex)

$$\cdots \to \mathcal{B}_2(W, L) \to \mathcal{B}_1(W, L) \to \mathcal{B}_0(W, L).$$

Here $\mathcal{B}_0$ is linear combinations of fields on W, $\mathcal{B}_1$ is linear combinations of local relations on W, $\mathcal{B}_2$ is linear combinations of relations amongst relations on W, and so on. We now have a short exact sequence of chain complexes relating resolutions of the link L (c.f. Lemma 4.1.5 which shows exactness with respect to boundary conditions in the context of Hochschild homology).

1.3 Formal properties

The blob complex enjoys the following list of formal properties.

Property 1.3.1 (Functoriality)**.** *The blob complex is functorial with respect to homeomorphisms. That is, for a fixed n-dimensional system of fields $\mathcal{F}$, the association*

$$X \mapsto \mathcal{B}_*(X; \mathcal{F})$$

is a functor from n-manifolds and homeomorphisms between them to chain complexes and isomorphisms between them.

As a consequence, there is an action of $\mathrm{Homeo}(X)$ on the chain complex $\mathcal{B}_*(X; \mathcal{F})$; this action is extended to all of $C_*(\mathrm{Homeo}(X))$ in Theorem 5.2.1 below.

The blob complex is also functorial with respect to $\mathcal{F}$, although we will not address this in detail here.

Property 1.3.2 (Disjoint union)**.** *The blob complex of a disjoint union is naturally isomorphic to the tensor product of the blob complexes.*

$$\mathcal{B}_*(X_1 \sqcup X_2) \cong \mathcal{B}_*(X_1) \otimes \mathcal{B}_*(X_2)$$

If an n-manifold X contains $Y \sqcup Y^{\text{op}}$ as a codimension 0 submanifold of its boundary, write $X_{\text{gl}} = X \bigcup_{Y} \circlearrowleft$ for the manifold obtained by gluing together Y and Y^{op}. Note that this includes the case of gluing two disjoint manifolds together.

Property 1.3.3 (Gluing map)**.** *Given a gluing $X \to X_{\text{gl}}$, there is a natural map*

$$\mathcal{B}_*(X) \to \mathcal{B}_*(X_{\text{gl}})$$

(natural with respect to homeomorphisms, and also associative with respect to iterated gluings).

Property 1.3.4 (Contractibility)**.** *With field coefficients, the blob complex on an n-ball is contractible in the sense that it is homotopic to its 0-th homology. Moreover, the 0-th homology of balls can be canonically identified with the vector spaces associated by the system of fields $\mathcal{F}$ to balls.*

$$\mathcal{B}_*(B^n; \mathcal{F}) \xrightarrow[qi]{\cong} H_0(\mathcal{B}_*(B^n; \mathcal{F})) \xrightarrow{\cong} A_{\mathcal{F}}(B^n)$$

Properties 1.3.1 will be immediate from the definition given in §3.1, and we'll recall it at the appropriate point there. Properties 1.3.2, 1.3.3 and 1.3.4 are established in §3.2.

1.4 Specializations

The blob complex is a simultaneous generalization of the TQFT skein module construction and of Hochschild homology.

Proposition 3.1.1 (Skein modules)**.** *The 0-th blob homology of X is the usual (dual) TQFT Hilbert space (a.k.a. skein module) associated to X by $\mathcal{F}$. (See §2.3.)*

$$H_0(\mathcal{B}_*(X; \mathcal{F})) \cong A_{\mathcal{F}}(X)$$

Theorem 4.1.1 (Hochschild homology when $X = S^1$)**.** *The blob complex for a 1-category $\mathcal{C}$ on the circle is quasi-isomorphic to the Hochschild complex.*

$$\mathcal{B}_*(S^1; \mathcal{C}) \xrightarrow[qi]{\cong} \text{Hoch}_*(\mathcal{C}).$$

Proposition 3.1.1 is immediate from the definition, and Theorem 4.1.1 is established in §4.

1.5 Structure of the blob complex

In the following $C_*(\text{Homeo}(X))$ is the singular chain complex of the space of homeomorphisms of X, fixed on ∂X.

Theorem 5.2.1 ($C_*(\text{Homeo}(-))$ action)**.** *There is a chain map*

$$e_X : C_*(\text{Homeo}(X)) \otimes \mathcal{B}_*(X) \to \mathcal{B}_*(X).$$

such that

1. *Restricted to $C_0(\mathrm{Homeo}(X))$ this is the action of homeomorphisms described in Property 1.3.1.*
2. *For any codimension 0-submanifold $Y \sqcup Y^{op} \subset \partial X$ the following diagram (using the gluing maps described in Property 1.3.3) commutes (up to homotopy).*

$$\begin{array}{ccc} C_*(\mathrm{Homeo}(X)) \otimes \mathcal{B}_*(X) & \xrightarrow{\quad e_X \quad} & \mathcal{B}_*(X) \\ \big\downarrow {\scriptstyle \mathrm{gl}_Y^{\mathrm{Homeo}} \otimes \mathrm{gl}_Y} & & \big\downarrow {\scriptstyle \mathrm{gl}_Y} \\ C_*(\mathrm{Homeo}(X \bigcup_Y \circlearrowleft)) \otimes \mathcal{B}_*(X \bigcup_Y \circlearrowleft) & \xrightarrow{e_{(X \bigcup_Y \circlearrowleft)}} & \mathcal{B}_*(X \bigcup_Y \circlearrowleft) \end{array}$$

Further,

Theorem 5.2.2. *The chain map of Theorem 5.2.1 is associative, in the sense that the following diagram commutes (up to homotopy).*

$$\begin{array}{ccc} C_*(\mathrm{Homeo}(X)) \otimes C_*(\mathrm{Homeo}(X)) \otimes \mathcal{B}_*(X) & \xrightarrow{\mathbf{1} \otimes e_X} & C_*(\mathrm{Homeo}(X)) \otimes \mathcal{B}_*(X) \\ \big\downarrow {\scriptstyle \circ \otimes \mathbf{1}} & & \big\downarrow {\scriptstyle e_X} \\ C_*(\mathrm{Homeo}(X)) \otimes \mathcal{B}_*(X) & \xrightarrow{\quad e_X \quad} & \mathcal{B}_*(X) \end{array}$$

Since the blob complex is functorial in the manifold X, this is equivalent to having chain maps

$$ev_{X \to Y} : C_*(\mathrm{Homeo}(X \to Y)) \otimes \mathcal{B}_*(X) \to \mathcal{B}_*(Y)$$

for any homeomorphic pair X and Y, satisfying corresponding conditions.

In §6 we introduce the notion of topological n-categories, from which we can construct systems of fields. Below, when we talk about the blob complex for a topological n-category, we are implicitly passing first to this associated system of fields. Further, in §6 we also have the notion of an A_∞ n-category. In that section we describe how to use the blob complex to construct A_∞ n-categories from topological n-categories:

Example 6.2.8 (Blob complexes of products with balls form an A_∞ n-category)**.** *Let $\mathcal{C}$ be a topological n-category. Let Y be an $n{-}k$-manifold. There is an A_∞ k-category $\mathcal{B}_*(Y;\mathcal{C})$, defined on each m-ball D, for $0 \le m < k$, to be the set*

$$\mathcal{B}_*(Y;\mathcal{C})(D) = \mathcal{C}(Y \times D)$$

and on k-balls D to be the set

$$\mathcal{B}_*(Y;\mathcal{C})(D) = \mathcal{B}_*(Y \times D;\mathcal{C}).$$

(When $m = k$ the subsets with fixed boundary conditions form a chain complex.) These sets have the structure of an A_∞ k-category, with compositions coming from the gluing map in Property 1.3.3 and with the action of families of homeomorphisms given in Theorem 5.2.1.

Remark. Perhaps the most interesting case is when Y is just a point; then we have a way of building an A_∞ n-category from a topological n-category. We think of this A_∞ n-category as a free resolution.

There is a version of the blob complex for $\mathcal{C}$ an A_∞ n-category instead of a topological n-category; this is described in §7. The definition is in fact simpler, almost tautological, and we use a different notation, $\underrightarrow{\mathcal{C}}(M)$. The next theorem describes the blob complex for product manifolds, in terms of the A_∞ blob complex of the A_∞ n-categories constructed as in the previous example.

Theorem 7.1.1 (Product formula)**.** *Let W be a k-manifold and Y be an $n-k$ manifold. Let $\mathcal{C}$ be an n-category. Let $\mathcal{B}_*(Y;\mathcal{C})$ be the A_∞ k-category associated to Y via blob homology (see Example 6.2.8). Then*

$$\mathcal{B}_*(Y \times W; \mathcal{C}) \simeq \underrightarrow{\mathcal{B}_*(Y;\mathcal{C})}(W).$$

The statement can be generalized to arbitrary fibre bundles, and indeed to arbitrary maps (see §7.1).

Fix a topological n-category $\mathcal{C}$, which we'll omit from the notation. Recall that for any $(n-1)$-manifold Y, the blob complex $\mathcal{B}_*(Y)$ is naturally an A_∞ category. (See Appendix C.3 for the translation between topological A_∞ 1-categories and the usual algebraic notion of an A_∞ category.)

Theorem 7.2.1 (Gluing formula)**.**

- *For any n-manifold X, with Y a codimension 0-submanifold of its boundary, the blob complex of X is naturally an A_∞ module for $\mathcal{B}_*(Y)$.*
- *For any n-manifold $X_{gl} = X \bigcup_Y \circlearrowleft$, the blob complex $\mathcal{B}_*(X_{gl})$ is the A_∞ self-tensor product of $\mathcal{B}_*(X)$ as an $\mathcal{B}_*(Y)$-bimodule:*

$$\mathcal{B}_*(X_{gl}) \simeq \mathcal{B}_*(X) \bigotimes_{\mathcal{B}_*(Y)}^{A_\infty} \circlearrowleft$$

Theorem 7.1.1 is proved in §7.1, and Theorem 7.2.1 in §7.2.

1.6 Applications

Finally, we give two applications of the above machinery.

Theorem 7.3.1 (Mapping spaces)**.** *Let $\pi^\infty_{\leq n}(T)$ denote the A_∞ n-category based on maps $B^n \to T$. (The case $n = 1$ is the usual A_∞-category of paths in T.) Then*

$$\mathcal{B}_*(X; \pi^\infty_{\leq n}(T)) \simeq C_*(\mathrm{Maps}(X \to T)).$$

This says that we can recover (up to homotopy) the space of maps to T via blob homology from local data. Note that there is no restriction on the connectivity of T. The proof appears in §7.3.

Theorem 8.0.2 (Higher dimensional Deligne conjecture)**.** *The singular chains of the n-dimensional surgery cylinder operad act on blob cochains. Since the little $n{+}1$-balls operad is a suboperad of the n-dimensional surgery cylinder operad, this implies that the little $n{+}1$-balls operad acts on blob cochains of the n-ball.*

See §8 for a full explanation of the statement, and the proof.

1.7 Thanks and acknowledgements

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2 TQFTs via fields

In this section we review the construction of TQFTs from fields and local relations. For more details see [14]. For our purposes, a TQFT is *defined* to be something which arises from this construction. This is an alternative to the more common definition of a TQFT as a functor on cobordism categories satisfying various conditions. A fully local ("down to points") version of the cobordism-functor TQFT definition should be equivalent to the fields-and-local-relations definition.

A system of fields is very closely related to an n-category. In one direction, Example 2.1.2 shows how to construct a system of fields from a (traditional) n-category. We do this in detail for $n = 1, 2$ (§2.2) and more informally for general n. In the other direction, our preferred definition of an n-category in §6 is essentially just a system of fields restricted to balls of dimensions 0 through n; one could call this the "local" part of a system of fields.

Since this section is intended primarily to motivate the blob complex construction of §3.1, we suppress some technical details. In §6 the analogous details are treated more carefully.

We only consider compact manifolds, so if $Y \subset X$ is a closed codimension 0 submanifold of X, then $X \setminus Y$ implicitly means the closure $\overline{X \setminus Y}$.

2.1 Systems of fields

Let $\mathcal{M}_k$ denote the category with objects unoriented PL manifolds of dimension k and morphisms homeomorphisms. (We could equally well work with a different category of manifolds — oriented, topological, smooth, spin, etc. — but for simplicity we will stick with unoriented PL.)

Fix a symmetric monoidal category $\mathcal{S}$. Fields on n-manifolds will be enriched over $\mathcal{S}$. Good examples to keep in mind are $\mathcal{S} = \mathbf{Set}$ or $\mathcal{S} = \mathbf{Vect}$. The presentation here requires that the objects of $\mathcal{S}$ have an underlying set, but this could probably be avoided if desired.

A n-dimensional *system of fields* in $\mathcal{S}$ is a collection of functors $\mathcal{C}_k : \mathcal{M}_k \to \mathbf{Set}$ for $0 \le k \le n$ together with some additional data and satisfying some additional conditions, all specified below.

Before finishing the definition of fields, we give two motivating examples of systems of fields.

Example 2.1.1. *Fix a target space T, and let $\mathcal{C}(X)$ be the set of continuous maps from X to T.*

Example 2.1.2. *Fix an n-category C, and let $\mathcal{C}(X)$ be the set of embedded cell complexes in X with codimension-j cells labeled by j-morphisms of C. One can think of such embedded cell complexes as dual to pasting diagrams for C. This is described in more detail in §2.2.*

Now for the rest of the definition of system of fields. (Readers desiring a more precise definition should refer to §6.1 and replace k-balls with k-manifolds.)

1. There are boundary restriction maps $\mathcal{C}_k(X) \to \mathcal{C}_{k-1}(\partial X)$, and these maps comprise a natural transformation between the functors $\mathcal{C}_k$ and $\mathcal{C}_{k-1} \circ \partial$. For $c \in \mathcal{C}_{k-1}(\partial X)$, we will denote by $\mathcal{C}_k(X; c)$ the subset of $\mathcal{C}(X)$ which restricts to c. In this context, we will call c a boundary condition.

2. The subset $\mathcal{C}_n(X; c)$ of top-dimensional fields with a given boundary condition is an object in our symmetric monoidal category $\mathcal{S}$. (This condition is of course trivial when $\mathcal{S} = \mathbf{Set}$.) If the objects are sets with extra structure (e.g. $\mathcal{S} = \mathbf{Vect}$ or $\mathbf{Kom}$), then this extra structure is considered part of the definition of $\mathcal{C}_n$. Any maps mentioned below between fields on n-manifolds must be morphisms in $\mathcal{S}$.

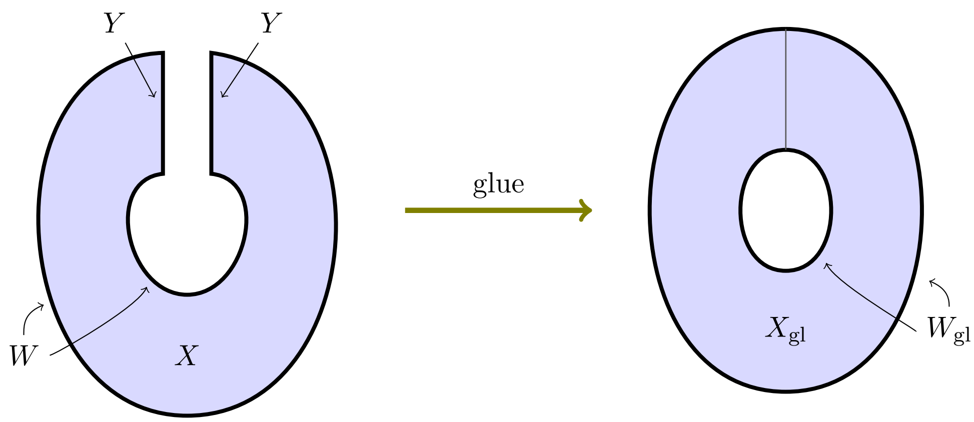


Figure 2: Gluing with corners

3. $\mathcal{C}_k$ is compatible with the symmetric monoidal structures on $\mathcal{M}_k$, **Set** and $\mathcal{S}$: $\mathcal{C}_k(X \sqcup W) \cong \mathcal{C}_k(X) \times \mathcal{C}_k(W)$, compatibly with homeomorphisms and restriction to boundary. We will call the projections $\mathcal{C}(X_1 \sqcup X_2) \to \mathcal{C}(X_i)$ restriction maps.

4. Gluing without corners. Let $\partial X = Y \sqcup Y \sqcup W$, where Y and W are closed $k{-}1$-manifolds. Let X_{gl} denote X glued to itself along the two copies of Y. Using the boundary restriction and disjoint union maps, we get two maps $\mathcal{C}_k(X) \to \mathcal{C}(Y)$, corresponding to the two copies of Y in ∂X. Let $\mathrm{Eq}_Y(\mathcal{C}_k(X))$ denote the equalizer of these two maps. Then (here's the axiom/definition part) there is an injective "gluing" map

$$\mathrm{Eq}_Y(\mathcal{C}_k(X)) \hookrightarrow \mathcal{C}_k(X_{\mathrm{gl}}),$$

and this gluing map is compatible with all of the above structure (actions of homeomorphisms, boundary restrictions, disjoint union). Furthermore, up to homeomorphisms of X_{gl} isotopic to the identity and collaring maps, the gluing map is surjective. We say that fields on X_{gl} in the image of the gluing map are transverse to Y or splittable along Y.

5. Gluing with corners. Let $\partial X = (Y \sqcup Y) \cup W$, where the two copies of Y are disjoint from each other and $\partial(Y \sqcup Y) = \partial W$. Let X_{gl} denote X glued to itself along the two copies of Y (Figure 2). Note that $\partial X_{\mathrm{gl}} = W_{\mathrm{gl}}$, where W_{gl} denotes W glued to itself (without corners) along two copies of ∂Y. Let $c_{\mathrm{gl}} \in \mathcal{C}_{k-1}(W_{\mathrm{gl}})$ be a be a splittable field on W_{gl} and let $c \in \mathcal{C}_{k-1}(W)$ be the cut open version of c_{gl}. Let $\mathcal{C}_k^c(X)$ denote the subset of $\mathcal{C}(X)$ which restricts to c on W. (This restriction map uses the gluing without corners map above.) Using the boundary restriction and gluing without corners maps, we get two maps $\mathcal{C}_k^c(X) \to \mathcal{C}(Y)$, corresponding to the two copies of Y in ∂X. Let $\mathrm{Eq}_Y^c(\mathcal{C}_k(X))$ denote the equalizer of these two maps. Then (here's the axiom/definition part) there is an injective "gluing" map

$$\mathrm{Eq}_Y^c(\mathcal{C}_k(X)) \hookrightarrow \mathcal{C}_k(X_{\mathrm{gl}}, c_{\mathrm{gl}}),$$

and this gluing map is compatible with all of the above structure (actions of homeomorphisms, boundary restrictions, disjoint union). Furthermore, up to homeomorphisms of X_{gl} isotopic to the identity and collaring maps, the gluing map is surjective. We say that fields in the image of the gluing map are transverse to Y or splittable along Y.

6. Product fields. There are maps $\mathcal{C}_{k-1}(Y) \to \mathcal{C}_k(Y \times I)$, denoted $c \mapsto c \times I$. These maps comprise a natural transformation of functors, and commute appropriately with all the structure maps above (disjoint union, boundary restriction, etc.). Furthermore, if $f : Y \times I \to Y \times I$ is a fiber-preserving homeomorphism covering $\bar{f} : Y \to Y$, then $f(c \times I) = \bar{f}(c) \times I$.

There are two notations we commonly use for gluing. One is

$$x_{\mathrm{gl}} \stackrel{\mathrm{def}}{=} \mathrm{gl}(x) \in \mathcal{C}(X_{\mathrm{gl}}),$$

for $x \in \mathcal{C}(X)$. The other is

$$x_1 \bullet x_2 \stackrel{\mathrm{def}}{=} \mathrm{gl}(x_1 \otimes x_2) \in \mathcal{C}(X_{\mathrm{gl}}),$$

in the case that $X = X_1 \sqcup X_2$, with $x_i \in \mathcal{C}(X_i)$.

Using the functoriality and product field properties above, together with boundary collar homeomorphisms of manifolds, we can define *collar maps* $\mathcal{C}(M) \to \mathcal{C}(M)$. Let M be an n-manifold and $Y \subset \partial M$ be a codimension zero submanifold of ∂M. Let $x \in \mathcal{C}(M)$ be a field on M and such that ∂x is splittable along ∂Y. Let c be x restricted to Y. Let $M \cup (Y \times I)$ denote M glued to $Y \times I$ along Y. Then we have the glued field $x \bullet (c \times I)$ on $M \cup (Y \times I)$. Let $f : M \cup (Y \times I) \to M$ be a collaring homeomorphism. Then we call the map $x \mapsto f(x \bullet (c \times I))$ a *collar map*. We call the equivalence relation generated by collar maps and homeomorphisms isotopic to the identity *extended isotopy*, since the collar maps can be thought of (informally) as the limit of homeomorphisms which expand an infinitesimally thin collar neighborhood of Y to a thicker collar neighborhood.

2.2 Systems of fields from n-categories

We now describe in more detail Example 2.1.2, systems of fields coming from embedded cell complexes labeled by n-category morphisms.

Given an n-category C with the right sort of duality (e.g. a pivotal 2-category, *-1-category), we can construct a system of fields as follows. Roughly speaking, $\mathcal{C}(X)$ will the set of all embedded cell complexes in X with codimension i cells labeled by i-morphisms of C. We'll spell this out for $n = 1, 2$ and then describe the general case.

This way of decorating an n-manifold with an n-category is sometimes referred to as a "string diagram". It can be thought of as (geometrically) dual to a pasting diagram. One of the advantages of string diagrams over pasting diagrams is that one has more flexibility in slicing them up in various ways. In addition, string diagrams are traditional in quantum topology. The diagrams predate by many years the terms "string diagram" and "quantum topology", e.g. [7, 8]

If X has boundary, we require that the cell decompositions are in general position with respect to the boundary — the boundary intersects each cell transversely, so cells meeting the boundary are mere half-cells. Put another way, the cell decompositions we consider are dual to standard cell decompositions of X.

We will always assume that our n-categories have linear n-morphisms.

For $n = 1$, a field on a 0-manifold P is a labeling of each point of P with an object (0-morphism) of the 1-category C. A field on a 1-manifold S consists of

- a cell decomposition of S (equivalently, a finite collection of points in the interior of S);
- a labeling of each 1-cell (and each half 1-cell adjacent to ∂S) by an object (0-morphism) of C;
- a transverse orientation of each 0-cell, thought of as a choice of "domain" and "range" for the two adjacent 1-cells; and
- a labeling of each 0-cell by a 1-morphism of C, with domain and range determined by the transverse orientation and the labelings of the 1-cells.

We want fields on 1-manifolds to be enriched over Vect, so we also allow formal linear combinations of the above fields on a 1-manifold X so long as these fields restrict to the same field on ∂X.

In addition, we mod out by the relation which replaces a 1-morphism label a of a 0-cell p with a^* and reverse the transverse orientation of p.

If C is a *-algebra (i.e. if C has only one 0-morphism) we can ignore the labels of 1-cells, so a field on a 1-manifold S is a finite collection of points in the interior of S, each transversely oriented and each labeled by an element (1-morphism) of the algebra.

For $n = 2$, fields are just the sort of pictures based on 2-categories (e.g. tensor categories) that are common in the literature. We describe these carefully here.

A field on a 0-manifold P is a labeling of each point of P with an object of the 2-category C. A field of a 1-manifold is defined as in the $n = 1$ case, using the 0- and 1-morphisms of C. A field on a 2-manifold Y consists of

- a cell decomposition of Y (equivalently, a graph embedded in Y such that each component of the complement is homeomorphic to a disk);
- a labeling of each 2-cell (and each partial 2-cell adjacent to ∂Y) by a 0-morphism of C;
- a transverse orientation of each 1-cell, thought of as a choice of "domain" and "range" for the two adjacent 2-cells;
- a labeling of each 1-cell by a 1-morphism of C, with domain and range determined by the transverse orientation of the 1-cell and the labelings of the 2-cells;
- for each 0-cell, a homeomorphism of the boundary R of a small neighborhood of the 0-cell to S^1 such that the intersections of the 1-cells with R are not mapped to $\pm 1 \in S^1$ (this amounts to splitting of the link of the 0-cell into domain and range); and
- a labeling of each 0-cell by a 2-morphism of C, with domain and range determined by the labelings of the 1-cells and the parameterizations of the previous bullet.

As in the $n = 1$ case, we allow formal linear combinations of fields on 2-manifolds, so long as their restrictions to the boundary coincide.

In addition, we regard the labelings as being equivariant with respect to the * structure on 1-morphisms and pivotal structure on 2-morphisms. That is, we mod out by the relation which flips the transverse orientation of a 1-cell and replaces its label a by a^*, as well as the relation which changes the parameterization of the link of a 0-cell and replaces its label by the appropriate pivotal conjugate.

For general n, a field on a k-manifold X^k consists of

- A cell decomposition of X;
- an explicit general position homeomorphism from the link of each j-cell to the boundary of the standard $(k-j)$-dimensional bihedron; and
- a labeling of each j-cell by a $(k-j)$-dimensional morphism of C, with domain and range determined by the labelings of the link of j-cell.

2.3 Local relations

For convenience we assume that fields are enriched over Vect.

Local relations are subspaces $U(B;c) \subset \mathcal{C}(B;c)$ of the fields on balls which form an ideal under gluing. Again, we give the examples first.

Example 2.3.-1 (contd.)**.** *For maps into spaces, $U(B;c)$ is generated by fields of the form $a-b \in \mathcal{C}(B;c)$, where a and b are maps (fields) which are homotopic rel boundary.*

Example 2.3.0 (contd.)**.** *For n-category pictures, $U(B;c)$ is equal to the kernel of the evaluation map $\mathcal{C}(B;c) \to \mathrm{mor}(c', c'')$, where (c', c'') is some (any) division of c into domain and range.*

These motivate the following definition.

Definition 2.3.1. *A local relation is a collection subspaces $U(B;c) \subset \mathcal{C}(B;c)$, for all n-manifolds B which are homeomorphic to the standard n-ball and all $c \in \mathcal{C}(\partial B)$, satisfying the following properties.*

1. *Functoriality: $f(U(B;c)) = U(B', f(c))$ for all homeomorphisms $f : B \to B'$*
2. *Local relations imply extended isotopy: if $x, y \in \mathcal{C}(B;c)$ and x is extended isotopic to y, then $x - y \in U(B;c)$.*
3. *Ideal with respect to gluing: if $B = B' \cup B''$, $x \in U(B')$, and $c \in \mathcal{C}(B'')$, then $x \bullet r \in U(B)$*

See [14] for further details.

2.4 Constructing a TQFT

In this subsection we briefly review the construction of a TQFT from a system of fields and local relations. As usual, see [14] for more details.

We can think of a path integral $Z(W)$ of an $n+1$-manifold (which we're not defining in this context; this is just motivation) as assigning to each boundary condition $x \in \mathcal{C}(\partial W)$ a complex number $Z(W)(x)$. In other words, $Z(W)$ lies in $\mathbb{C}^{\mathcal{C}(\partial W)}$, the vector space of linear maps $\mathcal{C}(\partial W) \to \mathbb{C}$.

The locality of the TQFT implies that $Z(W)$ in fact lies in a subspace $Z(\partial W) \subset \mathbb{C}^{\mathcal{C}(\partial W)}$ defined by local projections. The linear dual to this subspace, $A(\partial W) = Z(\partial W)^*$, can be thought of as finite linear combinations of fields modulo local relations. (In other words, $A(\partial W)$ is a sort of generalized skein module.) This is the motivation behind the definition of fields and local relations above.

In more detail, let X be an n-manifold.

Definition 2.4.1. *The TQFT invariant of X associated to a system of fields $\mathcal{F}$ and local relations $\mathcal{U}$ is*

$$A(X) \stackrel{\text{def}}{=} \mathcal{C}(X)/U(X),$$

where $\mathcal{U}(X) \subset \mathcal{C}(X)$ is the space of local relations in $\mathcal{C}(X)$: $\mathcal{U}(X)$ is generated by fields of the form $u \bullet r$, where $u \in U(B)$ for some embedded n-ball $B \subset X$ and $r \in \mathcal{C}(X \setminus B)$.

The blob complex, defined in the next section, is in some sense the derived version of $A(X)$. If X has boundary we can similarly define $A(X; c)$ for each boundary condition $c \in \mathcal{C}(\partial X)$.

The above construction can be extended to higher codimensions, assigning a k-category $A(Y)$ to an $n-k$-manifold Y, for $0 \le k \le n$. These invariants fit together via actions and gluing formulas. We describe only the case $k = 1$ below. The construction of the n+1-dimensional part of the theory (the path integral) requires that the starting data (fields and local relations) satisfy additional conditions. We do not assume these conditions here, so when we say "TQFT" we mean a decapitated TQFT that lacks its n+1-dimensional part. Such a "decapitated" TQFT is sometimes also called an $n + \epsilon$ or $n + \frac{1}{2}$ dimensional TQFT, referring to the fact that it assigns maps to mapping cylinders between n-manifolds, but nothing to arbitrary n+1-manifolds.

Let Y be an $n-1$-manifold. Define a linear 1-category $A(Y)$ as follows. The set of objects of $A(Y)$ is $\mathcal{C}(Y)$. The morphisms from a to b are $A(Y \times I; a, b)$, where a and b label the two boundary components of the cylinder $Y \times I$. Composition is given by gluing of cylinders.

Let X be an n-manifold with boundary and consider the collection of vector spaces $A(X; -) \stackrel{\text{def}}{=} \{A(X; c)\}$ where c ranges through $\mathcal{C}(\partial X)$. This collection of vector spaces affords a representation of the category $A(\partial X)$, where the action is given by gluing a collar $\partial X \times I$ to X.

Given a splitting $X = X_1 \cup_Y X_2$ of a closed n-manifold X along an $n-1$-manifold Y, we have left and right actions of $A(Y)$ on $A(X_1; -)$ and $A(X_2; -)$. The gluing theorem for n-manifolds states that there is a natural isomorphism

$$A(X) \cong A(X_1; -) \otimes_{A(Y)} A(X_2; -).$$

A proof of this gluing formula appears in [14], but it also becomes a special case of Theorem 7.2.1 by taking 0-th homology.

3 The blob complex

3.1 Definitions

Let X be an n-manifold. Let $(\mathcal{F}, U)$ be a fixed system of fields and local relations. We'll assume it is enriched over **Vect**; if it is not we can make it so by allowing finite linear combinations of elements of $\mathcal{F}(X; c)$, for fixed $c \in \mathcal{F}(\partial X)$.

We want to replace the quotient

$$A(X) \stackrel{\text{def}}{=} \mathcal{F}(X)/U(X)$$

of Definition 2.4.1 with a resolution

$$\cdots \to \mathcal{B}_2(X) \to \mathcal{B}_1(X) \to \mathcal{B}_0(X).$$

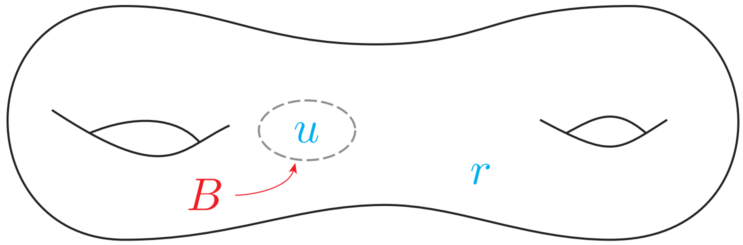


Figure 3: A 1-blob diagram.

We will define $\mathcal{B}_0(X)$, $\mathcal{B}_1(X)$ and $\mathcal{B}_2(X)$, then give the general case $\mathcal{B}_k(X)$. In fact, on the first pass we will intentionally describe the definition in a misleadingly simple way, then explain the technical difficulties, and finally give a cumbersome but complete definition in Definition 3.1.6. If (we don't recommend it) you want to keep track of the ways in which this initial description is misleading, or you're reading through a second time to understand the technical difficulties, keep note that later we will give precise meanings to "a ball in X", "nested" and "disjoint", that are not quite the intuitive ones. Moreover some of the pieces into which we cut manifolds below are not themselves manifolds, and it requires special attention to define fields on these pieces.

We of course define $\mathcal{B}_0(X) = \mathcal{F}(X)$. (If X has nonempty boundary, instead define $\mathcal{B}_0(X;c) = \mathcal{F}(X;c)$ for each $c \in \mathcal{F}(\partial X)$. We'll omit such boundary conditions from the notation in the rest of this section.) In other words, $\mathcal{B}_0(X)$ is just the vector space of all fields on X.

We want the vector space $\mathcal{B}_1(X)$ to capture "the space of all local relations that can be imposed on $\mathcal{B}_0(X)$". Thus we say a 1-blob diagram consists of:

- An closed ball in X ("blob") $B \subset X$.
- A boundary condition $c \in \mathcal{F}(\partial B) = \mathcal{F}(\partial(X \setminus B))$.
- A field $r \in \mathcal{F}(X \setminus B; c)$.
- A local relation field $u \in U(B; c)$.

(See Figure 3.) Since c is implicitly determined by u or r, we usually omit it from the notation. In order to get the linear structure correct, we define

$$\mathcal{B}_1(X) \stackrel{\text{def}}{=} \bigoplus_B \bigoplus_c U(B; c) \otimes \mathcal{F}(X \setminus B; c).$$

The first direct sum is indexed by all blobs $B \subset X$, and the second by all boundary conditions $c \in \mathcal{F}(\partial B)$. Note that $\mathcal{B}_1(X)$ is spanned by 1-blob diagrams (B, u, r).

Define the boundary map $\partial : \mathcal{B}_1(X) \to \mathcal{B}_0(X)$ by

$$(B, u, r) \mapsto u \bullet r,$$

where $u \bullet r$ denotes the field on X obtained by gluing u to r. In other words $\partial : \mathcal{B}_1(X) \to \mathcal{B}_0(X)$ is given by just erasing the blob from the picture (but keeping the blob label u).

Note that directly from the definition we have

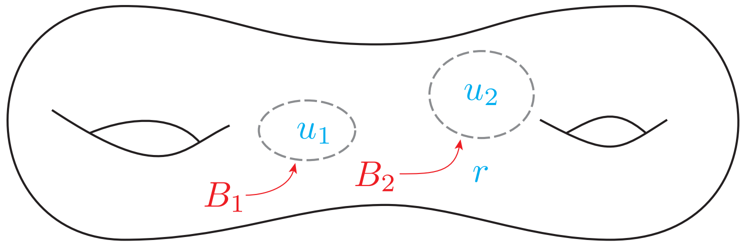


Figure 4: A disjoint 2-blob diagram.

Proposition 3.1.1. *The skein module $A(X)$ is naturally isomorphic to $\mathcal{B}_0(X)/\partial(\mathcal{B}_1(X)) = H_0(\mathcal{B}_*(X))$.*

This also establishes the second half of Property 1.3.4.

Next, we want the vector space $\mathcal{B}_2(X)$ to capture "the space of all relations (redundancies, syzygies) among the local relations encoded in $\mathcal{B}_1(X)$". A 2-blob diagram, comes in one of two types, disjoint and nested. A disjoint 2-blob diagram consists of

- A pair of closed balls (blobs) $B_1, B_2 \subset X$ with disjoint interiors.
- A field $r \in \mathcal{F}(X \setminus (B_1 \cup B_2); c_1, c_2)$ (where $c_i \in \mathcal{F}(\partial B_i)$).
- Local relation fields $u_i \in U(B_i; c_i)$, $i = 1, 2$.

(See Figure 4.) We also identify (B_1, B_2, u_1, u_2, r) with $-(B_2, B_1, u_2, u_1, r)$; reversing the order of the blobs changes the sign. Define $\partial(B_1, B_2, u_1, u_2, r) = (B_2, u_2, u_1 \bullet r) - (B_1, u_1, u_2 \bullet r) \in \mathcal{B}_1(X)$. In other words, the boundary of a disjoint 2-blob diagram is the sum (with alternating signs) of the two ways of erasing one of the blobs. It's easy to check that $\partial^2 = 0$.

A nested 2-blob diagram consists of

- A pair of nested balls (blobs) $B_1 \subseteq B_2 \subseteq X$.
- A field $r' \in \mathcal{F}(B_2 \setminus B_1; c_1, c_2)$ (for some $c_1 \in \mathcal{F}(\partial B_1)$ and $c_2 \in \mathcal{F}(\partial B_2)$).
- A field $r \in \mathcal{F}(X \setminus B_2; c_2)$.
- A local relation field $u \in U(B_1; c_1)$.

(See Figure 5.) Define $\partial(B_1, B_2, u, r', r) = (B_2, u \bullet r', r) - (B_1, u, r' \bullet r)$. As in the disjoint 2-blob case, the boundary of a nested 2-blob is the alternating sum of the two ways of erasing one of the blobs. When we erase the inner blob, the outer blob inherits the label $u \bullet r'$. It is again easy to check that $\partial^2 = 0$. Note that the requirement that local relations are an ideal with respect to gluing guarantees that $u \bullet r' \in U(B_2)$.

As with the 1-blob diagrams, in order to get the linear structure correct the actual definition is

$$\mathcal{B}_2(X) \stackrel{\text{def}}{=} \left(\bigoplus_{B_1, B_2 \text{ disjoint}} \bigoplus_{c_1, c_2} U(B_1; c_1) \otimes U(B_2; c_2) \otimes \mathcal{F}(X \setminus (B_1 \cup B_2); c_1, c_2) \right) \bigoplus \left(\bigoplus_{B_1 \subset B_2} \bigoplus_{c_1, c_2} U(B_1; c_1) \otimes \mathcal{F}(B_2 \setminus B_1; c_1, c_2) \otimes \mathcal{F}(X \setminus B_2; c_2) \right).$$

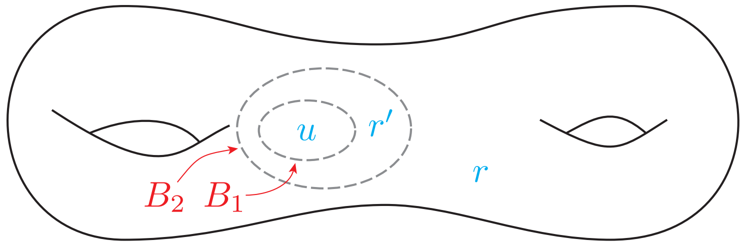


Figure 5: A nested 2-blob diagram.

Roughly, $\mathcal{B}_k(X)$ is generated by configurations of k blobs, pairwise disjoint or nested, along with fields on all the components that the blobs divide X into. Blobs which have no other blobs inside are called 'twig blobs', and the fields on the twig blobs must be local relations. The boundary is the alternating sum of erasing one of the blobs. In order to describe this general case in full detail, we must give a more precise description of which configurations of balls inside X we permit. These configurations are generated by two operations:

- For any (possibly empty) configuration of blobs on an n-ball D, we can add D itself as an outermost blob. (This is used in the proof of Proposition 3.2.1.)
- If X' is obtained from X by gluing, then any permissible configuration of blobs on X gives rise to a permissible configuration on X'. (This is necessary for Proposition 3.2.4.)

Combining these two operations can give rise to configurations of blobs whose complement in X is not a manifold. Thus will need to be more careful when speaking of a field r on the complement of the blobs.

Example 3.1.2. *Consider the four subsets of* $\mathbb{R}^3$*,*

$$\begin{aligned} A &= [0,1] \times [0,1] \times [0,1] \\ B &= [0,1] \times [-1,0] \times [0,1] \\ C &= [-1,0] \times \{\, (y,z) \mid z\sin(1/z) \leq y \leq 1, z \in [0,1] \} \\ D &= [-1,0] \times \{\, (y,z) \mid -1 \leq y \leq z\sin(1/z), z \in [0,1] \} \,. \end{aligned}$$

Here $A \cup B = [0,1] \times [-1,1] \times [0,1]$ *and* $C \cup D = [-1,0] \times [-1,1] \times [0,1]$*. Now,* $\{A\}$ *is a valid configuration of blobs in* $A \cup B$*, and* $\{C\}$ *is a valid configuration of blobs in* $C \cup D$*, so we must allow* $\{A, C\}$ *as a configuration of blobs in* $[-1,1]^2 \times [0,1]$*. Note however that the complement is not a manifold.*

Definition 3.1.3. *A* gluing decomposition *of an* n*-manifold* X *is a sequence of manifolds* $M_0 \to M_1 \to \cdots \to M_m = X$ *such that each* M_k *is obtained from* M_{k-1} *by gluing together some disjoint pair of homeomorphic* $n{-}1$*-manifolds in the boundary of* M_{k-1}*. If, in addition,* M_0 *is a disjoint union of balls, we call it a* ball decomposition*.*

Given a gluing decomposition $M_0 \to M_1 \to \cdots \to M_m = X$, we say that a field is splittable along it if it is the image of a field on M_0.

In the example above, note that

$$A \sqcup B \sqcup C \sqcup D \to (A \cup B) \sqcup (C \cup D) \to A \cup B \cup C \cup D$$

is a ball decomposition, but other sequences of gluings starting from $A \sqcup B \sqcup C \sqcup D$ have intermediate steps which are not manifolds.

We'll now slightly restrict the possible configurations of blobs.

Definition 3.1.4. *A configuration of k blobs in X is an ordered collection of k subsets $\{B_1, \ldots B_k\}$ of X such that there exists a gluing decomposition $M_0 \to \cdots \to M_m = X$ of X and for each subset B_i there is some $0 \le r \le m$ and some connected component M'_r of M_r which is a ball, so B_i is the image of M'_r in X. We say that such a gluing decomposition is* compatible *with the configuration. A blob B_i is a twig blob if no other blob B_j is a strict subset of it.*

In particular, this implies what we said about blobs above: that for any two blobs in a configuration of blobs in X, they either have disjoint interiors, or one blob is contained in the other. We describe these as disjoint blobs and nested blobs. Note that nested blobs may have boundaries that overlap, or indeed coincide. Blobs may meet the boundary of X. Further, note that blobs need not actually be embedded balls in X, since parts of the boundary of the ball M'_r may have been glued together.

Note that often the gluing decomposition for a configuration of blobs may just be the trivial one: if the boundaries of all the blobs cut X into pieces which are all manifolds, we can just take M_0 to be these pieces, and $M_1 = X$.

In the informal description above, in the definition of a k-blob diagram we asked for any collection of k balls which were pairwise disjoint or nested. We now further insist that the balls are a configuration in the sense of Definition 3.1.4. Also, we asked for a local relation on each twig blob, and a field on the complement of the twig blobs; this is unsatisfactory because that complement need not be a manifold. Thus, the official definitions are

Definition 3.1.5. *A k-blob diagram on X consists of*

- *a configuration $\{B_1, \ldots B_k\}$ of k blobs in X,*
- *and a field $r \in \mathcal{F}(X)$ which is splittable along some gluing decomposition compatible with that configuration,*

such that the restriction u_i of r to each twig blob B_i lies in the subspace $U(B_i) \subset \mathcal{F}(B_i)$. (See Figure 6.) More precisely, each twig blob B_i is the image of some ball M'_r as above, and it is really the restriction to M'_r that must lie in the subspace $U(M'_r)$.

and

Definition 3.1.6. *The k-th vector space $\mathcal{B}_k(X)$ of the* blob complex *of X is the direct sum over all configurations of k blobs in X of the vector space of k-blob diagrams with that configuration, modulo identifying the vector spaces for configurations that only differ by a permutation of the balls by the sign of that permutation. The differential $\mathcal{B}_k(X) \to \mathcal{B}_{k-1}(X)$ is, as above, the signed sum of ways of forgetting one blob from the configuration, preserving the field r:*

$$\partial(\{B_1, \ldots B_k\}, r) = \sum_{i=1}^{k} (-1)^{i+1} (\{B_1, \ldots, \widehat{B_i}, \ldots, B_k\}, r)$$

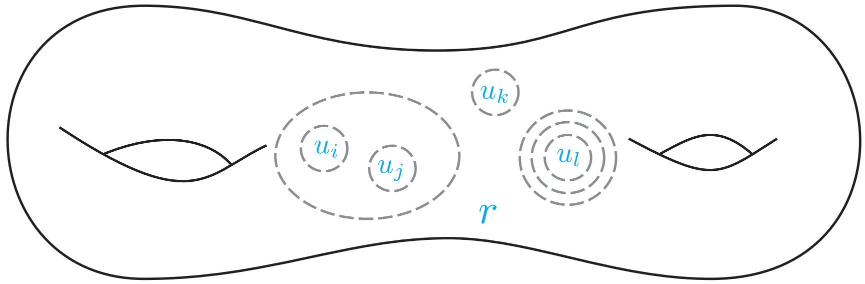


Figure 6: A k-blob diagram.

We readily see that if a gluing decomposition is compatible with some configuration of blobs, then it is also compatible with any configuration obtained by forgetting some blobs, ensuring that the differential in fact lands in the space of $k-1$-blob diagrams. A slight compensation to the complication of the official definition arising from attention to splitting is that the differential now just preserves the entire field r without having to say anything about gluing together fields on smaller components.

Note that Property 1.3.1, that the blob complex is functorial with respect to homeomorphisms, is immediately obvious from the definition. A homeomorphism acts in an obvious way on blobs and on fields.

We define the *support* of a blob diagram b, $\mathrm{supp}(b) \subset X$, to be the union of the blobs of b. For $y \in \mathcal{B}_*(X)$ with $y = \sum c_i b_i$ (c_i a non-zero number, b_i a blob diagram), we define $\mathrm{supp}(y) \stackrel{\mathrm{def}}{=} \bigcup_i \mathrm{supp}(b_i)$.

Remark 3.1.7. We note that blob diagrams in X have a structure similar to that of a simplicial set, but with simplices replaced by a more general class of combinatorial shapes. Let P be the minimal set of (isomorphisms classes of) polyhedra which is closed under products and cones, and which contains the point. We can associate an element $p(b)$ of P to each blob diagram b (equivalently, to each rooted tree) according to the following rules:

- $p(\emptyset) = pt$, where $\emptyset$ denotes a 0-blob diagram or empty tree;
- $p(a \sqcup b) = p(a) \times p(b)$, where $a \sqcup b$ denotes the distant (non-overlapping) union of two blob diagrams (equivalently, join two trees at the roots); and
- $p(\bar{b}) = \mathrm{cone}(p(b))$, where $\bar{b}$ is obtained from b by adding an outer blob which encloses all the others (equivalently, add a new edge to the root, with the new vertex becoming the root).

For example, a diagram of k strictly nested blobs corresponds to a k-simplex, while a diagram of k disjoint blobs corresponds to a k-cube. (When the fields come from an n-category, this correspondence works best if we think of each twig label u_i as having the form $x - s(e(x))$, where x is an arbitrary field on B_i, $e : \mathcal{F}(B_i) \to C$ is the evaluation map, and $s : C \to \mathcal{F}(B_i)$ is some fixed section of e.)

For lack of a better name, we'll call elements of P cone-product polyhedra, and say that blob diagrams have the structure of a cone-product set (analogous to simplicial set).

3.2 Basic properties

In this section we complete the proofs of Properties 1.3.2–1.3.4. Throughout the paper, where possible, we prove results using Properties 1.3.1–1.3.4, rather than the actual definition of blob homology. This allows the possibility of future improvements on or alternatives to our definition. In fact, we hope that there may be a characterization of the blob complex in terms of Properties 1.3.1–1.3.4, but at this point we are unaware of one.

Recall Property 1.3.2, that there is a natural isomorphism $\mathcal{B}_*(X \sqcup Y) \cong \mathcal{B}_*(X) \otimes \mathcal{B}_*(Y)$.

Proof of Property 1.3.2. Given blob diagrams b_1 on X and b_2 on Y, we can combine them (putting the b_1 blobs before the b_2 blobs in the ordering) to get a blob diagram (b_1, b_2) on $X \sqcup Y$. Because of the blob reordering relations, all blob diagrams on $X \sqcup Y$ arise this way. In the other direction, any blob diagram on $X \sqcup Y$ is equal (up to sign) to one that puts X blobs before Y blobs in the ordering, and so determines a pair of blob diagrams on X and Y. These two maps are compatible with our sign conventions. (We follow the usual convention for tensors products of complexes, as in e.g. [2]: $d(a \otimes b) = da \otimes b + (-1)^{\deg(a)} a \otimes db$.) The two maps are inverses of each other. □

For the next proposition we will temporarily restore n-manifold boundary conditions to the notation.

Suppose that for all $c \in \mathcal{C}(\partial B^n)$ we have a splitting $s : H_0(\mathcal{B}_*(B^n, c)) \to \mathcal{B}_0(B^n; c)$ of the quotient map $p : \mathcal{B}_0(B^n; c) \to H_0(\mathcal{B}_*(B^n, c))$. For example, this is always the case if the coefficient ring is a field. Then

Proposition 3.2.1. *For all $c \in \mathcal{C}(\partial B^n)$ the natural map $p : \mathcal{B}_*(B^n, c) \to H_0(\mathcal{B}_*(B^n, c))$ is a chain homotopy equivalence with inverse $s : H_0(\mathcal{B}_*(B^n, c)) \to \mathcal{B}_*(B^n; c)$. Here we think of $H_0(\mathcal{B}_*(B^n, c))$ as a 1-step complex concentrated in degree 0.*

Proof. By assumption $p \circ s = \mathbf{1}$, so all that remains is to find a degree 1 map $h : \mathcal{B}_*(B^n; c) \to \mathcal{B}_*(B^n; c)$ such that $\partial h + h\partial = \mathbf{1} - s \circ p$. For $i \geq 1$, define $h_i : \mathcal{B}_i(B^n; c) \to \mathcal{B}_{i+1}(B^n; c)$ by adding an $(i+1)$-st blob equal to all of B^n. In other words, add a new outermost blob which encloses all of the others. Define $h_0 : \mathcal{B}_0(B^n; c) \to \mathcal{B}_1(B^n; c)$ by setting $h_0(x)$ equal to the 1-blob with blob B^n and label $x - s(p(x)) \in U(B^n; c)$. □

This proves Property 1.3.4 (the second half of the statement of this Property was immediate from the definitions). Note that even when there is no splitting s, we can let $h_0 = 0$ and get a homotopy equivalence to the 2-step complex $U(B^n; c) \to \mathcal{C}(B^n; c)$.

For fields based on n-categories, $H_0(\mathcal{B}_*(B^n; c)) \cong \text{mor}(c', c'')$, where (c', c'') is some (any) splitting of c into domain and range.

Corollary 3.2.2. *If X is a disjoint union of n-balls, then $\mathcal{B}_*(X; c)$ is contractible.*

Proof. This follows from Properties 1.3.2 and 1.3.4. □

Recall the definition of the support of a blob diagram as the union of all the blobs of the diagram. For future use we prove the following lemma.

Lemma 3.2.3. *Let $L_* \subset \mathcal{B}_*(X)$ be a subcomplex generated by some subset of the blob diagrams on X, and let $f : L_* \to L_*$ be a chain map which does not increase supports and which induces an isomorphism on $H_0(L_*)$. Then f is homotopic (in $\mathcal{B}_*(X)$) to the identity $L_* \to L_*$.*

Proof. We will use the method of acyclic models. Let b be a blob diagram of L_*, let $S \subset X$ be the support of b, and let r be the restriction of b to $X \setminus S$. Note that S is a disjoint union of balls. Assign to b the acyclic (in positive degrees) subcomplex $T(b) \stackrel{\text{def}}{=} r \bullet \mathcal{B}_*(S)$. Note that if a diagram b' is part of ∂b then $T(B') \subset T(b)$. Both f and the identity are compatible with T (in the sense of acyclic models, §A), so f and the identity map are homotopic. □

For the next proposition we will temporarily restore n-manifold boundary conditions to the notation. Let X be an n-manifold, with $\partial X = Y \cup Y \cup Z$. Gluing the two copies of Y together yields an n-manifold X_{gl} with boundary Z_{gl}. Given compatible fields (boundary conditions) a, b and c on Y, Y and Z, we have the blob complex $\mathcal{B}_*(X; a, b, c)$. If $b = a$, then we can glue up blob diagrams on X to get blob diagrams on X_{gl}. This proves Property 1.3.3, which we restate here in more detail.

Proposition 3.2.4. *There is a natural chain map*

$$\text{gl} : \bigoplus_a \mathcal{B}_*(X; a, a, c) \to \mathcal{B}_*(X_{\text{gl}}; c_{\text{gl}}).$$

The sum is over all fields a on Y compatible at their ($n-2$-dimensional) boundaries with c. "Natural" means natural with respect to the actions of diffeomorphisms.

This map is very far from being an isomorphism, even on homology. We fix this deficit in §7.2 below.

4 Hochschild homology when $n = 1$

4.1 Outline

So far we have provided no evidence that blob homology is interesting in degrees greater than zero. In this section we analyze the blob complex in dimension $n = 1$. We find that $\mathcal{B}_*(S^1, \mathcal{C})$ is homotopy equivalent to the Hochschild complex of the 1-category $\mathcal{C}$. (Recall from §2.2 that a 1-category gives rise to a 1-dimensional system of fields; as usual, talking about the blob complex with coefficients in a n-category means first passing to the corresponding n dimensional system of fields.) Thus the blob complex is a natural generalization of something already known to be interesting in higher homological degrees.

It is also worth noting that the original idea for the blob complex came from trying to find a more "local" description of the Hochschild complex.

Let C be a *-1-category. Then specializing the definition of the associated system of fields from §2.2 above to the case $n = 1$ we have:

- $\mathcal{C}(pt) = \text{ob}(C)$.
- Let R be a 1-manifold and $c \in \mathcal{C}(\partial R)$. Then an element of $\mathcal{C}(R; c)$ is a collection of (transversely oriented) points in the interior of R, each labeled by a morphism of C. The intervals between the points are labeled by objects of C, consistent with the boundary condition c and the domains and ranges of the point labels.

- There is an evaluation map $e : \mathcal{C}(I;a,b) \to \mathrm{mor}(a,b)$ given by composing the morphism labels of the points. Note that we also need the * of *-1-category here in order to make all the morphisms point the same way.

- For $x \in \mathrm{mor}(a,b)$ let $\chi(x) \in \mathcal{C}(I;a,b)$ be the field with a single point (at some standard location) labeled by x. Then the kernel of the evaluation map $U(I;a,b)$ is generated by things of the form $y - \chi(e(y))$. Thus we can, if we choose, restrict the blob twig labels to things of this form.

We want to show that $\mathcal{B}_*(S^1)$ is homotopy equivalent to the Hochschild complex of C. In order to prove this we will need to extend the definition of the blob complex to allow points to also be labeled by elements of C-C-bimodules. (See Subsections 6.5 and 6.7 for a more general version of this construction that applies in all dimensions.)

Fix points $p_1, \ldots, p_k \in S^1$ and C-C-bimodules $M_1, \ldots M_k$. We define a blob-like complex $K_*(S^1, (p_i), (M_i))$. The fields have elements of M_i labeling the fixed points p_i and elements of C labeling other (variable) points. As before, the regions between the marked points are labeled by objects of $\mathcal{C}$. The blob twig labels lie in kernels of evaluation maps. (The range of these evaluation maps is a tensor product (over C) of M_i's, corresponding to the p_i's that lie within the twig blob.) Let $K_*(M) = K_*(S^1, (*), (M))$, where $* \in S^1$ is some standard base point. In other words, fields for $K_*(M)$ have an element of M at the fixed point $*$ and elements of C at variable other points.

In the theorems, propositions and lemmas below we make various claims about complexes being homotopy equivalent. In all cases the complexes in question are free (and hence projective), so it suffices to show that they are quasi-isomorphic.

We claim that

Theorem 4.1.1. *The blob complex $\mathcal{B}_*(S^1; C)$ on the circle is homotopy equivalent to the usual Hochschild complex for C.*

This follows from two results. First, we see that

Lemma 4.1.2. *The complex $K_*(C)$ (here C is being thought of as a C-C-bimodule, not a category) is homotopy equivalent to the blob complex $\mathcal{B}_*(S^1; C)$.*

The proof appears below.

Next, we show that for any C-C-bimodule M,

Proposition 4.1.3. *The complex $K_*(M)$ is homotopy equivalent to* $\mathrm{Hoch}_*(M)$*, the usual Hochschild complex of M.*

Proof. Recall that the usual Hochschild complex of M is uniquely determined, up to quasi-isomorphism, by the following properties:

1. $\mathrm{Hoch}_*(M_1 \oplus M_2) \cong \mathrm{Hoch}_*(M_1) \oplus \mathrm{Hoch}_*(M_2)$.

2. An exact sequence $0 \to M_1 \hookrightarrow M_2 \twoheadrightarrow M_3 \to 0$ gives rise to an exact sequence $0 \to \mathrm{Hoch}_*(M_1) \hookrightarrow \mathrm{Hoch}_*(M_2) \twoheadrightarrow \mathrm{Hoch}_*(M_3) \to 0$.

3. $\mathrm{HH}_0(M)$ is isomorphic to the coinvariants of M, $\mathrm{coinv}(M) = M/\langle cm - mc\rangle$.

4. $\mathrm{Hoch}_*(C \otimes C)$ is contractible. (Here $C \otimes C$ denotes the free C-C-bimodule with one generator.) That is, $\mathrm{Hoch}_*(C \otimes C)$ is quasi-isomorphic to its 0-th homology (which in turn, by 3 above, is just C) via the quotient map $\mathrm{Hoch}_0 \twoheadrightarrow \mathrm{HH}_0$.

(Together, these just say that Hochschild homology is "the derived functor of coinvariants".) We'll first recall why these properties are characteristic.

Take some C-C bimodule M, and choose a free resolution

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0.$$

We will show that for any functor $\mathcal{P}$ satisfying properties 1, 2, 3 and 4, there is a quasi-isomorphism

$$\mathcal{P}_*(M) \cong \mathrm{coinv}(F_*).$$

Observe that there's a quotient map $\pi : F_0 \twoheadrightarrow M$, and by construction the cone of the chain map $\pi : F_* \to M$ is acyclic. Now construct the total complex $\mathcal{P}_i(F_j)$, with $i, j \geq 0$, graded by $i + j$. We have two chain maps

$$\mathcal{P}_i(F_*) \xrightarrow{\mathcal{P}_i(\pi)} \mathcal{P}_i(M)$$

and

$$\mathcal{P}_*(F_j) \xrightarrow{\mathcal{P}_0(F_j) \twoheadrightarrow H_0(\mathcal{P}_*(F_j))} \mathrm{coinv}(F_j).$$

The cone of each chain map is acyclic. In the first case, this is because the "rows" indexed by i are acyclic since $\mathcal{P}_i$ is exact. In the second case, this is because the "columns" indexed by j are acyclic, since F_j is free. Because the cones are acyclic, the chain maps are quasi-isomorphisms. Composing one with the inverse of the other, we obtain the desired quasi-isomorphism

$$\mathcal{P}_*(M) \xrightarrow[\text{q.i.}]{\cong} \mathrm{coinv}(F_*).$$

Proposition 4.1.3 then follows from the following lemmas, establishing that K_* has precisely these required properties.

Lemma 4.1.4. *Directly from the definition,* $K_*(M_1 \oplus M_2) \cong K_*(M_1) \oplus K_*(M_2)$.

Lemma 4.1.5. *An exact sequence* $0 \to M_1 \hookrightarrow M_2 \twoheadrightarrow M_3 \to 0$ *gives rise to an exact sequence* $0 \to K_*(M_1) \hookrightarrow K_*(M_2) \twoheadrightarrow K_*(M_3) \to 0$.

Lemma 4.1.6. $H_0(K_*(M))$ *is isomorphic to the coinvariants of* M.

Lemma 4.1.7. $K_*(C \otimes C)$ *is quasi-isomorphic to* $H_0(K_*(C \otimes C)) \cong C$.

The remainder of this section is devoted to proving Lemmas 4.1.2, 4.1.5, 4.1.6 and 4.1.7. □

4.2 Technical details

Proof of Lemma 4.1.2. We show that $K_*(C)$ is quasi-isomorphic to $\mathcal{B}_*(S^1)$. $K_*(C)$ differs from $\mathcal{B}_*(S^1)$ only in that the base point $*$ is always a labeled point in $K_*(C)$, while in $\mathcal{B}_*(S^1)$ it may or may not be. In particular, there is an inclusion map $i : K_*(C) \to \mathcal{B}_*(S^1)$.

We want to define a homotopy inverse to the above inclusion, but before doing so we must replace $\mathcal{B}_*(S^1)$ with a homotopy equivalent subcomplex. Let $J_* \subset \mathcal{B}_*(S^1)$ be the subcomplex where $*$ does not lie on the boundary of any blob. Note that the image of i is contained in J_*. Note also that in $\mathcal{B}_*(S^1)$ (away from J_*) a blob diagram could have multiple (nested) blobs whose boundaries contain $*$, on both the right and left of $*$.

We claim that J_* is homotopy equivalent to $\mathcal{B}_*(S^1)$. Let $F_*^\epsilon \subset \mathcal{B}_*(S^1)$ be the subcomplex where either (a) the point $*$ is not on the boundary of any blob or (b) there are no labeled points or blob boundaries within distance ϵ of $*$, other than blob boundaries at $*$ itself. Note that all blob diagrams are in F_*^ϵ for ϵ sufficiently small. Let b be a blob diagram in F_*^ϵ. Define $f(b)$ to be the result of moving any blob boundary points which lie on $*$ to distance ϵ from $*$. (Move right or left so as to shrink the blob.) Extend to get a chain map $f : F_*^\epsilon \to F_*^\epsilon$. By Lemma 3.2.3, f is homotopic to the identity. Since the image of f is in J_*, and since any blob chain is in F_*^ϵ for ϵ sufficiently small, we have that J_* is homotopic to all of $\mathcal{B}_*(S^1)$.

We now define a homotopy inverse $s : J_* \to K_*(C)$ to the inclusion i. If y is a field defined on a neighborhood of $*$, define $s(y) = y$ if $*$ is a labeled point in y. Otherwise, define $s(y)$ to be the result of adding a label 1 (identity morphism) at $*$. Extending linearly, we get the desired map $s : J_* \to K_*(C)$. It is easy to check that s is a chain map and $s \circ i = \mathbf{1}$.

Let N_ϵ denote the ball of radius ϵ around $*$. Let $L_*^\epsilon \subset J_*$ be the subcomplex spanned by blob diagrams where there are no labeled points in N_ϵ, except perhaps $*$, and N_ϵ is either disjoint from or contained in every blob in the diagram. Note that for any chain $x \in J_*$, $x \in L_*^\epsilon$ for sufficiently small ϵ.

We define a degree 1 map $j_\epsilon : L_*^\epsilon \to L_*^\epsilon$ as follows. Let $x \in L_*^\epsilon$ be a blob diagram. If $*$ is not contained in any twig blob, we define $j_\epsilon(x)$ by adding N_ϵ as a new twig blob, with label $y - s(y)$ where y is the restriction of x to N_ϵ. If $*$ is contained in a twig blob B with label $u = \sum z_i$, write y_i for the restriction of z_i to N_ϵ, and let x_i be equal to x on $S^1 \setminus B$, equal to z_i on $B \setminus N_\epsilon$, and have an additional blob N_ϵ with label $y_i - s(y_i)$. Define $j_\epsilon(x) = \sum x_i$.

It is not hard to show that on L_*^ϵ

$$\partial j_\epsilon + j_\epsilon \partial = \mathbf{1} - i \circ s.$$

(To get the signs correct here, we add N_ϵ as the first blob.) Since for ϵ small enough L_*^ϵ captures all of the homology of J_*, it follows that the mapping cone of $i \circ s$ is acyclic and therefore (using the fact that these complexes are free) $i \circ s$ is homotopic to the identity. □

Proof of Lemma 4.1.5. We now prove that K_* is an exact functor.

As a warm-up, we prove that the functor on C-C bimodules

$$M \mapsto \ker(C \otimes M \otimes C \xrightarrow{c_1 \otimes m \otimes c_2 \mapsto c_1 m c_2} M)$$

is exact. Suppose we have a short exact sequence of C-C bimodules

$$0 \longrightarrow K \overset{f}{\hookrightarrow} E \overset{g}{\twoheadrightarrow} Q \longrightarrow 0 .$$

We'll write $\hat{f}$ and $\hat{g}$ for the image of f and g under the functor, so

$$\hat{f}(\sum_i a_i \otimes k_i \otimes b_i) = \sum_i a_i \otimes f(k_i) \otimes b_i,$$

and similarly for $\hat{g}$. Most of what we need to check is easy. Suppose we have $\sum_i (a_i \otimes k_i \otimes b_i) \in \ker(C \otimes K \otimes C \to K)$, assuming without loss of generality that $\{a_i \otimes b_i\}_i$ is linearly independent in $C \otimes C$, and $\hat{f}(a \otimes k \otimes b) = 0 \in \ker(C \otimes E \otimes C \to E)$. We must then have $f(k_i) = 0 \in E$ for each i, which implies $k_i = 0$ itself. If $\sum_i (a_i \otimes e_i \otimes b_i) \in \ker(C \otimes E \otimes C \to E)$ is in the image of $\ker(C \otimes K \otimes C \to K)$ under $\hat{f}$, again by assuming the set $\{a_i \otimes b_i\}_i$ is linearly independent we can deduce that each e_i is in the image of the original f, and so is in the kernel of the original g, and so $\hat{g}(\sum_i a_i \otimes e_i \otimes b_i) = 0$. If $\hat{g}(\sum_i a_i \otimes e_i \otimes b_i) = 0$, then each $g(e_i) = 0$, so $e_i = f(\widetilde{e_i})$ for some $\widetilde{e_i} \in K$, and $\sum_i a_i \otimes e_i \otimes b_i = \hat{f}(\sum_i a_i \otimes \widetilde{e_i} \otimes b_i)$. Finally, the interesting step is in checking that any $q = \sum_i a_i \otimes q_i \otimes b_i$ such that $\sum_i a_i q_i b_i = 0$ is in the image of $\ker(C \otimes E \otimes C \to C)$ under $\hat{g}$. For each i, we can find $\widetilde{q_i}$ so $g(\widetilde{q_i}) = q_i$. However $\sum_i a_i \widetilde{q_i} b_i$ need not be zero. Consider then

$$\widetilde{q} = \sum_i (a_i \otimes \widetilde{q_i} \otimes b_i) - 1 \otimes \left(\sum_i a_i \widetilde{q_i} b_i \right) \otimes 1.$$

Certainly $\widetilde{q} \in \ker(C \otimes E \otimes C \to E)$. Further,

$$\begin{aligned} \hat{g}(\widetilde{q}) &= \sum_i (a_i \otimes g(\widetilde{q_i}) \otimes b_i) - 1 \otimes \left(\sum_i a_i g(\widetilde{q_i} \right) b_i) \otimes 1 \\ &= q - 0 \end{aligned}$$

(here we used that g is a map of C-C bimodules, and that $\sum_i a_i q_i b_i = 0$).

Similar arguments show that the functors

$$M \mapsto \ker(C^{\otimes k} \otimes M \otimes C^{\otimes l} \to M) \tag{4.1}$$

are all exact too. Moreover, tensor products of such functors with each other and with C or $\ker(C^{\otimes k} \to C)$ (e.g., producing the functor $M \mapsto \ker(M \otimes C \to M) \otimes C \otimes \ker(C \otimes C \to M)$) are all still exact.

Finally, then we see that the functor K_* is simply an (infinite) direct sum of copies of this sort of functor. The direct sum is indexed by configurations of nested blobs and of labels; for each such configuration, we have one of the above tensor product functors, with the labels of twig blobs corresponding to tensor factors as in (4.1) or $\ker(C^{\otimes k} \to C)$ (depending on whether they contain a marked point p_i), and all other labelled points corresponding to tensor factors of C and M. □

Proof of Lemma 4.1.6. We show that $H_0(K_*(M))$ is isomorphic to the coinvariants of M.

We define a map $\mathrm{ev} : K_0(M) \to M$. If $x \in K_0(M)$ has the label $m \in M$ at $*$, and labels $c_i \in C$ at the other labeled points of S^1, reading clockwise from $*$, we set $\mathrm{ev}(x) = mc_1 \cdots c_k$. We can think of this as $\mathrm{ev} : M \otimes C^{\otimes k} \to M$, for each direct summand of $K_0(M)$ indexed by a configuration of labeled points.

There is a quotient map $\pi : M \to \operatorname{coinv} M$. We claim that the composition $\pi \circ \mathrm{ev}$ is well-defined on the quotient $H_0(K_*(M))$; i.e. that $\pi(\mathrm{ev}(\partial y)) = 0$ for all $y \in K_1(M)$. There are two cases, depending on whether the blob of y contains the point *. If it doesn't, then suppose y has label

m at $*$, labels c_i at other labeled points outside the blob, and the field inside the blob is a sum, with the j-th term having labeled points $d_{j,i}$. Then $\sum_j d_{j,1} \otimes \cdots \otimes d_{j,k_j} \in \ker(\bigoplus_k C^{\otimes k} \to C)$, and so $\mathrm{ev}(\partial y) = 0$, because

$$C^{\otimes \ell_1} \otimes \ker(\bigoplus_k C^{\otimes k} \to C) \otimes C^{\otimes \ell_2} \subset \ker(\bigoplus_k C^{\otimes k} \to C).$$

Similarly, if $*$ is contained in the blob, then the blob label is a sum, with the j-th term have labelled points $d_{j,i}$ to the left of $*$, m_j at $*$, and $d'_{j,i}$ to the right of $*$, and there are labels c_i at the labeled points outside the blob. We know that

$$\sum_j d_{j,1} \otimes \cdots \otimes d_{j,k_j} \otimes m_j \otimes d'_{j,1} \otimes \cdots \otimes d'_{j,k'_j} \in \ker(\bigoplus_{k,k'} C^{\otimes k} \otimes M \otimes C^{\otimes k'} \otimes \to M),$$

and so

$$\begin{aligned} \mathrm{ev}(\partial y) &= \sum_j m_j d'_{j,1} \cdots d'_{j,k'_j} c_1 \cdots c_k d_{j,1} \cdots d_{j,k_j} \\ &= \sum_j d_{j,1} \cdots d_{j,k_j} m_j d'_{j,1} \cdots d'_{j,k'_j} c_1 \cdots c_k \\ &= 0 \end{aligned}$$

where this time we use the fact that we're mapping to $\mathrm{coinv}\, M$, not just M.

The map $\pi \circ \mathrm{ev} : H_0(K_*(M)) \to \mathrm{coinv}\, M$ is clearly surjective (ev surjects onto M); we now show that it's injective. This is equivalent to showing that

$$\mathrm{ev}^{-1}(\ker(\pi)) \subset \partial K_1(M).$$

The above inclusion follows from

$$\ker(\mathrm{ev}) \subset \partial K_1(M)$$

and

$$\ker(\pi) \subset \mathrm{ev}(\partial K_1(M)).$$

Let $x = \sum x_i$ be in the kernel of ev, where each x_i is a configuration of labeled points in S^1. Since the sum is finite, we can find an interval (blob) B in S^1 such that for each i the C-labeled points of x_i all lie to the right of the base point *. Let y_i be the restriction of x_i to B and $y = \sum y_i$. Let r be the "empty" field on $S^1 \setminus B$. It follows that $y \in U(B)$ and

$$\partial(B, y, r) = x.$$

$\ker(\pi)$ is generated by elements of the form $cm - mc$. As shown in Figure 8, $cm - mc$ lies in $\mathrm{ev}(\partial K_1(M))$. □

Proof of Lemma 4.1.7. We show that $K_*(C \otimes C)$ is quasi-isomorphic to the 0-step complex C. We'll do this in steps, establishing quasi-isomorphisms and homotopy equivalences

$$K_*(C \otimes C) \xrightarrow[\text{q.i.}]{\cong} K'_* \xrightarrow[\text{htpy}]{\simeq} K''_* \xrightarrow[\text{q.i.}]{\cong} C.$$

Figure 7: Defining s_ϵ.

Let $K'_* \subset K_*(C \otimes C)$ be the subcomplex where the label of the point $*$ is $1 \otimes 1 \in C \otimes C$. We will show that the inclusion $i : K'_* \to K_*(C \otimes C)$ is a quasi-isomorphism.

Fix a small $\epsilon > 0$. Let N_ϵ be the ball of radius ϵ around $* \in S^1$. Let $K^\epsilon_* \subset K_*(C \otimes C)$ be the subcomplex generated by blob diagrams b such that N_ϵ is either disjoint from or contained in each blob of b, and the only labeled point inside N_ϵ is $*$. For a field y on N_ϵ, let $s_\epsilon(y)$ be the equivalent picture with $*$ labeled by $1 \otimes 1$ and the only other labeled points at distance $\pm\epsilon/2$ from $*$. (See Figure 7.) Note that $y - s_\epsilon(y) \in U(N_\epsilon)$. Let $\sigma_\epsilon : K^\epsilon_* \to K^\epsilon_*$ be the chain map given by replacing the restriction y to N_ϵ of each field appearing in an element of K^ϵ_* with $s_\epsilon(y)$. Note that $\sigma_\epsilon(x) \in K'_*$.

Define a degree 1 map $j_\epsilon : K^\epsilon_* \to K^\epsilon_*$ as follows. Let $x \in K^\epsilon_*$ be a blob diagram. If $*$ is not contained in any twig blob, $j_\epsilon(x)$ is obtained by adding N_ϵ to x as a new twig blob, with label $y - s_\epsilon(y)$, where y is the restriction of x to N_ϵ. If $*$ is contained in a twig blob B with label $u = \sum z_i$, $j_\epsilon(x)$ is obtained as follows. Let y_i be the restriction of z_i to N_ϵ. Let x_i be equal to x outside of B, equal to z_i on $B \setminus N_\epsilon$, and have an additional blob N_ϵ with label $y_i - s_\epsilon(y_i)$. Define $j_\epsilon(x) = \sum x_i$. Note that if $x \in K'_* \cap K^\epsilon_*$ then $j_\epsilon(x) \in K'_*$ also.

The key property of j_ϵ is

$$\partial j_\epsilon + j_\epsilon \partial = \mathbf{1} - \sigma_\epsilon.$$

(Again, to get the correct signs, N_ϵ must be added as the first blob.) If j_ϵ were defined on all of $K_*(C \otimes C)$, this would show that σ_ϵ is a homotopy inverse to the inclusion $K'_* \to K_*(C \otimes C)$. One strategy would be to try to stitch together various j_ϵ for progressively smaller ϵ and show that K'_* is homotopy equivalent to $K_*(C \otimes C)$. Instead, we'll be less ambitious and just show that K'_* is quasi-isomorphic to $K_*(C \otimes C)$.

If x is a cycle in $K_*(C \otimes C)$, then for sufficiently small ϵ we have $x \in K^\epsilon_*$. (This is true for any chain in $K_*(C \otimes C)$, since chains are sums of finitely many blob diagrams.) Then x is homologous to $\sigma_\epsilon(x)$, which is in K'_*, so the inclusion map $K'_* \subset K_*(C \otimes C)$ is surjective on homology. If $y \in K_*(C \otimes C)$ and $\partial y = x \in K_*(C \otimes C)$, then $y \in K^\epsilon_*$ for some ϵ and

$$\partial y = \partial(\sigma_\epsilon(y) + j_\epsilon(x)).$$

Since $\sigma_\epsilon(y) + j_\epsilon(x) \in K'_*$, it follows that the inclusion map is injective on homology. This completes the proof that K'_* is quasi-isomorphic to $K_*(C \otimes C)$.

Let $K''_* \subset K'_*$ be the subcomplex of K'_* where $*$ is not contained in any blob. We will show that the inclusion $i : K''_* \to K'_*$ is a homotopy equivalence.

First, a lemma: Let G''_* and G'_* be defined similarly to K''_* and K'_*, except with S^1 replaced by some neighborhood N of $* \in S^1$. (G''_* and G'_* depend on N, but that is not reflected in the notation.) Then G''_* and G'_* are both contractible and the inclusion $G''_* \subset G'_*$ is a homotopy equivalence. For G'_* the proof is the same as in Lemma 3.2.1, except that the splitting $G'_0 \to H_0(G'_*)$ concentrates the point labels at two points to the right and left of $*$. For G''_* we note that any cycle is supported away from $*$. Thus any cycle lies in the image of the normal blob complex of a disjoint union of two intervals, which is contractible by Lemma 3.2.1 and Corollary 3.2.2. Finally, it is easy to see that the inclusion $G''_* \to G'_*$ induces an isomorphism on H_0.

Next we construct a degree 1 map (homotopy) $h : K'_* \to K'_*$ such that for all $x \in K'_*$ we have

$$x - \partial h(x) - h(\partial x) \in K''_*.$$

Since $K'_0 = K''_0$, we can take $h_0 = 0$. Let $x \in K'_1$, with single blob $B \subset S^1$. If $* \notin B$, then $x \in K''_1$ and we define $h_1(x) = 0$. If $* \in B$, then we work in the image of G'_* and G''_* (with B playing the role of N above). Choose $x'' \in G''_1$ such that $\partial x'' = \partial x$. Since G'_* is contractible, there exists $y \in G'_2$ such that $\partial y = x - x''$. Define $h_1(x) = y$. The general case is similar, except that we have to take lower order homotopies into account. Let $x \in K'_k$. If $*$ is not contained in any of the blobs of x, then define $h_k(x) = 0$. Otherwise, let B be the outermost blob of x containing $*$. We can decompose $x = x' \bullet p$, where x' is supported on B and p is supported away from B. So $x' \in G'_l$ for some $l \leq k$. Choose $x'' \in G''_l$ such that $\partial x'' = \partial(x' - h_{l-1}\partial x')$. Choose $y \in G'_{l+1}$ such that $\partial y = x' - x'' - h_{l-1}\partial x'$. Define $h_k(x) = y \bullet p$. This completes the proof that $i : K''_* \to K'_*$ is a homotopy equivalence.

Finally, we show that K''_* is contractible with $H_0 \cong C$. This is similar to the proof of Proposition 3.2.1, but a bit more complicated since there is no single blob which contains the support of all blob diagrams in K''_*. Let x be a cycle of degree greater than zero in K''_*. The union of the supports of the diagrams in x does not contain $*$, so there exists a ball $B \subset S^1$ containing the union of the supports and not containing $*$. Adding B as an outermost blob to each summand of x gives a chain y with $\partial y = x$. Thus $H_i(K''_*) \cong 0$ for $i > 0$ and K''_* is contractible.

To see that $H_0(K''_*) \cong C$, consider the map $p : K''_0 \to C$ which sends a 0-blob diagram to the product of its labeled points. p is clearly surjective. It's also easy to see that $p(\partial K''_1) = 0$. Finally, if $p(y) = 0$ then there exists a blob $B \subset S^1$ which contains all of the labeled points (other than $*$) of all of the summands of y. This allows us to construct $x \in K''_1$ such that $\partial x = y$. (The label of B is the restriction of y to B.) It follows that $H_0(K''_*) \cong C$. □

4.3 An explicit chain map in low degrees

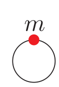
For purposes of illustration, we describe an explicit chain map $\mathrm{Hoch}_*(M) \to K_*(M)$ between the Hochschild complex and the blob complex (with bimodule point) for degree ≤ 2. This map can be completed to a homotopy equivalence, though we will not prove that here. There are of course many such maps; what we describe here is one of the simpler possibilities.

Recall that in low degrees $\mathrm{Hoch}_*(M)$ is

$$\cdots \xrightarrow{\partial} M \otimes C \otimes C \xrightarrow{\partial} M \otimes C \xrightarrow{\partial} M$$

with

$$\begin{aligned} \partial(m \otimes a) &= ma - am \\ \partial(m \otimes a \otimes b) &= ma \otimes b - m \otimes ab + bm \otimes a. \end{aligned}$$

In degree 0, we send $m \in M$ to the 0-blob diagram 

; the base point in S^1 is labeled by m and there are no other labeled points. In degree 1, we send $m \otimes a$ to the sum of two 1-blob diagrams as shown in Figure 8.

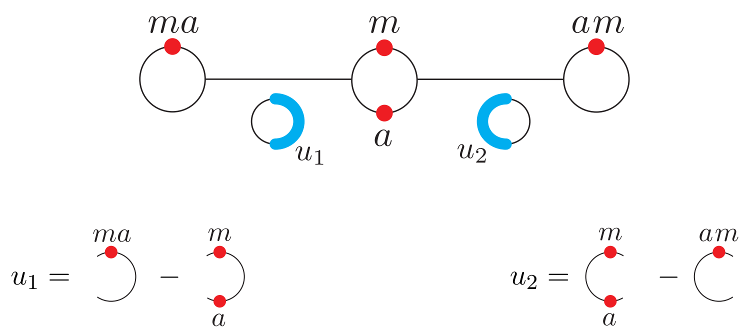


Figure 8: The image of $m \otimes a$ in the blob complex.

In degree 2, we send $m \otimes a \otimes b$ to the sum of 24 $(= 6 \cdot 4)$ 2-blob diagrams as shown in Figures 9 and 10. In Figure 10 the 1- and 2-blob diagrams are indicated only by their support. We leave it to the reader to determine the labels of the 1-blob diagrams. Each 2-cell in the figure is labeled by a ball V in S^1 which contains the support of all 1-blob diagrams in its boundary. Such a 2-cell corresponds to a sum of the 2-blob diagrams obtained by adding V as an outer (non-twig) blob to each of the 1-blob diagrams in the boundary of the 2-cell. Figure 11 shows this explicitly for the 2-cell labeled A in Figure 10. Note that the (blob complex) boundary of this sum of 2-blob diagrams is precisely the sum of the 1-blob diagrams corresponding to the boundary of the 2-cell. (Compare with the proof of 3.2.1.)

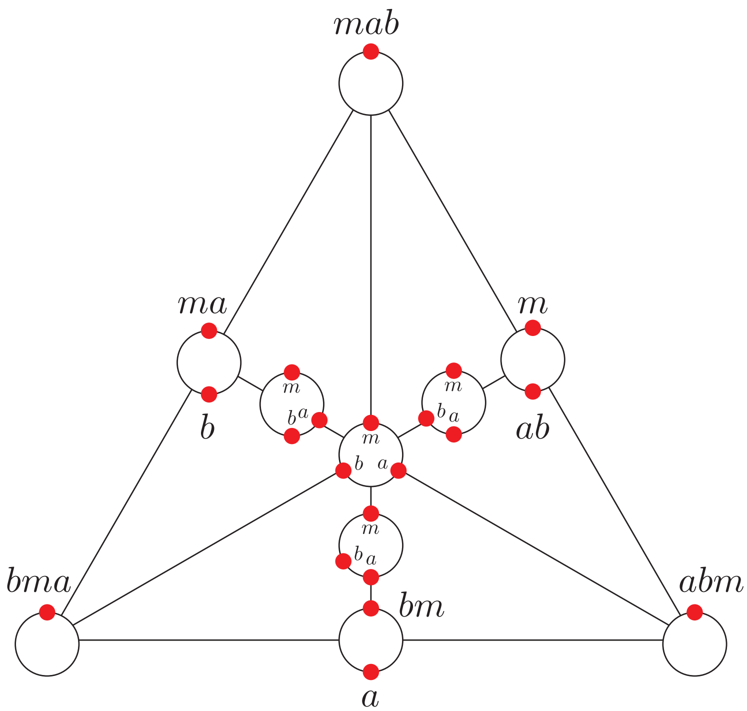


Figure 9: The 0-chains in the image of $m \otimes a \otimes b$.

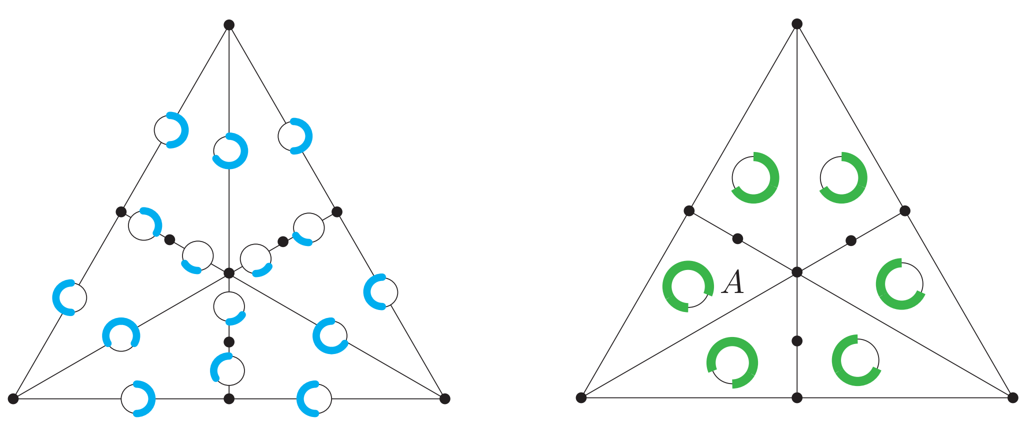


Figure 10: The 1- and 2-chains in the image of $m \otimes a \otimes b$. Only the supports of the blobs are shown, but see Figure 11 for an example of a 2-cell label.

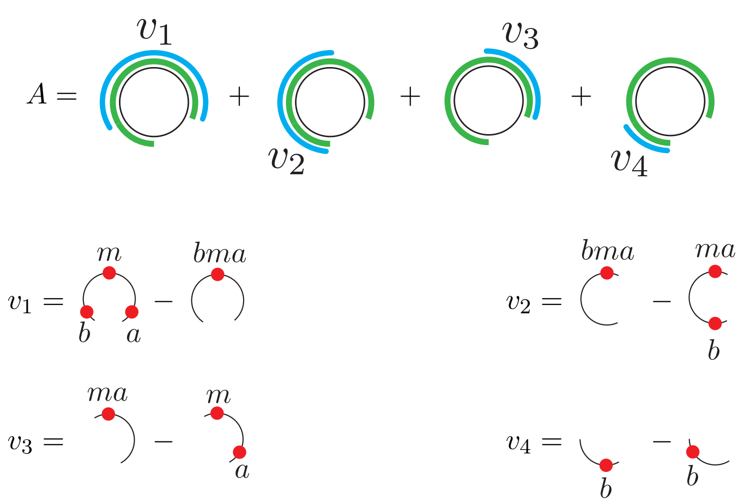


Figure 11: One of the 2-cells from Figure 10.

5 Action of $C_*(\mathrm{Homeo}(X))$

In this section we extend the action of homeomorphisms on $\mathcal{B}_*(X)$ to an action of *families* of homeomorphisms. That is, for each pair of homeomorphic manifolds X and Y we define a chain map

$$e_{XY} : CH_*(X,Y) \otimes \mathcal{B}_*(X) \to \mathcal{B}_*(Y),$$

where $CH_*(X,Y) = C_*(\mathrm{Homeo}(X,Y))$, the singular chains on the space of homeomorphisms from X to Y. (If X and Y have non-empty boundary, these families of homeomorphisms are required to restrict to a fixed homeomorphism on the boundaries.) These actions (for various X and Y) are compatible with gluing. See §5.2 for a more precise statement.

The most convenient way to prove that maps e_{XY} with the desired properties exist is to introduce a homotopy equivalent alternate version of the blob complex, $\mathcal{BT}_*(X)$, which is more amenable to this sort of action. Recall from Remark 3.1.7 that blob diagrams have the structure of a cone-product set. Blob diagrams can also be equipped with a natural topology, which converts this cone-product set into a cone-product space. Taking singular chains of this space we get $\mathcal{BT}_*(X)$. The details are in §5.1. We also prove a useful result (Lemma 5.1.1) which says that we can assume that blobs are small with respect to any fixed open cover.

5.1 Alternative definitions of the blob complex

In this subsection we define a subcomplex (small blobs) and supercomplex (families of blobs) of the blob complex, and show that they are both homotopy equivalent to $\mathcal{B}_*(X)$.

If b is a blob diagram in $\mathcal{B}_*(X)$, define the *support* of b, denoted $\mathrm{supp}(b)$ or $|b|$, to be the union of the blobs of b. More generally, we say that a chain $a \in \mathcal{B}_k(X)$ is supported on S if $a = a' \bullet r$, where $a' \in \mathcal{B}_k(S)$ and $r \in \mathcal{B}_0(X \setminus S)$.

Similarly, if $f : P \times X \to X$ is a family of homeomorphisms and $Y \subset X$, we say that f is *supported on* Y if $f(p,x) = f(p',x)$ for all $x \in X \setminus Y$ and all $p, p' \in P$. We will sometimes abuse language and talk about "the" support of f, again denoted $\mathrm{supp}(f)$ or $|f|$, to mean some particular choice of Y such that f is supported on Y.

If $f : M \cup (Y \times I) \to M$ is a collaring homeomorphism (cf. end of §2.1), we say that f is supported on $S \subset M$ if $f(x) = x$ for all $x \in M \setminus S$.

Fix $\mathcal{U}$, an open cover of X. Define the "small blob complex" $\mathcal{B}^{\mathcal{U}}_*(X)$ to be the subcomplex of $\mathcal{B}_*(X)$ of all blob diagrams in which every blob is contained in some open set of $\mathcal{U}$, and moreover each field labeling a region cut out by the blobs is splittable into fields on smaller regions, each of which is contained in some open set of $\mathcal{U}$.

Lemma 5.1.1 (Small blobs)**.** *The inclusion $i : \mathcal{B}^{\mathcal{U}}_*(X) \hookrightarrow \mathcal{B}_*(X)$ is a homotopy equivalence.*

Proof. Since both complexes are free, it suffices to show that the inclusion induces an isomorphism of homotopy groups. To show that it suffices to show that for any finitely generated pair (C_*, D_*), with D_* a subcomplex of C_* such that

$$(C_*, D_*) \subset (\mathcal{B}_*(X), \mathcal{B}^{\mathcal{U}}_*(X))$$

we can find a homotopy $h : C_* \to \mathcal{B}_*(X)$ such that $h(D_*) \subset \mathcal{B}^{\mathcal{U}}_*(X)$ and

$$h\partial(x) + \partial h(x) + x \in \mathcal{B}^{\mathcal{U}}_*(X)$$

for all $x \in C_*$.

For simplicity we will assume that all fields are splittable into small pieces, so that $\mathcal{B}_0^{\mathcal{U}}(X) = \mathcal{B}_0(X)$. (This is true for all of the examples presented in this paper.) Accordingly, we define $h_0 = 0$.

Next we define h_1. Let $b \in C_1$ be a 1-blob diagram. Let B be the blob of b. We will construct a 1-chain $s(b) \in \mathcal{B}_1^{\mathcal{U}}(X)$ such that $\partial(s(b)) = \partial b$ and the support of $s(b)$ is contained in B. (If B is not embedded in X, then we implicitly work in some stage of a decomposition of X where B is embedded. See Definition 3.1.4 and preceding discussion.) It then follows from Corollary 3.2.2 that we can choose $h_1(b) \in \mathcal{B}_2(X)$ such that $\partial(h_1(b)) = s(b) - b$.

Roughly speaking, $s(b)$ consists of a series of 1-blob diagrams implementing a series of small collar maps, plus a shrunken version of b. The composition of all the collar maps shrinks B to a ball which is small with respect to $\mathcal{U}$.

Let $\mathcal{V}_1$ be an auxiliary open cover of X, subordinate to $\mathcal{U}$ and fine enough that a condition stated later in the proof is satisfied. Let $b = (B, u, r)$, with $u = \sum a_i$ the label of B, and $a_i \in \mathcal{B}_0(B)$. Choose a sequence of collar maps $\bar{f}_j : B \cup \text{collar} \to B$ satisfying conditions specified at the end of this paragraph. Let $f_j : B \to B$ be the restriction of $\bar{f}_j$ to B; f_j maps B homeomorphically to a slightly smaller submanifold of B. Let $g_j = f_1 \circ f_2 \circ \cdots \circ f_j$. Let g be the last of the g_j's. Choose the sequence $\bar{f}_j$ so that $g(B)$ is contained is an open set of $\mathcal{V}_1$ and $g_{j-1}(|f_j|)$ is also contained is an open set of $\mathcal{V}_1$.

There are 1-blob diagrams $c_{ij} \in \mathcal{B}_1(B)$ such that c_{ij} is compatible with $\mathcal{V}_1$ (more specifically, $|c_{ij}| = g_{j-1}(B)$) and $\partial c_{ij} = g_{j-1}(a_i) - g_j(a_i)$. Define

$$s(b) = \sum_{i,j} c_{ij} + g(b)$$

and choose $h_1(b) \in \mathcal{B}_2(X)$ such that

$$\partial(h_1(b)) = s(b) - b.$$

Next we define h_2. Let $b \in C_2$ be a 2-blob diagram. Let $B = |b|$, either a ball or a union of two balls. By possibly working in a decomposition of X, we may assume that the ball(s) of B are disjointly embedded. We will construct a 2-chain $s(b) \in \mathcal{B}_2^{\mathcal{U}}(X)$ such that

$$\partial(s(b)) = \partial(h_1(\partial b) + b) = s(\partial b)$$

and the support of $s(b)$ is contained in B. It then follows from Corollary 3.2.2 that we can choose $h_2(b) \in \mathcal{B}_2(X)$ such that $\partial(h_2(b)) = s(b) - b - h_1(\partial b)$.

Similarly to the construction of h_1 above, $s(b)$ consists of a series of 2-blob diagrams implementing a series of small collar maps, plus a shrunken version of b. The composition of all the collar maps shrinks B to a sufficiently small disjoint union of balls.

Let $\mathcal{V}_2$ be an auxiliary open cover of X, subordinate to $\mathcal{U}$ and fine enough that a condition stated later in the proof is satisfied. As before, choose a sequence of collar maps f_j such that each has support contained in an open set of $\mathcal{V}_1$ and the composition of the corresponding collar homeomorphisms yields an embedding $g : B \to B$ such that $g(B)$ is contained in an open set of $\mathcal{V}_1$. Let $g_j : B \to B$ be the embedding at the j-th stage.

Fix j. We will construct a 2-chain d_j such that $\partial d_j = g_{j-1}(s(\partial b)) - g_j(s(\partial b))$. Let $s(\partial b) = \sum e_k$, and let $\{p_m\}$ be the 0-blob diagrams appearing in the boundaries of the e_k. As in the construction of h_1, we can choose 1-blob diagrams q_m such that $\partial q_m = g_{j-1}(p_m) - g_j(p_m)$ and $|q_m|$ is contained in an open set of $\mathcal{V}_1$. If x is a sum of p_m's, we denote the corresponding sum of q_m's by $q(x)$.

Now consider, for each k, $g_{j-1}(e_k) - q(\partial e_k)$. This is a 1-chain whose boundary is $g_j(\partial e_k)$. The support of e_k is $g_{j-1}(V)$ for some $V \in \mathcal{V}_1$, and the support of $q(\partial e_k)$ is contained in a union V' of finitely many open sets of $\mathcal{V}_1$, all of which contain the support of f_j. We now reveal the mysterious condition (mentioned above) which $\mathcal{V}_1$ satisfies: the union of $g_{j-1}(V)$ and V', for all of the finitely many instances arising in the construction of h_2, lies inside a disjoint union of balls U such that each individual ball lies in an open set of $\mathcal{V}_2$. (In this case there are either one or two balls in the disjoint union.) For any fixed open cover $\mathcal{V}_2$ this condition can be satisfied by choosing $\mathcal{V}_1$ to be a sufficiently fine cover. It follows from Corollary 3.2.2 that we can choose $x_k \in \mathcal{B}_2(X)$ with $\partial x_k = g_{j-1}(e_k) - g_j(e_k) - q(\partial e_k)$ and with $\mathrm{supp}(x_k) = U$. We can now take $d_j \stackrel{\mathrm{def}}{=} \sum x_k$. It is clear that $\partial d_j = \sum(g_{j-1}(e_k) - g_j(e_k)) = g_{j-1}(s(\partial b)) - g_j(s(\partial b))$, as desired.

We now define

$$s(b) = \sum d_j + g(b),$$

where g is the composition of all the f_j's. It is easy to verify that $s(b) \in \mathcal{B}_2^{\mathcal{U}}$, $\mathrm{supp}(s(b)) = \mathrm{supp}(b)$, and $\partial(s(b)) = s(\partial b)$. If follows that we can choose $h_2(b) \in \mathcal{B}_2(X)$ such that $\partial(h_2(b)) = s(b) - b - h_1(\partial b)$. This completes the definition of h_2.

The general case h_l is similar. When constructing the analogue of x_k above, we will need to find a disjoint union of balls U which contains finitely many open sets from $\mathcal{V}_{l-1}$ such that each ball is contained in some open set of $\mathcal{V}_l$. For sufficiently fine $\mathcal{V}_{l-1}$ this will be possible. Since C_* is finite, the process terminates after finitely many, say r, steps. We take $\mathcal{V}_r = \mathcal{U}$. □

Next we define the cone-product space version of the blob complex, $\mathcal{BT}_*(X)$. First we must specify a topology on the set of k-blob diagrams, BD_k. We give BD_k the finest topology such that

- For any $b \in BD_k$ the action map $\mathrm{Homeo}(X) \to BD_k$, $f \mapsto f(b)$ is continuous.
- The gluing maps $BD_k(M) \to BD_k(M_{\mathrm{gl}})$ are continuous.
- For balls B, the map $U(B) \to BD_1(B)$, $u \mapsto (B, u, \emptyset)$, is continuous, where $U(B) \subset \mathcal{B}_0(B)$ inherits its topology from $\mathcal{B}_0(B)$ and the topology on $\mathcal{B}_0(B)$ comes from the generating set $BD_0(B)$.

We can summarize the above by saying that in the typical continuous family $P \to BD_k(X)$, $p \mapsto (B_i(p), u_i(p), r(p))$, $B_i(p)$ and $r(p)$ are induced by a map $P \to \mathrm{Homeo}(X)$, with the twig blob labels $u_i(p)$ varying independently. We note that while we've decided not to allow the blobs $B_i(p)$ to vary independently of the field $r(p)$, if we did allow this it would not affect the truth of the claims we make below. In particular, such a definition of $\mathcal{BT}_*(X)$ would result in a homotopy equivalent complex.

Next we define $\mathcal{BT}_*(X)$ to be the total complex of the double complex (denoted $\mathcal{BT}_{**}$) whose (i, j) entry is $C_j(BD_i)$, the singular j-chains on the space of i-blob diagrams. The vertical boundary of the double complex, denoted ∂_t, is the singular boundary, and the horizontal boundary, denoted ∂_b, is the blob boundary. Following the usual sign convention, we have $\partial = \partial_b + (-1)^i \partial_t$.

We will regard $\mathcal{B}_*(X)$ as the subcomplex $\mathcal{BT}_{*0}(X) \subset \mathcal{BT}_{**}(X)$. The main result of this subsection is

Lemma 5.1.2. *The inclusion $\mathcal{B}_*(X) \subset \mathcal{BT}_*(X)$ is a homotopy equivalence*

Before giving the proof we need a few preliminary results.

Lemma 5.1.3. *$\mathcal{BT}_*(B^n)$ is contractible (acyclic in positive degrees).*

Proof. We will construct a contracting homotopy $h : \mathcal{BT}_*(B^n) \to \mathcal{BT}_{*+1}(B^n)$.

We will assume a splitting $s : H_0(\mathcal{BT}_*(B^n)) \to \mathcal{BT}_0(B^n)$ of the quotient map $q : \mathcal{BT}_0(B^n) \to H_0(\mathcal{BT}_*(B^n))$. Let $\rho = s \circ q$.

For $x \in \mathcal{BT}_{ij}$ with $i \geq 1$ define

$$h(x) = e(x),$$

where

$$e : \mathcal{BT}_{ij} \to \mathcal{BT}_{i+1,j}$$

adds an outermost blob, equal to all of B^n, to the j-parameter family of blob diagrams. Note that for fixed i, e is a chain map, i.e. $\partial_t e = e\partial_t$.

A generator $y \in \mathcal{BT}_{0j}$ is a map $y : P \to BD_0$, where P is some j-dimensional polyhedron. We define $r(y) \in \mathcal{BT}_{0j}$ to be the constant function $\rho \circ y : P \to BD_0$. Let $c(r(y)) \in \mathcal{BT}_{0,j+1}$ be the constant map from the cone of P to BD_0 taking the same value (namely $r(y(p))$, for any $p \in P$). Let $e(y - r(y)) \in \mathcal{BT}_{1j}$ denote the j-parameter family of 1-blob diagrams whose value at $p \in P$ is the blob B^n with label $y(p) - r(y(p))$. Now define, for $y \in \mathcal{BT}_{0j}$,

$$h(y) = e(y - r(y)) - c(r(y)).$$

We must now verify that h does the job it was intended to do. For $x \in \mathcal{BT}_{ij}$ with $i \geq 2$ we have

$$\begin{aligned}
\partial h(x) + h(\partial x) &= \partial(e(x)) + e(\partial x) \\
&= \partial_b(e(x)) + (-1)^{i+1}\partial_t(e(x)) + e(\partial_b x) + (-1)^i e(\partial_t x) \\
&= \partial_b(e(x)) + e(\partial_b x) && \text{(since } \partial_t(e(x)) = e(\partial_t x)\text{)} \\
&= x.
\end{aligned}$$

For $x \in \mathcal{BT}_{1j}$ we have

$$\begin{aligned}
\partial h(x) + h(\partial x) &= \partial_b(e(x)) + \partial_t(e(x)) + e(\partial_b x - r(\partial_b x)) - c(r(\partial_b x)) - e(\partial_t x) \\
&= \partial_b(e(x)) + e(\partial_b x) && \text{(since } r(\partial_b x) = 0\text{)} \\
&= x.
\end{aligned}$$

For $x \in \mathcal{BT}_{0j}$ with $j \geq 1$ we have

$$\begin{aligned}
\partial h(x) + h(\partial x) &= \partial_b(e(x - r(x))) - \partial_t(e(x - r(x))) + \partial_t(c(r(x))) + e(\partial_t x - r(\partial_t x)) - c(r(\partial_t x)) \\
&= x - r(x) + \partial_t(c(r(x))) - c(r(\partial_t x)) \\
&= x - r(x) + r(x) \\
&= x.
\end{aligned}$$

Here we have used the fact that $\partial_b(c(r(x))) = 0$ since $c(r(x))$ is a 0-blob diagram, as well as that $\partial_t(e(r(x))) = e(r(\partial_t x))$ and $\partial_t(c(r(x))) - c(r(\partial_t x)) = r(x)$.

For $x \in \mathcal{BT}_{00}$ we have

$$\begin{aligned}\partial h(x) + h(\partial x) &= \partial_b(e(x - r(x))) + \partial_t(c(r(x))) \\ &= x - r(x) + r(x) - r(x) \\ &= x - r(x).\end{aligned}$$ □

Lemma 5.1.4. *For manifolds X and Y, we have $\mathcal{BT}_*(X \sqcup Y) \simeq \mathcal{BT}_*(X) \otimes \mathcal{BT}_*(Y)$.*

Proof. This follows from the Eilenberg-Zilber theorem and the fact that

$$BD_k(X \sqcup Y) \cong \coprod_{i+j=k} BD_i(X) \times BD_j(Y).$$ □

For $S \subset X$, we say that $a \in \mathcal{BT}_k(X)$ is *supported on S* if there exists $a' \in \mathcal{BT}_k(S)$ and $r \in \mathcal{BT}_0(X \setminus S)$ such that $a = a' \bullet r$.

Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{BT}^{\mathcal{U}}_*(X) \subset \mathcal{BT}_*(X)$ be the subcomplex generated by $a \in \mathcal{BT}_*(X)$ such that there is a decomposition $X = \cup_i D_i$ such that each D_i is a ball contained in some open set of $\mathcal{U}$ and a is splittable along this decomposition. In other words, a can be obtained by gluing together pieces, each of which is small with respect to $\mathcal{U}$.

Lemma 5.1.5. *For any open cover $\mathcal{U}$ of X, the inclusion $\mathcal{BT}^{\mathcal{U}}_*(X) \subset \mathcal{BT}_*(X)$ is a homotopy equivalence.*

Proof. This follows from a combination of Lemma B.0.5 and the techniques of the proof of Lemma 5.1.1.

It suffices to show that we can deform a finite subcomplex C_* of $\mathcal{BT}_*(X)$ into $\mathcal{BT}^{\mathcal{U}}_*(X)$ (relative to any designated subcomplex of C_* already in $\mathcal{BT}^{\mathcal{U}}_*(X)$). The first step is to replace families of general blob diagrams with families that are small with respect to $\mathcal{U}$. This is done as in the proof of Lemma 5.1.1; the technique of the proof works in families. Each such family is homotopic to a sum families which can be a "lifted" to Homeo(X). That is, $f : P \to BD_k$ has the form $f(p) = g(p)(b)$ for some $g : P \to \text{Homeo}(X)$ and $b \in BD_k$. (We are ignoring a complication related to twig blob labels, which might vary independently of g, but this complication does not affect the conclusion we draw here.) We now apply Lemma B.0.5 to get families which are supported on balls D_i contained in open sets of $\mathcal{U}$. □

Proof of Lemma 5.1.2. Armed with the above lemmas, we can now proceed similarly to the proof of Lemma 5.1.1.

It suffices to show that for any finitely generated pair of subcomplexes $(C_*, D_*) \subset (\mathcal{BT}_*(X), \mathcal{B}_*(X))$ we can find a homotopy $h : C_* \to \mathcal{BT}_{*+1}(X)$ such that $h(D_*) \subset \mathcal{B}_{*+1}(X)$ and $x + h\partial(x) + \partial h(x) \in \mathcal{B}_*(X)$ for all $x \in C_*$.

By Lemma 5.1.5, we may assume that $C_* \subset \mathcal{BT}^{\mathcal{U}}_*(X)$ for some cover $\mathcal{U}$ of our choosing. We choose $\mathcal{U}$ fine enough so that each generator of C_* is supported on a disjoint union of balls. (This is possible since the original C_* was finite and therefore had bounded dimension.)

Since $\mathcal{B}_0(X) = \mathcal{BT}_0(X)$, we can take $h_0 = 0$.

Let $b \in C_1$ be a generator. Since b is supported in a disjoint union of balls, we can find $s(b) \in \mathcal{B}_1$ with $\partial(s(b)) = \partial b$ (by Corollary 3.2.2), and also $h_1(b) \in \mathcal{BT}_2(X)$ such that $\partial(h_1(b)) = s(b) - b$ (by Lemmas 5.1.3 and 5.1.4).

Now let b be a generator of C_2. If $\mathcal{U}$ is fine enough, there is a disjoint union of balls V on which $b + h_1(\partial b)$ is supported. Since $\partial(b + h_1(\partial b)) = s(\partial b) \in \mathcal{B}_2(X)$, we can find $s(b) \in \mathcal{B}_2(X)$ with $\partial(s(b)) = \partial(b + h_1(\partial b))$ (by Corollary 3.2.2). By Lemmas 5.1.3 and 5.1.4, we can now find $h_2(b) \in \mathcal{BT}_3(X)$, also supported on V, such that $\partial(h_2(b)) = s(b) - b - h_1(\partial b)$

The general case, h_k, is similar. □

The proof of Lemma 5.1.2 constructs a homotopy inverse to the inclusion $\mathcal{B}_*(X) \subset \mathcal{BT}_*(X)$. One might ask for more: a contractible set of possible homotopy inverses, or at least an m-connected set for arbitrarily large m. The latter can be achieved with finer control over the various choices of disjoint unions of balls in the above proofs, but we will not pursue this here.

5.2 Action of $C_*(\mathrm{Homeo}(X))$

Let $CH_*(X, Y)$ denote $C_*(\mathrm{Homeo}(X \to Y))$, the singular chain complex of the space of homeomorphisms between the n-manifolds X and Y (any given singular chain extends a fixed homeomorphism $\partial X \to \partial Y$). We also will use the abbreviated notation $CH_*(X) \stackrel{\text{def}}{=} CH_*(X, X)$. (For convenience, we will permit the singular cells generating $CH_*(X, Y)$ to be more general than simplices — they can be based on any cone-product polyhedron (see Remark 3.1.7).)

Theorem 5.2.1. *For n-manifolds X and Y there is a chain map*

$$e_{XY} : CH_*(X, Y) \otimes \mathcal{B}_*(X) \to \mathcal{B}_*(Y),$$

well-defined up to homotopy, such that

1. *on $CH_0(X, Y) \otimes \mathcal{B}_*(X)$ it agrees with the obvious action of* $\mathrm{Homeo}(X, Y)$ *on $\mathcal{B}_*(X)$ described in Property 1.3.1, and*

2. *for any compatible splittings $X \to X_{\mathrm{gl}}$ and $Y \to Y_{\mathrm{gl}}$, the following diagram commutes up to homotopy*

$$\begin{array}{ccc} CH_*(X, Y) \otimes \mathcal{B}_*(X) & \xrightarrow{\quad e_{XY} \quad} & \mathcal{B}_*(Y) \\ \Big\downarrow {\scriptstyle \mathrm{gl} \otimes \mathrm{gl}} & & \Big\downarrow {\scriptstyle \mathrm{gl}} \\ CH_*(X_{\mathrm{gl}}, Y_{\mathrm{gl}}) \otimes \mathcal{B}_*(X_{\mathrm{gl}}) & \xrightarrow{\quad e_{X_{\mathrm{gl}} Y_{\mathrm{gl}}} \quad} & \mathcal{B}_*(Y_{\mathrm{gl}}) \end{array}$$

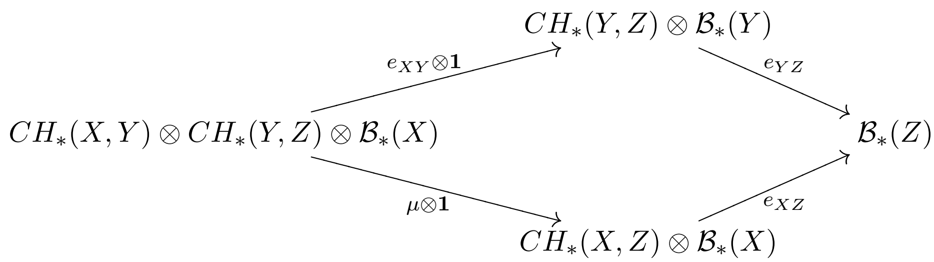
Proof. In light of Lemma 5.1.2, it suffices to prove the theorem with $\mathcal{B}_*$ replaced by $\mathcal{BT}_*$. In fact, for $\mathcal{BT}_*$ we get a sharper result: we can omit the "up to homotopy" qualifiers.

Let $f \in CH_k(X, Y)$, $f : P^k \to \mathrm{Homeo}(X \to Y)$ and $a \in \mathcal{BT}_{ij}(X)$, $a : Q^j \to BD_i(X)$. Define $e_{XY}(f \otimes a) \in \mathcal{BT}_{i,j+k}(Y)$ by

$$\begin{aligned} e_{XY}(f \otimes a) : P \times Q &\to BD_i(Y) \\ (p, q) &\mapsto f(p)(a(q)). \end{aligned}$$

It is clear that this agrees with the previously defined $CH_0(X, Y)$ action on $\mathcal{BT}_*$, and it is also easy to see that the diagram in item 2 of the statement of the theorem commutes on the nose. □

Theorem 5.2.2. *The $CH_*(X,Y)$ actions defined above are associative. That is, the following diagram commutes up to homotopy:*



Here $\mu : CH_*(X,Y) \otimes CH_*(Y,Z) \to CH_*(X,Z)$ is the map induced by composition of homeomorphisms.

Proof. The corresponding diagram for $\mathcal{BT}_*$ commutes on the nose. □

6 n-categories and their modules

6.1 Definition of n-categories

Before proceeding, we need more appropriate definitions of n-categories, A_∞ n-categories, as well as modules for these, and tensor products of these modules. (As is the case throughout this paper, by "n-category" we mean some notion of a "weak" n-category with "strong duality".)

The definitions presented below tie the categories more closely to the topology and avoid combinatorial questions about, for example, the minimal sufficient collections of generalized associativity axioms; we prefer maximal sets of axioms to minimal sets. It is easy to show that examples of topological origin (e.g. categories whose morphisms are maps into spaces or decorated balls), satisfy our axioms. For examples of a more purely algebraic origin, one would typically need the combinatorial results that we have avoided here.

There are many existing definitions of n-categories, with various intended uses. In any such definition, there are sets of k-morphisms for each $0 \le k \le n$. Generally, these sets are indexed by instances of a certain typical shape. Some n-category definitions model k-morphisms on the standard bihedron (interval, bigon, and so on). Other definitions have a separate set of 1-morphisms for each interval $[0,l] \subset \mathbb{R}$, a separate set of 2-morphisms for each rectangle $[0,l_1] \times [0,l_2] \subset \mathbb{R}^2$, and so on. (This allows for strict associativity.) Still other definitions (see, for example, [5]) model the k-morphisms on more complicated combinatorial polyhedra.

For our definition, we will allow our k-morphisms to have any shape, so long as it is homeomorphic to the standard k-ball. Thus we associate a set of k-morphisms $\mathcal{C}_k(X)$ to any k-manifold X homeomorphic to the standard k-ball. By "a k-ball" we mean any k-manifold which is homeomorphic to the standard k-ball. We *do not* assume that it is equipped with a preferred homeomorphism to the standard k-ball, and the same applies to "a k-sphere" below.

The axioms for an n-category are spread throughout this section. Collecting these together, an n-category is a gadget satisfying Axioms 6.1.1, 6.1.3, 6.1.5, 6.1.6, 6.1.8 and 6.1.10; for an A_∞ n-category, we replace Axiom 6.1.10 with Axiom 6.1.11.

Given a homeomorphism $f : X \to Y$ between k-balls (not necessarily fixed on the boundary), we want a corresponding bijection of sets $f : \mathcal{C}(X) \to \mathcal{C}(Y)$. (This will imply "strong duality", among other things.) Putting these together, we have

Axiom 6.1.1 (Morphisms)**.** *For each $0 \le k \le n$, we have a functor $\mathcal{C}_k$ from the category of k-balls and homeomorphisms to the category of sets and bijections.*

(Note: We often omit the subscript k.)

We are being deliberately vague about what flavor of k-balls we are considering. They could be unoriented or oriented or Spin or $\text{Pin}_\pm$. They could be topological or PL or smooth. (If smooth, "homeomorphism" should be read "diffeomorphism", and we would need to be fussier about corners and boundaries.) For each flavor of manifold there is a corresponding flavor of n-category. For simplicity, we will concentrate on the case of PL unoriented manifolds.

An ambitious reader may want to keep in mind two other classes of balls. The first is balls equipped with a map to some other space Y (c.f. [10]). This will be used below (see the end of §7.1) to describe the blob complex of a fiber bundle with base space Y. The second is balls equipped with a section of the tangent bundle, or the frame bundle (i.e. framed balls), or more generally some partial flag bundle associated to the tangent bundle. These can be used to define categories with less than the "strong" duality we assume here, though we will not develop that idea fully in this paper.

Next we consider domains and ranges of morphisms (or, as we prefer to say, boundaries of morphisms). The 0-sphere is unusual among spheres in that it is disconnected. Correspondingly, for 1-morphisms it makes sense to distinguish between domain and range. (Actually, this is only true in the oriented case, with 1-morphisms parameterized by *oriented* 1-balls.) For $k > 1$ and in the presence of strong duality the division into domain and range makes less sense. For example, in a pivotal tensor category, there are natural isomorphisms $\text{Hom}\,(A, B \otimes C) \xrightarrow{\cong} \text{Hom}\,(B^* \otimes A, C)$, etc. (sometimes called "Frobenius reciprocity"), which canonically identify all the morphism spaces which have the same boundary. We prefer not to make the distinction in the first place.

Instead, we will combine the domain and range into a single entity which we call the boundary of a morphism. Morphisms are modeled on balls, so their boundaries are modeled on spheres. In other words, we need to extend the functors $\mathcal{C}_{k-1}$ from balls to spheres, for $1 \le k \le n$. At first it might seem that we need another axiom for this, but in fact once we have all the axioms in this subsection for 0 through $k-1$ we can use a colimit construction, as described in §6.3 below, to extend $\mathcal{C}_{k-1}$ to spheres (and any other manifolds):

Lemma 6.1.2. *For each $1 \le k \le n$, we have a functor $\underrightarrow{\mathcal{C}}_{k-1}$ from the category of $k{-}1$-spheres and homeomorphisms to the category of sets and bijections.*

We postpone the proof of this result until after we've actually given all the axioms. Note that defining this functor for some k only requires the data described in Axiom 6.1.1 at level k, along with the data described in the other axioms at lower levels.

Axiom 6.1.3 (Boundaries)**.** *For each k-ball X, we have a map of sets $\partial : \mathcal{C}_k(X) \to \underrightarrow{\mathcal{C}}_{k-1}(\partial X)$. These maps, for various X, comprise a natural transformation of functors.*

Note that the first "∂" above is part of the data for the category, while the second is the ordinary boundary of manifolds. Given $c \in \underrightarrow{\mathcal{C}}(\partial(X))$, we will write $\mathcal{C}(X; c)$ for $\partial^{-1}(c)$, those morphisms with specified boundary c.

Most of the examples of n-categories we are interested in are enriched in the following sense. The various sets of n-morphisms $\mathcal{C}(X; c)$, for all n-balls X and all $c \in \underrightarrow{\mathcal{C}}(\partial X)$, have the structure of an object in some auxiliary symmetric monoidal category with sufficient limits and colimits (e.g.

vector spaces, or modules over some ring, or chain complexes), and all the structure maps of the n-category should be compatible with the auxiliary category structure. Note that this auxiliary structure is only in dimension n; if $\dim(Y) < n$ then $\mathcal{C}(Y;c)$ is just a plain set.

In order to simplify the exposition we have concentrated on the case of unoriented PL manifolds and avoided the question of what exactly we mean by the boundary of a manifold with extra structure, such as an oriented manifold. In general, all manifolds of dimension less than n should be equipped with the germ of a thickening to dimension n, and this germ should carry whatever structure we have on n-manifolds. In addition, lower dimensional manifolds should be equipped with a framing of their normal bundle in the thickening; the framing keeps track of which side (iterated) bounded manifolds lie on. For example, the boundary of an oriented n-ball should be an $n{-}1$-sphere equipped with an orientation of its once stabilized tangent bundle and a choice of direction in this bundle indicating which side the n-ball lies on.

We have just argued that the boundary of a morphism has no preferred splitting into domain and range, but the converse meets with our approval. That is, given compatible domain and range, we should be able to combine them into the full boundary of a morphism. The following lemma will follow from the colimit construction used to define $\underrightarrow{\mathcal{C}}_{k-1}$ on spheres.

Lemma 6.1.4 (Boundary from domain and range)**.** *Let $S = B_1 \cup_E B_2$, where S is a $k{-}1$-sphere ($1 \le k \le n$), B_i is a $k{-}1$-ball, and $E = B_1 \cap B_2$ is a $k{-}2$-sphere (Figure 12). Let $\mathcal{C}(B_1) \times_{\underrightarrow{\mathcal{C}}(E)} \mathcal{C}(B_2)$ denote the fibered product of the two maps $\partial : \mathcal{C}(B_i) \to \underrightarrow{\mathcal{C}}(E)$. Then we have an injective map*

$$\mathrm{gl}_E : \mathcal{C}(B_1) \times_{\underrightarrow{\mathcal{C}}(E)} \mathcal{C}(B_2) \hookrightarrow \underrightarrow{\mathcal{C}}(S)$$

which is natural with respect to the actions of homeomorphisms. (When $k = 1$ we stipulate that $\underrightarrow{\mathcal{C}}(E)$ is a point, so that the above fibered product becomes a normal product.)

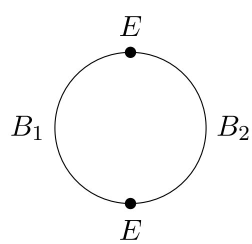


Figure 12: Combining two balls to get a full boundary.

Note that we insist on injectivity above. The lemma follows from Definition 6.3.3 and Lemma 6.3.5.

Let $\underrightarrow{\mathcal{C}}(S)_E$ denote the image of gl_E. We will refer to elements of $\underrightarrow{\mathcal{C}}(S)_E$ as "splittable along E" or "transverse to E".

If X is a k-ball and $E \subset \partial X$ splits ∂X into two $k{-}1$-balls B_1 and B_2 as above, then we define $\mathcal{C}(X)_E = \partial^{-1}(\underrightarrow{\mathcal{C}}(\partial X)_E)$.

We will call the projection $\underrightarrow{\mathcal{C}}(S)_E \to \mathcal{C}(B_i)$ a *restriction* map and write $\mathrm{res}_{B_i}(a)$ (or simply $\mathrm{res}(a)$ when there is no ambiguity), for $a \in \underrightarrow{\mathcal{C}}(S)_E$. More generally, we also include under the rubric "restriction map" the boundary maps of Axiom 6.1.3 above, another class of maps introduced after

Axiom 6.1.6 below, as well as any composition of restriction maps. In particular, we have restriction maps $\mathcal{C}(X)_E \to \mathcal{C}(B_i)$ ($i = 1, 2$, notation from previous paragraph). These restriction maps can be thought of as domain and range maps, relative to the choice of splitting $\partial X = B_1 \cup_E B_2$.

Next we consider composition of morphisms. For n-categories which lack strong duality, one usually considers k different types of composition of k-morphisms, each associated to a different direction. (For example, vertical and horizontal composition of 2-morphisms.) In the presence of strong duality, these k distinct compositions are subsumed into one general type of composition which can be in any "direction".

Axiom 6.1.5 (Composition)**.** *Let $B = B_1 \cup_Y B_2$, where B, B_1 and B_2 are k-balls ($0 \le k \le n$) and $Y = B_1 \cap B_2$ is a $k{-}1$-ball (Figure 13). Let $E = \partial Y$, which is a $k{-}2$-sphere. Note that each of B, B_1 and B_2 has its boundary split into two $k{-}1$-balls by E. We have restriction (domain or range) maps $\mathcal{C}(B_i)_E \to \mathcal{C}(Y)$. Let $\mathcal{C}(B_1)_E \times_{\mathcal{C}(Y)} \mathcal{C}(B_2)_E$ denote the fibered product of these two maps. We have a map*

$$\mathrm{gl}_Y : \mathcal{C}(B_1)_E \times_{\mathcal{C}(Y)} \mathcal{C}(B_2)_E \to \mathcal{C}(B)_E$$

which is natural with respect to the actions of homeomorphisms, and also compatible with restrictions to the intersection of the boundaries of B and B_i. If $k < n$, or if $k = n$ and we are in the A_∞ case, we require that gl_Y is injective. (For $k = n$ in the plain (non-A_∞) case, see below.)

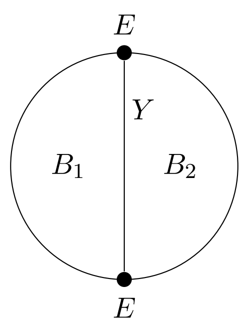


Figure 13: From two balls to one ball.

Axiom 6.1.6 (Strict associativity)**.** *The composition (gluing) maps above are strictly associative. Given any splitting of a ball B into smaller balls*

$$\bigsqcup B_i \to B,$$

any sequence of gluings (in the sense of Definition 3.1.3, where all the intermediate steps are also disjoint unions of balls) yields the same result.

We'll use the notation $a \bullet b$ for the glued together field $\mathrm{gl}_Y(a, b)$. In the other direction, we will call the projection from $\mathcal{C}(B)_E$ to $\mathcal{C}(B_i)_E$ a restriction map (one of many types of map so called) and write $\mathrm{res}_{B_i}(a)$ for $a \in \mathcal{C}(B)_E$.

We will write $\mathcal{C}(B)_Y$ for the image of gl_Y in $\mathcal{C}(B)$. We will call elements of $\mathcal{C}(B)_Y$ morphisms which are "splittable along Y" or "transverse to Y". We have $\mathcal{C}(B)_Y \subset \mathcal{C}(B)_E \subset \mathcal{C}(B)$.

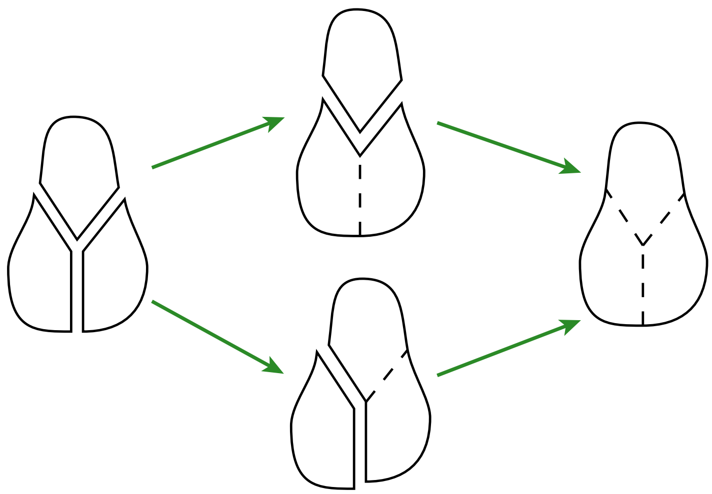


Figure 14: An example of strict associativity.

More generally, let α be a splitting of X into smaller balls. Let $\mathcal{C}(X)_\alpha \subset \mathcal{C}(X)$ denote the image of the iterated gluing maps from the smaller balls to X. We say that elements of $\mathcal{C}(X)_\alpha$ are morphisms which are "splittable along α". In situations where the splitting is notationally anonymous, we will write $\mathcal{C}(X)_\pitchfork$ for the morphisms which are splittable along (a.k.a. transverse to) the unnamed splitting. If β is a ball decomposition of ∂X, we define $\mathcal{C}(X)_\beta \stackrel{\text{def}}{=} \partial^{-1}(\underrightarrow{\mathcal{C}}(\partial X)_\beta)$; this can also be denoted $\mathcal{C}(X)_\pitchfork$ if the context contains an anonymous decomposition of ∂X and no competing splitting of X.

The above two composition axioms are equivalent to the following one, which we state in slightly vague form.

Multi-composition: *Given any splitting $B_1 \sqcup \cdots \sqcup B_m \to B$ of a k-ball into small k-balls, there is a map from an appropriate subset (like a fibered product) of $\mathcal{C}(B_1)_\pitchfork \times \cdots \times \mathcal{C}(B_m)_\pitchfork$ to $\mathcal{C}(B)_\pitchfork$, and these various m-fold composition maps satisfy an operad-type strict associativity condition (Figure 15).*

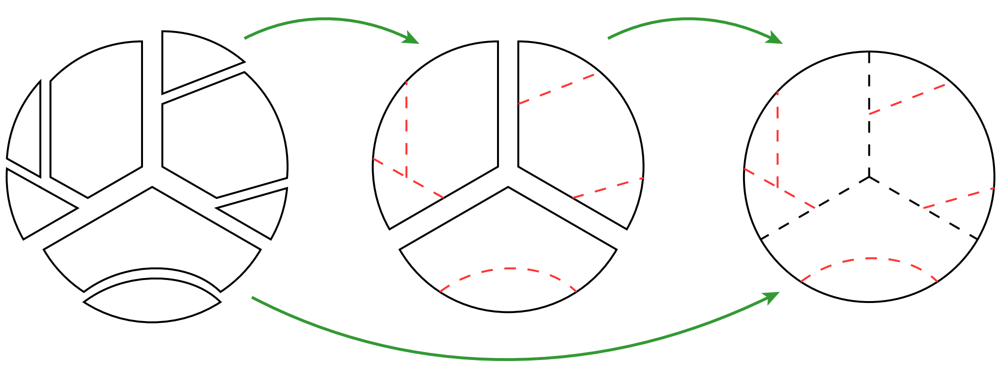


Figure 15: Operad composition and associativity

The next axiom is related to identity morphisms, though that might not be immediately obvious.

Axiom 6.1.7 (Product (identity) morphisms, preliminary version)**.** *For each k-ball X and m-ball D, with $k + m \le n$, there is a map $\mathcal{C}(X) \to \mathcal{C}(X \times D)$, usually denoted $a \mapsto a \times D$ for $a \in \mathcal{C}(X)$. These maps must satisfy the following conditions.*

1. *If $f : X \to X'$ and $\tilde{f} : X \times D \to X' \times D'$ are homeomorphisms such that the diagram*

$$\begin{CD} X \times D @>{\tilde{f}}>> X' \times D' \\ @V{\pi}VV @VV{\pi}V \\ X @>{f}>> X' \end{CD}$$

   commutes, then we have

$$\tilde{f}(a \times D) = f(a) \times D'.$$

2. *Product morphisms are compatible with gluing (composition) in both factors:*

$$(a' \times D) \bullet (a'' \times D) = (a' \bullet a'') \times D$$

   and

$$(a \times D') \bullet (a \times D'') = a \times (D' \bullet D'').$$

3. *Product morphisms are associative:*

$$(a \times D) \times D' = a \times (D \times D').$$

   (Here we are implicitly using functoriality and the obvious homeomorphism $(X \times D) \times D' \to X \times (D \times D')$.)

4. *Product morphisms are compatible with restriction:*

$$\mathrm{res}_{X \times E}(a \times D) = a \times E$$

   for $E \subset \partial D$ and $a \in \mathcal{C}(X)$.

We will need to strengthen the above preliminary version of the axiom to allow for products which are "pinched" in various ways along their boundary. (See Figure 16.) (The need for a strengthened version will become apparent in Appendix C where we construct a traditional category from a topological category.) Define a *pinched product* to be a map

$$\pi : E \to X$$

such that E is a $k{+}m$-ball, X is a k-ball ($m \ge 1$), and π is locally modeled on a standard iterated degeneracy map

$$d : \Delta^{k+m} \to \Delta^k.$$

(We thank Kevin Costello for suggesting this approach.)

Note that for each interior point $x \in X$, $\pi^{-1}(x)$ is an m-ball, and for each boundary point $x \in \partial X$, $\pi^{-1}(x)$ is a ball of dimension $l \le m$, with l depending on x. It is easy to see that a

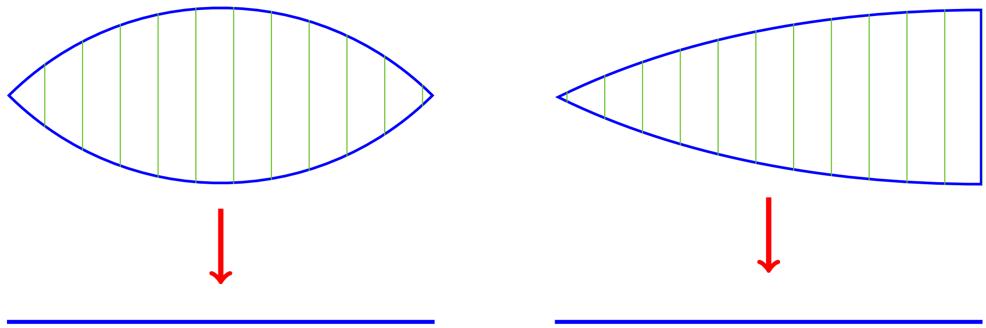


Figure 16: Examples of pinched products

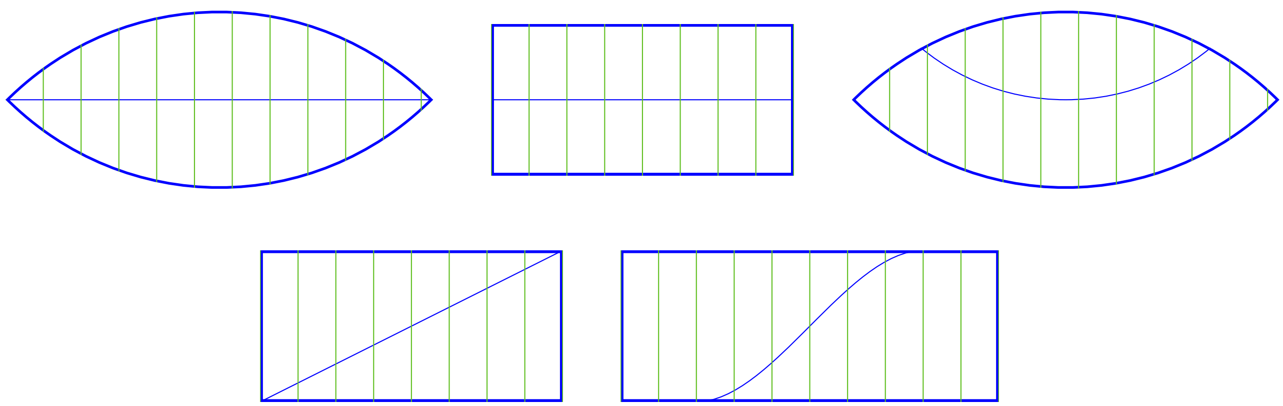


Figure 17: Five examples of unions of pinched products

composition of pinched products is again a pinched product. A *sub pinched product* is a sub-m-ball $E' \subset E$ such that the restriction $\pi : E' \to \pi(E')$ is again a pinched product. A union of pinched products is a decomposition $E = \cup_i E_i$ such that each $E_i \subset E$ is a sub pinched product. (See Figure 17.)

The product axiom will give a map $\pi^* : \mathcal{C}(X) \to \mathcal{C}(E)$ for each pinched product $\pi : E \to X$. Morphisms in the image of π^* will be called product morphisms. Before stating the axiom, we illustrate it in our two motivating examples of n-categories. In the case where $\mathcal{C}(X) = \{f : X \to T\}$, we define $\pi^*(f) = f \circ \pi$. In the case where $\mathcal{C}(X)$ is the set of all labeled embedded cell complexes K in X, define $\pi^*(K) = \pi^{-1}(K)$, with each codimension i cell $\pi^{-1}(c)$ labeled by the same (traditional) i-morphism as the corresponding codimension i cell c.

Axiom 6.1.8 (Product (identity) morphisms)**.** *For each pinched product $\pi : E \to X$, with X a k-ball and E a $k{+}m$-ball ($m \geq 1$), there is a map $\pi^* : \mathcal{C}(X) \to \mathcal{C}(E)$. These maps must satisfy the following conditions.*

1. *If $\pi : E \to X$ and $\pi' : E' \to X'$ are pinched products, and if $f : X \to X'$ and $\tilde{f} : E \to E'$ are*

maps such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{f} & X' \end{array}$$

commutes, then we have

$$\pi'^* \circ f = \tilde{f} \circ \pi^*.$$

2. *Product morphisms are compatible with gluing (composition). Let $\pi : E \to X$, $\pi_1 : E_1 \to X_1$, and $\pi_2 : E_2 \to X_2$ be pinched products with $E = E_1 \cup E_2$. Let $a \in \mathcal{C}(X)$, and let a_i denote the restriction of a to $X_i \subset X$. Then*

$$\pi^*(a) = \pi_1^*(a_1) \bullet \pi_2^*(a_2).$$

3. *Product morphisms are associative. If $\pi : E \to X$ and $\rho : D \to E$ are pinched products then*

$$\rho^* \circ \pi^* = (\pi \circ \rho)^*.$$

4. *Product morphisms are compatible with restriction. If we have a commutative diagram*

$$\begin{array}{ccc} D & \hookrightarrow & E \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \hookrightarrow & X \end{array}$$

such that ρ and π are pinched products, then

$$\operatorname{res}_D \circ \pi^* = \rho^* \circ \operatorname{res}_Y .$$

All of the axioms listed above hold for both ordinary n-categories and A_∞ n-categories. The last axiom (below), concerning actions of homeomorphisms in the top dimension n, distinguishes the two cases.

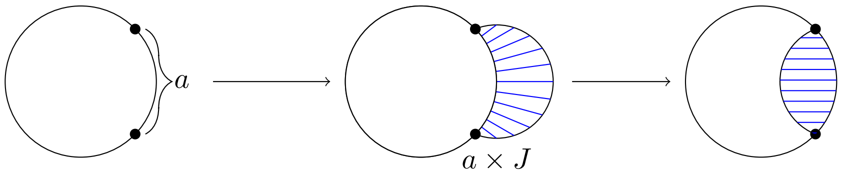
We start with the plain n-category case.

Axiom 6.1.9 ([**preliminary**] Isotopy invariance in dimension n)**.** *Let X be an n-ball and $f : X \to X$ be a homeomorphism which restricts to the identity on ∂X and is isotopic (rel boundary) to the identity. Then f acts trivially on $\mathcal{C}(X)$; that is $f(a) = a$ for all $a \in \mathcal{C}(X)$.*

This axiom needs to be strengthened to force product morphisms to act as the identity. Let X be an n-ball and $Y \subset \partial X$ be an $n{-}1$-ball. Let J be a 1-ball (interval). We have a collaring homeomorphism $s_{Y,J} : X \cup_Y (Y \times J) \to X$. (Here we use $Y \times J$ with boundary entirely pinched.) We define a map

$$\begin{aligned} \psi_{Y,J} : \mathcal{C}(X) &\to \mathcal{C}(X) \\ a &\mapsto s_{Y,J}(a \cup ((a|_Y) \times J)). \end{aligned}$$

(See Figure 18.) We call a map of this form a *collar map*. It can be thought of as the action



$$\mathcal{C}(X) \xrightarrow{\text{glue}} \mathcal{C}(X \cup \text{collar}) \xrightarrow{\text{homeo}} \mathcal{C}(X)$$

Figure 18: Extended homeomorphism.

of the inverse of a map which projects a collar neighborhood of Y onto Y, or as the limit of homeomorphisms $X \to X$ which expand a very thin collar of Y to a larger collar. We call the equivalence relation generated by collar maps and homeomorphisms isotopic (rel boundary) to the identity *extended isotopy*.

The revised axiom is

Axiom 6.1.10 (**[plain version]** Extended isotopy invariance in dimension n.)**.** *Let X be an n-ball and $f : X \to X$ be a homeomorphism which restricts to the identity on ∂X and isotopic (rel boundary) to the identity. Then f acts trivially on $\mathcal{C}(X)$. In addition, collar maps act trivially on $\mathcal{C}(X)$.*

For A_∞ n-categories, we replace isotopy invariance with the requirement that families of homeomorphisms act. For the moment, assume that our n-morphisms are enriched over chain complexes. Let $\mathrm{Homeo}_\partial(X)$ denote homeomorphisms of X which fix ∂X and $C_*(\mathrm{Homeo}_\partial(X))$ denote the singular chains on this space.

Axiom 6.1.11 (**[A_∞ version]** Families of homeomorphisms act in dimension n.)**.** *For each n-ball X and each $c \in \underrightarrow{\mathcal{C}}(\partial X)$ we have a map of chain complexes*

$$C_*(\mathrm{Homeo}_\partial(X)) \otimes \mathcal{C}(X; c) \to \mathcal{C}(X; c).$$

These action maps are required to be associative up to homotopy, and also compatible with composition (gluing) in the sense that a diagram like the one in Theorem 5.2.1 commutes.

We should strengthen the above A_∞ axiom to apply to families of collar maps. To do this we need to explain how collar maps form a topological space. Roughly, the set of collared $n-1$-balls in the boundary of an n-ball has a natural topology, and we can replace the class of all intervals J with intervals contained in $\mathbb{R}$. Having chains on the space of collar maps act gives rise to coherence maps involving weak identities. We will not pursue this in detail here.

Note that if we take homology of chain complexes, we turn an A_∞ n-category into a plain n-category (enriched over graded groups). In a different direction, if we enrich over topological spaces instead of chain complexes, we get a space version of an A_∞ n-category, with $\mathrm{Homeo}_\partial(X)$ acting instead of $C_*(\mathrm{Homeo}_\partial(X))$. Taking singular chains converts such a space type A_∞ n-category into a chain complex type A_∞ n-category.

The alert reader will have already noticed that our definition of a (plain) n-category is extremely similar to our definition of a system of fields. There are two differences. First, for the n-category definition we restrict our attention to balls (and their boundaries), while for fields we consider all manifolds. Second, in category definition we directly impose isotopy invariance in dimension n, while in the fields definition we instead remember a subspace of local relations which contain differences of isotopic fields. (Recall that the compensation for this complication is that we can demand that the gluing map for fields is injective.) Thus a system of fields and local relations $(\mathcal{F}, U)$ determines an n-category $\mathcal{C}_{\mathcal{F},U}$ simply by restricting our attention to balls and, at level n, quotienting out by the local relations:

$$\mathcal{C}_{\mathcal{F},U}(B^k) = \begin{cases} \mathcal{F}(B) & \text{when } k < n, \\ \mathcal{F}(B)/U(B) & \text{when } k = n. \end{cases}$$

This n-category can be thought of as the local part of the fields. Conversely, given a topological n-category we can construct a system of fields via a colimit construction; see §6.3 below.

6.2 Examples of n-categories

We now describe several classes of examples of n-categories satisfying our axioms. We typically specify only the morphisms; the rest of the data for the category (restriction maps, gluing, product morphisms, action of homeomorphisms) is usually obvious.

Example 6.2.1 (Maps to a space)**.** Let T be a topological space. We define $\pi_{\leq n}(T)$, the fundamental n-category of T, as follows. For X a k-ball with $k < n$, define $\pi_{\leq n}(T)(X)$ to be the set of all continuous maps from X to T. For X an n-ball define $\pi_{\leq n}(T)(X)$ to be continuous maps from X to T modulo homotopies fixed on ∂X. (Note that homotopy invariance implies isotopy invariance.) For $a \in \mathcal{C}(X)$ define the product morphism $a \times D \in \mathcal{C}(X \times D)$ to be $a \circ \pi_X$, where $\pi_X : X \times D \to X$ is the projection.

Example 6.2.2 (Maps to a space, with a fiber)**.** We can modify the example above, by fixing a closed m-manifold F, and defining $\pi^{\times F}_{\leq n}(T)(X) = \text{Maps}(X \times F \to T)$, otherwise leaving the definition in Example 6.2.1 unchanged. Taking F to be a point recovers the previous case.

Example 6.2.3 (Linearized, twisted, maps to a space)**.** We can linearize Examples 6.2.1 and 6.2.2 as follows. Let α be an $(n{+}m{+}1)$-cocycle on T with values in a ring R (have in mind the trivial cocycle). For X of dimension less than n define $\pi^{\alpha,\times F}_{\leq n}(T)(X)$ as before, ignoring α. For X an n-ball and $c \in \text{Maps}(\partial X \times F \to T)$ define $\pi^{\alpha,\times F}_{\leq n}(T)(X; c)$ to be the R-module of finite linear combinations of continuous maps from $X \times F$ to T, modulo the relation that if a is homotopic to b (rel boundary) via a homotopy $h : X \times F \times I \to T$, then $a = \alpha(h)b$. (In order for this to be well-defined we must choose α to be zero on degenerate simplices. Alternatively, we could equip the balls with fundamental classes.)

Example 6.2.4 (n-categories from TQFTs)**.** Let $\mathcal{F}$ be a TQFT in the sense of §2: an n-dimensional system of fields (also denoted $\mathcal{F}$) and local relations. Let W be an $n{-}j$-manifold. Define the j-category $\mathcal{F}(W)$ as follows. If X is a k-ball with $k < j$, let $\mathcal{F}(W)(X) \stackrel{\text{def}}{=} \mathcal{F}(W \times X)$. If X is a j-ball and $c \in \underrightarrow{\mathcal{F}(W)}(\partial X)$, let $\mathcal{F}(W)(X; c) \stackrel{\text{def}}{=} A_{\mathcal{F}}(W \times X; c)$.

The next example is only intended to be illustrative, as we don't specify which definition of a "traditional n-category" we intend. Further, most of these definitions don't even have an agreed-upon notion of "strong duality", which we assume here.

Example 6.2.5 (Traditional n-categories)**.** Given a "traditional n-category with strong duality" C define $\mathcal{C}(X)$, for X a k-ball with $k < n$, to be the set of all C-labeled embedded cell complexes of X (c.f. §2). For X an n-ball and $c \in \underrightarrow{\mathcal{C}}(\partial X)$, define $\mathcal{C}(X; c)$ to be finite linear combinations of C-labeled embedded cell complexes of X modulo the kernel of the evaluation map. Define a product morphism $a \times D$, for D an m-ball, to be the product of the cell complex of a with D, with each cell labelled according to the corresponding cell for a. (These two cells have the same codimension.) More generally, start with an $n+m$-category C and a closed m-manifold F. Define $\mathcal{C}(X)$, for $\dim(X) < n$, to be the set of all C-labeled embedded cell complexes of $X \times F$. Define $\mathcal{C}(X; c)$, for X an n-ball, to be the dual Hilbert space $A(X \times F; c)$. (See §2.4.)

Example 6.2.6 (The bordism n-category, plain version)**.** For a k-ball X, $k < n$, define $\mathrm{Bord}^n(X)$ to be the set of all k-dimensional submanifolds W of $X \times \mathbb{R}^\infty$ such that the projection $W \to X$ is transverse to ∂X. For an n-ball X define $\mathrm{Bord}^n(X)$ to be homeomorphism classes (rel boundary) of such n-dimensional submanifolds; we identify W and W' if $\partial W = \partial W'$ and there is a homeomorphism $W \to W'$ which restricts to the identity on the boundary.

Example 6.2.7 (Chains (or space) of maps to a space)**.** We can modify Example 6.2.1 above to define the fundamental A_∞ n-category $\pi^\infty_{\leq n}(T)$ of a topological space T. For a k-ball X, with $k < n$, the set $\pi^\infty_{\leq n}(T)(X)$ is just $\mathrm{Maps}(X \to T)$. Define $\pi^\infty_{\leq n}(T)(X; c)$ for an n-ball X and $c \in \pi^\infty_{\leq n}(T)(\partial X)$ to be the chain complex

$$C_*(\mathrm{Maps}_c(X \times F \to T)),$$

where Maps_c denotes continuous maps restricting to c on the boundary, and C_* denotes singular chains. Alternatively, if we take the n-morphisms to be simply $\mathrm{Maps}_c(X \times F \to T)$, we get an A_∞ n-category enriched over spaces.

See also Theorem 7.3.1 below, recovering $C_*(\mathrm{Maps}(M \to T))$ up to homotopy as the blob complex of M with coefficients in $\pi^\infty_{\leq n}(T)$.

Example 6.2.8 (Blob complexes of balls (with a fiber))**.** Fix an $n-k$-dimensional manifold F and an n-dimensional system of fields $\mathcal{E}$. We will define an A_∞ k-category $\mathcal{C}$. When X is a m-ball, with $m < k$, define $\mathcal{C}(X) = \mathcal{E}(X \times F)$. When X is an k-ball, define $\mathcal{C}(X; c) = \mathcal{B}^{\mathcal{E}}_*(X \times F; c)$ where $\mathcal{B}^{\mathcal{E}}_*$ denotes the blob complex based on $\mathcal{E}$.

This example will be used in Theorem 7.1.1 below, which allows us to compute the blob complex of a product. Notice that with F a point, the above example is a construction turning a topological n-category $\mathcal{C}$ into an A_∞ n-category. We think of this as providing a "free resolution" of the topological n-category. In fact, there is also a trivial, but mostly uninteresting, way to do this: we can think of each vector space associated to an n-ball as a chain complex concentrated in degree 0, and take $C_*(\mathrm{Diff}(B))$ to act trivially.

Beware that the "free resolution" of the topological n-category $\pi_{\leq n}(T)$ is not the A_∞ n-category $\pi^\infty_{\leq n}(T)$. It's easy to see that with $n = 0$, the corresponding system of fields is just linear combinations of connected components of T, and the local relations are trivial. There's no way for the blob complex to magically recover all the data of $\pi^\infty_{\leq 0}(T) \cong C_* T$.

Example 6.2.9 (The bordism n-category, A_∞ version)**.** As in Example 6.2.6, for X a k-ball, $k < n$, we define $\mathrm{Bord}^{n,\infty}(X)$ to be the set of all k-dimensional submanifolds W of $X \times \mathbb{R}^\infty$ such that the projection $W \to X$ is transverse to ∂X. For an n-ball X with boundary condition c define

$\mathrm{Bord}^{n,\infty}(X;c)$ to be the space of all k-dimensional submanifolds W of $X\times\mathbb{R}^\infty$ such that W coincides with c at $\partial X\times\mathbb{R}^\infty$. (The topology on this space is induced by ambient isotopy rel boundary. This is homotopy equivalent to a disjoint union of copies $\mathrm{BHomeo}(W')$, where W' runs though representatives of homeomorphism types of such manifolds.)

Let $\mathcal{EB}_n$ be the operad of smooth embeddings of k (little) copies of the standard n-ball B^n into another (big) copy of B^n. (We require that the interiors of the little balls be disjoint, but their boundaries are allowed to meet. Note in particular that the space for $k=1$ contains a copy of $\mathrm{Diff}(B^n)$, namely the embeddings of a "little" ball with image all of the big ball B^n. (But note also that this inclusion is not necessarily a homotopy equivalence.) The operad $\mathcal{EB}_n$ is homotopy equivalent to the standard framed little n-ball operad: by shrinking the little balls (precomposing them with dilations), we see that both operads are homotopic to the space of k framed points in B^n. It is easy to see that n-fold loop spaces $\Omega^n(T)$ have an action of $\mathcal{EB}_n$.

Example 6.2.10 (E_n algebras)**.** Let A be an $\mathcal{EB}_n$-algebra. Note that this implies a $\mathrm{Diff}(B^n)$ action on A, since $\mathcal{EB}_n$ contains a copy of $\mathrm{Diff}(B^n)$. We will define an A_∞ n-category $\mathcal{C}^A$. If X is a ball of dimension $k<n$, define $\mathcal{C}^A(X)$ to be a point. In other words, the k-morphisms are trivial for $k<n$. If X is an n-ball, we define $\mathcal{C}^A(X)$ via a colimit construction. (Plain colimit, not homotopy colimit.) Let J be the category whose objects are embeddings of a disjoint union of copies of the standard ball B^n into X, and who morphisms are given by engulfing some of the embedded balls into a single larger embedded ball. To each object of J we associate $A^{\times m}$ (where m is the number of balls), and to each morphism of J we associate a morphism coming from the $\mathcal{EB}_n$ action on A. Alternatively and more simply, we could define $\mathcal{C}^A(X)$ to be $\mathrm{Diff}(B^n\to X)\times A$ modulo the diagonal action of $\mathrm{Diff}(B^n)$. The remaining data for the A_∞ n-category — composition and $\mathrm{Diff}(X\to X')$ action — also comes from the $\mathcal{EB}_n$ action on A.

Conversely, one can show that a topological A_∞ n-category $\mathcal{C}$, where the k-morphisms $\mathcal{C}(X)$ are trivial (single point) for $k<n$, gives rise to an $\mathcal{EB}_n$-algebra.

If we apply the homotopy colimit construction of the next subsection to this example, we get an instance of Lurie's topological chiral homology construction.

6.3 From balls to manifolds

In this section we show how to extend an n-category $\mathcal{C}$ as described above (of either the plain or A_∞ variety) to an invariant of manifolds, which we denote by $\underrightarrow{\mathcal{C}}$. This extension is a certain colimit, and the arrow in the notation is intended as a reminder of this.

In the case of plain n-categories, this construction factors into a construction of a system of fields and local relations, followed by the usual TQFT definition of a vector space invariant of manifolds given as Definition 2.4.1. For an A_∞ n-category, $\underrightarrow{\mathcal{C}}$ is defined using a homotopy colimit instead. Recall that we can take a plain n-category $\mathcal{C}$ and pass to the "free resolution", an A_∞ n-category $\mathcal{B}_*(\mathcal{C})$, by computing the blob complex of balls (recall Example 6.2.8 above). We will show in Corollary 7.1.3 below that the homotopy colimit invariant for a manifold M associated to this A_∞ n-category is actually the same as the original blob complex for M with coefficients in $\mathcal{C}$.

Recall that we've already anticipated this construction in the previous section, inductively defining $\underrightarrow{\mathcal{C}}$ on k-spheres in terms of $\mathcal{C}$ on k-balls, so that we can state the boundary axiom for $\mathcal{C}$ on $k+1$-balls.

We will first define the "decomposition" poset $\mathfrak{D}(W)$ for any k-manifold W, for $1 \le k \le n$. An n-category $\mathcal{C}$ provides a functor from this poset to the category of sets, and we will define $\underrightarrow{\mathcal{C}}(W)$ as a suitable colimit (or homotopy colimit in the A_∞ case) of this functor. We'll later give a more explicit description of this colimit. In the case that the n-category $\mathcal{C}$ is enriched (e.g. associates vector spaces or chain complexes to n-balls with boundary data), then the resulting colimit is also enriched, that is, the set associated to W splits into subsets according to boundary data, and each of these subsets has the appropriate structure (e.g. a vector space or chain complex).

Recall (Definition 3.1.3) that a *ball decomposition* of W is a sequence of gluings $M_0 \to M_1 \to \cdots \to M_m = W$ such that M_0 is a disjoint union of balls $\sqcup_a X_a$. Abusing notation, we let X_a denote both the ball (component of M_0) and its image in W (which is not necessarily a ball — parts of ∂X_a may have been glued together). Define a *permissible decomposition* of W to be a map

$$\coprod_a X_a \to W,$$

which can be completed to a ball decomposition $\sqcup_a X_a = M_0 \to \cdots \to M_m = W$. Roughly, a permissible decomposition is like a ball decomposition where we don't care in which order the balls are glued up to yield W, so long as there is some (non-pathological) way to glue them.

Given permissible decompositions $x = \{X_a\}$ and $y = \{Y_b\}$ of W, we say that x is a refinement of y, or write $x \le y$, if there is a ball decomposition $\sqcup_a X_a = M_0 \to \cdots \to M_m = W$ with $\sqcup_b Y_b = M_i$ for some i.

Definition 6.3.1. *The poset $\mathfrak{D}(W)$ has objects the permissible decompositions of W, and a unique morphism from x to y if and only if x is a refinement of y. See Figure 19 for an example.*

An n-category $\mathcal{C}$ determines a functor $\psi_{\mathcal{C};W}$ from $\mathfrak{D}(W)$ to the category of sets (possibly with additional structure if $k = n$). Each k-ball X of a decomposition y of W has its boundary decomposed into $k{-}1$-balls, and, as described above, we have a subset $\mathcal{C}(X)_\pitchfork \subset \mathcal{C}(X)$ of morphisms whose boundaries are splittable along this decomposition.

Definition 6.3.2. *Define the functor $\psi_{\mathcal{C};W} : \mathfrak{D}(W) \to$ **Set** as follows. For a decomposition $x = \bigsqcup_a X_a$ in $\mathfrak{D}(W)$, $\psi_{\mathcal{C};W}(x)$ is the subset*

$$\psi_{\mathcal{C};W}(x) \subset \prod_a \mathcal{C}(X_a)_\pitchfork \tag{6.1}$$

where the restrictions to the various pieces of shared boundaries amongst the cells X_a all agree (this is a fibered product of all the labels of n-cells over the labels of $n-1$-cells). If x is a refinement of y, the map $\psi_{\mathcal{C};W}(x) \to \psi_{\mathcal{C};W}(y)$ is given by the composition maps of $\mathcal{C}$.

If $k = n$ in the above definition and we are enriching in some auxiliary category, we need to say a bit more. We can rewrite Equation 6.1 as

$$\psi_{\mathcal{C};W}(x) \stackrel{\text{def}}{=} \coprod_\beta \prod_a \mathcal{C}(X_a; \beta), \tag{6.2}$$

where β runs through labelings of the $k{-}1$-skeleton of the decomposition (which are compatible when restricted to the $k{-}2$-skeleton), and $\mathcal{C}(X_a; \beta)$ means the subset of $\mathcal{C}(X_a)$ whose restriction to

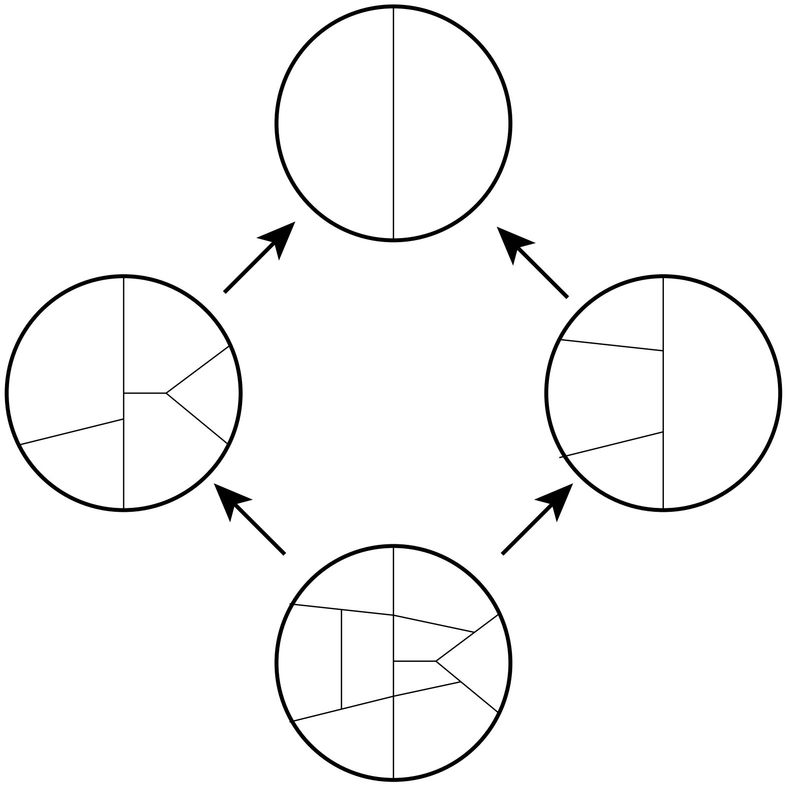


Figure 19: A small part of $\mathfrak{D}(W)$

∂X_a agress with β. If we are enriching over $\mathcal{S}$ and $k = n$, then $\mathcal{C}(X_a; \beta)$ is an object in $\mathcal{S}$ and the coproduct and product in Equation 6.2 should be replaced by the approriate operations in $\mathcal{S}$ (e.g. direct sum and tensor product if $\mathcal{S}$ is Vect).

Finally, we construct $\underrightarrow{\mathcal{C}}(W)$ as the appropriate colimit of $\psi_{\mathcal{C};W}$:

Definition 6.3.3 (System of fields functor)**.** *If $\mathcal{C}$ is an n-category enriched in sets or vector spaces, $\underrightarrow{\mathcal{C}}(W)$ is the usual colimit of the functor $\psi_{\mathcal{C};W}$. That is, for each decomposition x there is a map $\psi_{\mathcal{C};W}(x) \to \underrightarrow{\mathcal{C}}(W)$, these maps are compatible with the refinement maps above, and $\underrightarrow{\mathcal{C}}(W)$ is universal with respect to these properties.*

Definition 6.3.4 (System of fields functor, A_∞ case)**.** *When $\mathcal{C}$ is an A_∞ n-category, $\underrightarrow{\mathcal{C}}(W)$ for W a k-manifold with $k < n$ is defined as above, as the colimit of $\psi_{\mathcal{C};W}$. When W is an n-manifold, the chain complex $\underrightarrow{\mathcal{C}}(W)$ is the homotopy colimit of the functor $\psi_{\mathcal{C};W}$.*

We can specify boundary data $c \in \underrightarrow{\mathcal{C}}(\partial W)$, and define functors $\psi_{\mathcal{C};W,c}$ with values the subsets of those of $\psi_{\mathcal{C};W}$ which agree with c on the boundary of W.

We now give more concrete descriptions of the above colimits.

In the non-enriched case (e.g. $k < n$), where each $\mathcal{C}(X_a; \beta)$ is just a set, the colimit is

$$\underrightarrow{\mathcal{C}}(W, c) = \left(\coprod_x \coprod_\beta \prod_a \mathcal{C}(X_a; \beta) \right) \Big/ \sim,$$

where x runs through decomposition of W, and $\sim$ is the obvious equivalence relation induced by refinement and gluing. If $\mathcal{C}$ is enriched over vector spaces and W is an n-manifold, we can take

$$\underrightarrow{\mathcal{C}}(W,c) = \left(\bigoplus_x \bigoplus_\beta \bigotimes_a \mathcal{C}(X_a;\beta) \right) \Big/ K,$$

where K is the vector space spanned by elements $a - g(a)$, with $a \in \psi_{\mathcal{C};W,c}(x)$ for some decomposition x, and $g : \psi_{\mathcal{C};W,c}(x) \to \psi_{\mathcal{C};W,c}(y)$ is value of $\psi_{\mathcal{C};W,c}$ on some antirefinement $x \le y$.

In the A_∞ case, enriched over chain complexes, the concrete description of the homotopy colimit is more involved. We will describe two different (but homotopy equivalent) versions of the homotopy colimit of $\psi_{\mathcal{C};W}$. The first is the usual one, which works for any indexing category. The second construction, which we call the *local* homotopy colimit, is more closely related to the blob complex construction of §3.1 and takes advantage of local (gluing) properties of the indexing category $\mathfrak{D}(W)$.

Define an m-sequence in W to be a sequence $x_0 \le x_1 \le \cdots \le x_m$ of permissible decompositions of W. Such sequences (for all m) form a simplicial set in $\mathfrak{D}(W)$. Define $\underrightarrow{\mathcal{C}}(W)$ as a vector space via

$$\underrightarrow{\mathcal{C}}(W) = \bigoplus_{(x_i)} \psi_{\mathcal{C};W}(x_0)[m],$$

where the sum is over all m and all m-sequences (x_i), and each summand is degree shifted by m. Elements of a summand indexed by an m-sequence will be call m-simplices. We endow $\underrightarrow{\mathcal{C}}(W)$ with a differential which is the sum of the differential of the $\psi_{\mathcal{C};W}(x_0)$ summands plus another term using the differential of the simplicial set of m-sequences. More specifically, if $(a,\bar{x})$ denotes an element in the $\bar{x}$ summand of $\underrightarrow{\mathcal{C}}(W)$ (with $\bar{x} = (x_0,\ldots,x_k)$), define

$$\partial(a,\bar{x}) = (\partial a,\bar{x}) + (-1)^{\deg a}(g(a), d_0(\bar{x})) + (-1)^{\deg a} \sum_{j=1}^{k} (-1)^j (a, d_j(\bar{x})),$$

where $d_j(\bar{x}) = (x_0,\ldots,x_{j-1},x_{j+1},\ldots,x_k)$ and $g : \psi_{\mathcal{C}}(x_0) \to \psi_{\mathcal{C}}(x_1)$ is the usual gluing map coming from the antirefinement $x_0 \le x_1$.

We can think of this construction as starting with a disjoint copy of a complex for each permissible decomposition (the 0-simplices). Then we glue these together with mapping cylinders coming from gluing maps (the 1-simplices). Then we kill the extra homology we just introduced with mapping cylinders between the mapping cylinders (the 2-simplices), and so on.

Next we describe the local homotopy colimit. This is similar to the usual homotopy colimit, but using a cone-product set (Remark 3.1.7) in place of a simplicial set. The cone-product m-polyhedra for the set are pairs (x,E), where x is a decomposition of W and E is an m-blob diagram such that each blob is a union of balls of x. (Recall that this means that the interiors of each pair of blobs (i.e. balls) of E are either disjoint or nested.) To each (x,E) we associate the chain complex $\psi_{\mathcal{C};W}(x)$, shifted in degree by m. The boundary has a term for omitting each blob of E. If we omit an innermost blob then we replace x by the formal difference $x - \mathrm{gl}(x)$, where $\mathrm{gl}(x)$ is obtained from x by gluing together the balls of x contained in the blob we are omitting. The gluing maps of $\mathcal{C}$ give us a maps from $\psi_{\mathcal{C};W}(x)$ to $\psi_{\mathcal{C};W}(\mathrm{gl}(x))$.

One can show that the usual hocolimit and the local hocolimit are homotopy equivalent using an Eilenberg-Zilber type subdivision argument.

$\underset{\to}{\mathcal{C}}(W)$ is functorial with respect to homeomorphisms of k-manifolds. Restricting the k-spheres, we have now proved Lemma 6.1.2.

It is easy to see that there are well-defined maps $\underset{\to}{\mathcal{C}}(W) \to \underset{\to}{\mathcal{C}}(\partial W)$, and that these maps comprise a natural transformation of functors.

Lemma 6.3.5. *Let W be a manifold of dimension less than n. Then for each decomposition x of W the natural map $\psi_{\mathcal{C};W}(x) \to \underset{\to}{\mathcal{C}}(W)$ is injective.*

Proof. $\underset{\to}{\mathcal{C}}(W)$ is a colimit of a diagram of sets, and each of the arrows in the diagram is injective. Concretely, the colimit is the disjoint union of the sets (one for each decomposition of W), modulo the relation which identifies the domain of each of the injective maps with it's image.

To save ink and electrons we will simplify notation and write $\psi(x)$ for $\psi_{\mathcal{C};W}(x)$.

Suppose $a, \hat{a} \in \psi(x)$ have the same image in $\underset{\to}{\mathcal{C}}(W)$ but $a \neq \hat{a}$. Then there exist

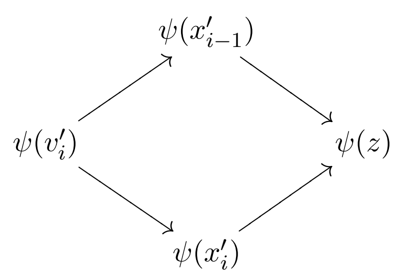
- decompositions $x = x_0, x_1, \ldots, x_{k-1}, x_k = x$ and $v_1, \ldots, v_k$ of W;
- anti-refinements $v_i \to x_i$ and $v_i \to x_{i-1}$; and
- elements $a_i \in \psi(x_i)$ and $b_i \in \psi(v_i)$, with $a_0 = a$ and $a_k = \hat{a}$, such that b_i and b_{i+1}both map to (glue up to) a_i.

In other words, we have a zig-zag of equivalences starting at a and ending at $\hat{a}$. The idea of the proof is to produce a similar zig-zag where everything antirefines to the same disjoint union of balls, and then invoke Axiom 6.1.6 which ensures associativity.

Let z be a decomposition of W which is in general position with respect to all of the x_i's and v_i's. There there decompositions x_i' and v_i' (for all i) such that

- x_i' antirefines to x_i and z;
- v_i' antirefines to x_i', x_{i-1}' and v_i;
- b_i is the image of some $b_i' \in \psi(v_i')$; and
- a_i is the image of some $a_i' \in \psi(x_i')$, which in turn is the image of b_i' and b_{i+1}'.

Now consider the diagrams



The associativity axiom applied to this diagram implies that a_{i-1}' and a_i' map to the same element $c \in \psi(z)$. Therefore a_0' and a_k' both map to c. But a_0' and a_k' are both elements of $\psi(x_0')$ (because $x_k' = x_0'$). So by the injectivity clause of the composition axiom, we must have that $a_0' = a_k'$. But this implies that $a = a_0 = a_k = \hat{a}$, contrary to our assumption that $a \neq \hat{a}$. □

6.4 Modules

Next we define plain and A_∞ n-category modules. The definition will be very similar to that of n-categories, but with k-balls replaced by *marked k-balls,* defined below.

Our motivating example comes from an $(m{-}n{+}1)$-dimensional manifold W with boundary in the context of an $m{+}1$-dimensional TQFT. Such a W gives rise to a module for the n-category associated to ∂W. This will be explained in more detail as we present the axioms.

Throughout, we fix an n-category $\mathcal{C}$. For all but one axiom, it doesn't matter whether $\mathcal{C}$ is a topological n-category or an A_∞ n-category. We state the final axiom, regarding actions of homeomorphisms, differently in the two cases.

Define a *marked k-ball* to be a pair (B, N) homeomorphic to the pair

$$(\text{standard } k\text{-ball}, \text{northern hemisphere in boundary of standard } k\text{-ball}).$$

We call B the ball and N the marking. A homeomorphism between marked k-balls is a homeomorphism of balls which restricts to a homeomorphism of markings.

Module Axiom 6.4.1 (Module morphisms)**.** *For each $0 \le k \le n$, we have a functor $\mathcal{M}_k$ from the category of marked k-balls and homeomorphisms to the category of sets and bijections.*

(As with n-categories, we will usually omit the subscript k.)

For example, let $\mathcal{D}$ be the TQFT which assigns to a k-manifold N the set of maps from N to T (for $k \le m$), modulo homotopy (and possibly linearized) if $k = m$. Let W be an $(m{-}n{+}1)$-dimensional manifold with boundary. Let $\mathcal{C}$ be the n-category with $\mathcal{C}(X) \overset{\text{def}}{=} \mathcal{D}(X \times \partial W)$. Let $\mathcal{M}(B, N) \overset{\text{def}}{=} \mathcal{D}((B \times \partial W) \cup (N \times W))$ (see Example 6.2.2). (The union is along $N \times \partial W$.)

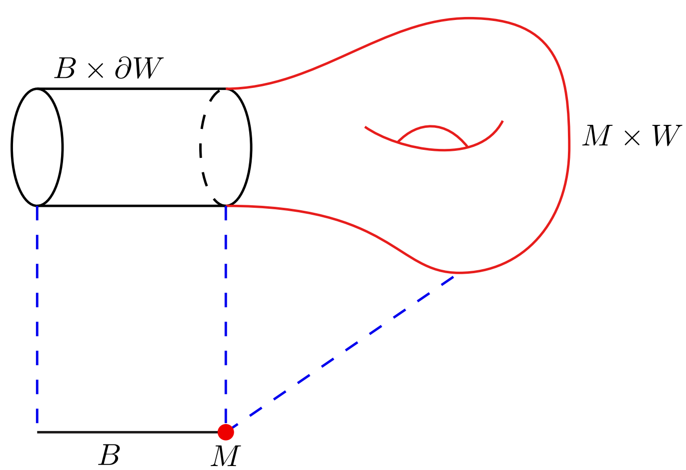


Figure 20: From manifold with boundary collar to marked ball

Define the boundary of a marked k-ball (B, N) to be the pair $(\partial B \setminus N, \partial N)$. Call such a thing a marked $k{-}1$-hemisphere.

Lemma 6.4.2. *For each $0 \le k \le n-1$, we have a functor $\underrightarrow{\mathcal{M}}_k$ from the category of marked k-hemispheres and homeomorphisms to the category of sets and bijections.*

The proof is exactly analogous to that of Lemma 6.1.2, and we omit the details. We use the same type of colimit construction.

In our example, $\underrightarrow{\mathcal{M}}(H) = \mathcal{D}(H \times \partial W \cup \partial H \times W)$.

Module Axiom 6.4.3 (Module boundaries (maps))**.** *For each marked k-ball M we have a map of sets $\partial : \mathcal{M}(M) \to \underrightarrow{\mathcal{M}}(\partial M)$. These maps, for various M, comprise a natural transformation of functors.*

Given $c \in \underrightarrow{\mathcal{M}}(\partial M)$, let $\mathcal{M}(M; c) \stackrel{\text{def}}{=} \partial^{-1}(c)$.

If the n-category $\mathcal{C}$ is enriched over some other category (e.g. vector spaces), then $\mathcal{M}(M; c)$ should be an object in that category for each marked n-ball M and $c \in \mathcal{C}(\partial M)$.

Lemma 6.4.4 (Boundary from domain and range)**.** *Let $H = M_1 \cup_E M_2$, where H is a marked $k{-}1$-hemisphere ($1 \le k \le n$), M_i is a marked $k{-}1$-ball, and $E = M_1 \cap M_2$ is a marked $k{-}2$-hemisphere. Let $\mathcal{M}(M_1) \times_{\mathcal{M}(E)} \mathcal{M}(M_2)$ denote the fibered product of the two maps $\partial : \mathcal{M}(M_i) \to \underrightarrow{\mathcal{M}}(E)$. Then we have an injective map*

$$\mathrm{gl}_E : \mathcal{M}(M_1) \times_{\underrightarrow{\mathcal{M}}(E)} \mathcal{M}(M_2) \hookrightarrow \underrightarrow{\mathcal{M}}(H)$$

which is natural with respect to the actions of homeomorphisms.

Again, this is in exact analogy with Lemma 6.1.4.

Let $\underrightarrow{\mathcal{M}}(H)_E$ denote the image of gl_E. We will refer to elements of $\underrightarrow{\mathcal{M}}(H)_E$ as "splittable along E" or "transverse to E".

Lemma 6.4.5 (Module to category restrictions)**.** *For each marked k-hemisphere H there is a restriction map $\underrightarrow{\mathcal{M}}(H) \to \mathcal{C}(H)$. ($\mathcal{C}(H)$ means apply $\mathcal{C}$ to the underlying k-ball of H.) These maps comprise a natural transformation of functors.*

Note that combining the various boundary and restriction maps above (for both modules and n-categories) we have for each marked k-ball (B, N) and each $k{-}1$-ball $Y \subset \partial B \setminus N$ a natural map from a subset of $\mathcal{M}(B, N)$ to $\mathcal{C}(Y)$. The subset is the subset of morphisms which are appropriately splittable (transverse to the cutting submanifolds). This fact will be used below.

In our example, the various restriction and gluing maps above come from restricting and gluing maps into T.

We require two sorts of composition (gluing) for modules, corresponding to two ways of splitting a marked k-ball into two (marked or plain) k-balls. (See Figure 21.)

First, we can compose two module morphisms to get another module morphism.

Module Axiom 6.4.6 (Module composition)**.** *Let $M = M_1 \cup_Y M_2$, where M, M_1 and M_2 are marked k-balls (with $0 \le k \le n$) and $Y = M_1 \cap M_2$ is a marked $k{-}1$-ball. Let $E = \partial Y$, which is a marked $k{-}2$-hemisphere. Note that each of M, M_1 and M_2 has its boundary split into two marked $k{-}1$-balls by E. We have restriction (domain or range) maps $\mathcal{M}(M_i)_E \to \mathcal{M}(Y)$. Let $\mathcal{M}(M_1)_E \times_{\mathcal{M}(Y)} \mathcal{M}(M_2)_E$ denote the fibered product of these two maps. Then (axiom) we have a map*

$$\mathrm{gl}_Y : \mathcal{M}(M_1)_E \times_{\mathcal{M}(Y)} \mathcal{M}(M_2)_E \to \mathcal{M}(M)_E$$

which is natural with respect to the actions of homeomorphisms, and also compatible with restrictions to the intersection of the boundaries of M and M_i. If $k < n$, or if $k = n$ and we are in the A_∞ case, we require that gl_Y is injective. (For $k = n$ in the plain (non-A_∞) case, see below.)

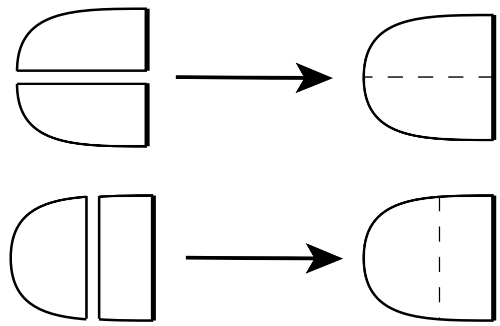


Figure 21: Module composition (top); n-category action (bottom).

Second, we can compose an n-category morphism with a module morphism to get another module morphism. We'll call this the action map to distinguish it from the other kind of composition.

Module Axiom 6.4.7 (n-category action)**.** *Let $M = X \cup_Y M'$, where M and M' are marked k-balls ($0 \le k \le n$), X is a plain k-ball, and $Y = X \cap M'$ is a $k{-}1$-ball. Let $E = \partial Y$, which is a $k{-}2$-sphere. We have restriction maps $\mathcal{M}(M')_E \to \mathcal{C}(Y)$ and $\mathcal{C}(X)_E \to \mathcal{C}(Y)$. Let $\mathcal{C}(X)_E \times_{\mathcal{C}(Y)} \mathcal{M}(M')_E$ denote the fibered product of these two maps. Then (axiom) we have a map*

$$\mathrm{gl}_Y : \mathcal{C}(X)_E \times_{\mathcal{C}(Y)} \mathcal{M}(M')_E \to \mathcal{M}(M)_E$$

which is natural with respect to the actions of homeomorphisms, and also compatible with restrictions to the intersection of the boundaries of X and M'. If $k < n$, or if $k = n$ and we are in the A_∞ case, we require that gl_Y is injective. (For $k = n$ in the plain (non-A_∞) case, see below.)

Module Axiom 6.4.8 (Strict associativity)**.** *The composition and action maps above are strictly associative. Given any decomposition of a large marked ball into smaller marked and unmarked balls any sequence of pairwise gluings yields (via composition and action maps) the same result.*

Note that the above associativity axiom applies to mixtures of module composition, action maps and n-category composition. See Figure 22.

The above three axioms are equivalent to the following axiom, which we state in slightly vague form.

Module multi-composition: *Given any splitting*

$$X_1 \sqcup \cdots \sqcup X_p \sqcup M_1 \sqcup \cdots \sqcup M_q \to M$$

of a marked k-ball M into small (marked and plain) k-balls M_i and X_j, there is a map from an appropriate subset (like a fibered product) of

$$\mathcal{C}(X_1) \times \cdots \times \mathcal{C}(X_p) \times \mathcal{M}(M_1) \times \cdots \times \mathcal{M}(M_q)$$

to $\mathcal{M}(M)$, and these various multifold composition maps satisfy an operad-type strict associativity condition.

The above operad-like structure is analogous to the swiss cheese operad [12].

We can define marked pinched products $\pi : E \to M$ of marked balls analogously to the plain ball case. Note that a marked pinched product can be decomposed into either two marked pinched products or a plain pinched product and a marked pinched product.

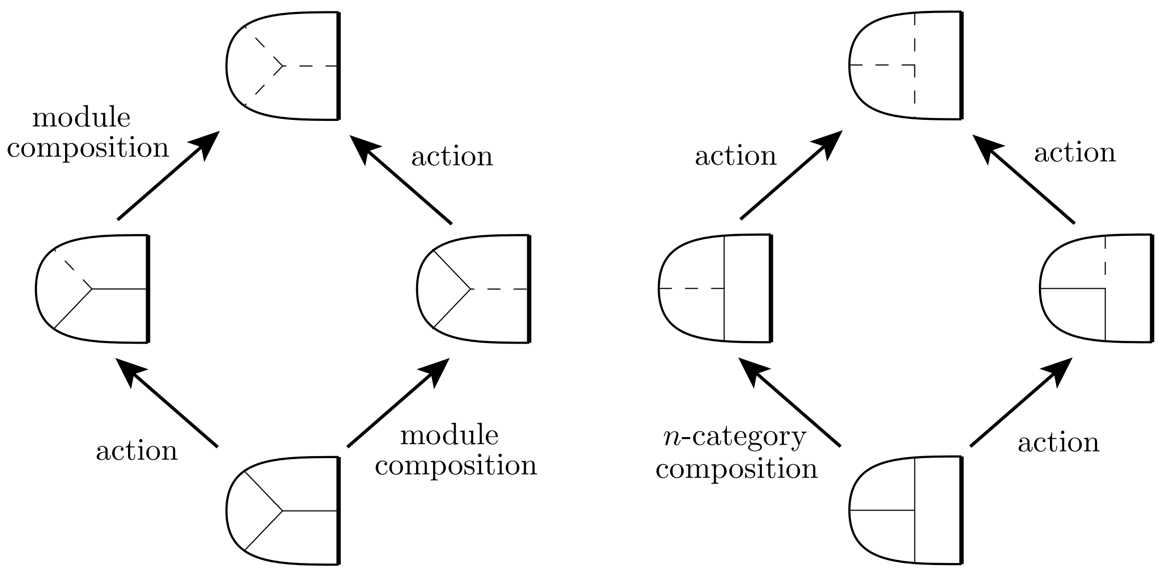


Figure 22: Two examples of mixed associativity

Module Axiom 6.4.9 (Product (identity) morphisms)**.** *For each pinched product $\pi : E \to M$, with M a marked k-ball and E a marked $k{+}m$-ball ($m \geq 1$), there is a map $\pi^* : \mathcal{M}(M) \to \mathcal{M}(E)$. These maps must satisfy the following conditions.*

1. *If $\pi : E \to M$ and $\pi' : E' \to M'$ are marked pinched products, and if $f : M \to M'$ and $\tilde{f} : E \to E'$ are maps such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

commutes, then we have

$$\pi'^* \circ f = \tilde{f} \circ \pi^*.$$

2. *Product morphisms are compatible with module composition and module action. Let $\pi : E \to M$, $\pi_1 : E_1 \to M_1$, and $\pi_2 : E_2 \to M_2$ be pinched products with $E = E_1 \cup E_2$. Let $a \in \mathcal{M}(M)$, and let a_i denote the restriction of a to $M_i \subset M$. Then*

$$\pi^*(a) = \pi_1^*(a_1) \bullet \pi_2^*(a_2).$$

Similarly, if $\rho : D \to X$ is a pinched product of plain balls and $E = D \cup E_1$, then

$$\pi^*(a) = \rho^*(a') \bullet \pi_1^*(a_1),$$

where a' is the restriction of a to D.

3. *Product morphisms are associative. If* $\pi : E \to M$ *and* $\rho : D \to E$ *are marked pinched products then*
$$\rho^* \circ \pi^* = (\pi \circ \rho)^*.$$

4. *Product morphisms are compatible with restriction. If we have a commutative diagram*
$$\begin{array}{ccc} D & \hookrightarrow & E \\ \rho \downarrow & & \downarrow \pi \\ Y & \hookrightarrow & M \end{array}$$
such that ρ *and* π *are pinched products, then*
$$\operatorname{res}_D \circ \pi^* = \rho^* \circ \operatorname{res}_Y .$$
(Y *could be either a marked or plain ball.)*

As in the n-category definition, once we have product morphisms we can define collar maps $\mathcal{M}(M) \to \mathcal{M}(M)$. Note that there are two cases: the collar could intersect the marking of the marked ball M, in which case we use a product on a morphism of $\mathcal{M}$; or the collar could be disjoint from the marking, in which case we use a product on a morphism of $\mathcal{C}$.

In our example, elements a of $\mathcal{M}(M)$ maps to T, and $\pi^*(a)$ is the pullback of a along a map associated to π.

There are two alternatives for the next axiom, according whether we are defining modules for plain n-categories or A_∞ n-categories. In the plain case we require

Module Axiom 6.4.10 (**[plain version]** Extended isotopy invariance in dimension n)**.** *Let* M *be a marked* n*-ball and* $f : M \to M$ *be a homeomorphism which restricts to the identity on* ∂M *and is isotopic (rel boundary) to the identity. Then* f *acts trivially on* $\mathcal{M}(M)$*. In addition, collar maps act trivially on* $\mathcal{M}(M)$*.*

We emphasize that the ∂M above means boundary in the marked k-ball sense. In other words, if $M = (B, N)$ then we require only that isotopies are fixed on $\partial B \setminus N$.

For A_∞ modules we require

Module Axiom 6.4.11 (**[A_∞ version]** Families of homeomorphisms act)**.** *For each marked* n*-ball* M *and each* $c \in \mathcal{M}(\partial M)$ *we have a map of chain complexes*
$$C_*(\mathrm{Homeo}_\partial(M)) \otimes \mathcal{M}(M; c) \to \mathcal{M}(M; c).$$
Here C_* *means singular chains and* $\mathrm{Homeo}_\partial(M)$ *is the space of homeomorphisms of* M *which fix* ∂M*. These action maps are required to be associative up to homotopy, as in Theorem 5.2.2, and also compatible with composition (gluing) in the sense that a diagram like the one in Theorem 5.2.1 commutes.*

As with the n-category version of the above axiom, we should also have families of collar maps act.

Note that the above axioms imply that an n-category module has the structure of an $n{-}1$-category. More specifically, let J be a marked 1-ball, and define $\mathcal{E}(X) \stackrel{\text{def}}{=} \mathcal{M}(X \times J)$, where X is a

k-ball and in the product $X \times J$ we pinch above the non-marked boundary component of J. (More specifically, we collapse $X \times P$ to a single point, where P is the non-marked boundary component of J.) Then $\mathcal{E}$ has the structure of an $n{-}1$-category.

All marked k-balls are homeomorphic, unless $k = 1$ and our manifolds are oriented or Spin (but not unoriented or $\text{Pin}_\pm$). In this case ($k = 1$ and oriented or Spin), there are two types of marked 1-balls, call them left-marked and right-marked, and hence there are two types of modules, call them right modules and left modules. In all other cases ($k > 1$ or unoriented or $\text{Pin}_\pm$), there is no left/right module distinction.

We now give some examples of modules over topological and A_∞ n-categories.

Example 6.4.12 (Examples from TQFTs)**.** Continuing Example 6.2.4, with $\mathcal{F}$ a TQFT, W an $n{-}j$-manifold, and $\mathcal{F}(W)$ the j-category associated to W. Let Y be an $(n{-}j{+}1)$-manifold with $\partial Y = W$. Define a $\mathcal{F}(W)$ module $\mathcal{F}(Y)$ as follows. If $M = (B, N)$ is a marked k-ball with $k < j$ let $\mathcal{F}(Y)(M) \stackrel{\text{def}}{=} \mathcal{F}((B \times W) \cup (N \times Y))$. If $M = (B, N)$ is a marked j-ball and $c \in \underrightarrow{\mathcal{F}(Y)}(\partial M)$ let $\mathcal{F}(Y)(M) \stackrel{\text{def}}{=} A_{\mathcal{F}}((B \times W) \cup (N \times Y); c)$.

Example 6.4.13 (Examples from the blob complex)**.** In the previous example, we can instead define $\mathcal{F}(Y)(M) \stackrel{\text{def}}{=} \mathcal{B}_*((B \times W) \cup (N \times Y), c; \mathcal{F})$ (when $\dim(M) = n$) and get a module for the A_∞ n-category associated to $\mathcal{F}$ as in Example 6.2.8.

Example 6.4.14. Suppose S is a topological space, with a subspace T. We can define a module $\pi_{\leq n}(S, T)$ so that on each marked k-ball (B, N) for $k < n$ the set $\pi_{\leq n}(S, T)(B, N)$ consists of all continuous maps of pairs $(B, N) \to (S, T)$ and on each marked n-ball (B, N) it consists of all such maps modulo homotopies fixed on $\partial B \setminus N$. This is a module over the fundamental n-category $\pi_{\leq n}(S)$ of S, from Example 6.2.1.

Modifications corresponding to Examples 6.2.2 and 6.2.3 are also possible, and there is an A_∞ version analogous to Example 6.2.7 given by taking singular chains.

6.5 Modules as boundary labels (colimits for decorated manifolds)

Fix a topological n-category or A_∞ n-category $\mathcal{C}$. Let W be a k-manifold ($k \leq n$), let $\{Y_i\}$ be a collection of disjoint codimension 0 submanifolds of ∂W, and let $\mathcal{N} = (\mathcal{N}_i)$ be an assignment of a $\mathcal{C}$ module $\mathcal{N}_i$ to Y_i.

We will define a set $\mathcal{C}(W, \mathcal{N})$ using a colimit construction very similar to the one appearing in §6.3 above. (If $k = n$ and our n-categories are enriched, then $\mathcal{C}(W, \mathcal{N})$ will have additional structure; see below.)

Define a permissible decomposition of W to be a map

$$\left(\bigsqcup_a X_a\right) \sqcup \left(\bigsqcup_{i,b} M_{ib}\right) \to W,$$

where each X_a is a plain k-ball disjoint, in W, from $\cup Y_i$, and each M_{ib} is a marked k-ball intersecting Y_i (once mapped into W), with $M_{ib} \cap Y_i$ being the marking, which extends to a ball decomposition in the sense of Definition 3.1.3. (See Figure 23.) Given permissible decompositions x and y, we say

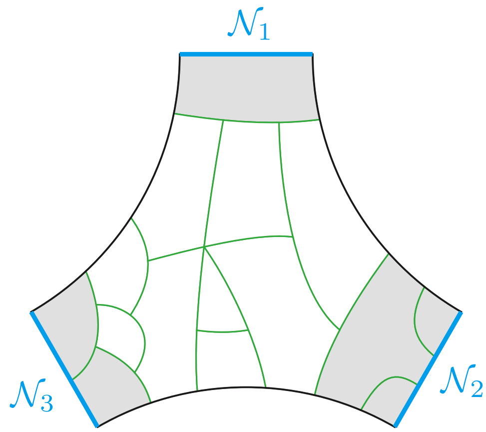


Figure 23: A permissible decomposition of a manifold whose boundary components are labeled by $\mathcal{C}$ modules $\{\mathcal{N}_i\}$. Marked balls are shown shaded, plain balls are unshaded.

that x is a refinement of y, or write $x \leq y$, if each ball of y is a union of balls of x. This defines a partial ordering $\mathfrak{D}(W)$, which we will think of as a category. (The objects of $\mathfrak{D}(D)$ are permissible decompositions of W, and there is a unique morphism from x to y if and only if x is a refinement of y.)

The collection of modules $\mathcal{N}$ determines a functor $\psi_\mathcal{N}$ from $\mathfrak{D}(W)$ to the category of sets (possibly with additional structure if $k = n$). For a decomposition $x = (X_a, M_{ib})$ in $\mathfrak{D}(W)$, define $\psi_\mathcal{N}(x)$ to be the subset

$$\psi_\mathcal{N}(x) \subset \left(\prod_a \mathcal{C}(X_a)\right) \times \left(\prod_{ib} \mathcal{N}_i(M_{ib})\right)$$

such that the restrictions to the various pieces of shared boundaries amongst the X_a and M_{ib} all agree. (That is, the fibered product over the boundary restriction maps.) If x is a refinement of y, define a map $\psi_\mathcal{N}(x) \to \psi_\mathcal{N}(y)$ via the gluing (composition or action) maps from $\mathcal{C}$ and the $\mathcal{N}_i$.

We now define the set $\mathcal{C}(W, \mathcal{N})$ to be the colimit of the functor $\psi_\mathcal{N}$. (As in §6.3, if $k = n$ we take a colimit in whatever category we are enriching over, and if additionally we are in the A_∞ case, then we use a homotopy colimit.)

If D is an m-ball, $0 \leq m \leq n - k$, then we can similarly define $\mathcal{C}(D \times W, \mathcal{N})$, where in this case $\mathcal{N}_i$ labels the submanifold $D \times Y_i \subset \partial(D \times W)$. It is not hard to see that the assignment $D \mapsto \mathcal{C}(D \times W, \mathcal{N})$ has the structure of an $n{-}k$-category.

We will use a simple special case of the above construction to define tensor products of modules. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be modules for an n-category $\mathcal{C}$. (If $k = 1$ and our manifolds are oriented, then one should be a left module and the other a right module.) Choose a 1-ball J, and label the two boundary points of J by $\mathcal{M}_1$ and $\mathcal{M}_2$. Define the tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ to be the $n{-}1$-category associated as above to J with its boundary labeled by $\mathcal{M}_1$ and $\mathcal{M}_2$. This of course depends (functorially) on the choice of 1-ball J.

We will define a more general self tensor product (categorified coend) below.

6.6 Morphisms of modules

Modules are collections of functors together with some additional data, so we define morphisms of modules to be collections of natural transformations which are compatible with this additional data.

More specifically, let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{C}$ modules, i.e. collections of functors $\{\mathcal{X}_k\}$ and $\{\mathcal{Y}_k\}$, for $0 \le k \le n$, from marked k-balls to sets as in Module Axiom 6.4.1. A morphism $g : \mathcal{X} \to \mathcal{Y}$ is a collection of natural transformations $g_k : \mathcal{X}_k \to \mathcal{Y}_k$ satisfying:

- Each g_k commutes with ∂.
- Each g_k commutes with gluing (module composition and $\mathcal{C}$ action).
- Each g_k commutes with taking products.
- In the top dimension $k = n$, g_n preserves whatever additional structure we are enriching over (e.g. vector spaces). In the A_∞ case (e.g. enriching over chain complexes) g_n should live in an appropriate derived hom space, as described below.

We will be mainly interested in the case $n = 1$ and enriched over chain complexes, since this is the case that's relevant to the generalized Deligne conjecture of §8. So we treat this case in more detail.

First we explain the remark about derived hom above. Let L be a marked 1-ball and let $\underrightarrow{\mathcal{X}}(L)$ denote the local homotopy colimit construction associated to L by $\mathcal{X}$ and $\mathcal{C}$. (See §6.3 and §6.5.) Define $\underrightarrow{\mathcal{Y}}(L)$ similarly. For K an unmarked 1-ball let $\underrightarrow{\mathcal{C}(K)}$ denote the local homotopy colimit construction associated to K by $\mathcal{C}$. Then we have an injective gluing map

$$\mathrm{gl} : \underrightarrow{\mathcal{X}}(L) \otimes \underrightarrow{\mathcal{C}}(K) \to \underrightarrow{\mathcal{X}}(L \cup K)$$

which is also a chain map. (For simplicity we are suppressing mention of boundary conditions on the unmarked boundary components of the 1-balls.) We define $\hom_{\mathcal{C}}(\mathcal{X} \to \mathcal{Y})$ to be a collection of (graded linear) natural transformations $g : \underrightarrow{\mathcal{X}}(L) \to \underrightarrow{\mathcal{Y}}(L)$ such that the following diagram commutes for all L and K:

$$\begin{array}{ccc} \underrightarrow{\mathcal{X}}(L) \otimes \underrightarrow{\mathcal{C}}(K) & \xrightarrow{\mathrm{gl}} & \underrightarrow{\mathcal{X}}(L \cup K) \\ \downarrow{\scriptstyle g\otimes \mathbf{1}} & & \downarrow{\scriptstyle g} \\ \underrightarrow{\mathcal{Y}}(L) \otimes \underrightarrow{\mathcal{C}}(K) & \xrightarrow{\mathrm{gl}} & \underrightarrow{\mathcal{Y}}(L \cup K) \end{array}$$

The usual differential on graded linear maps between chain complexes induces a differential on $\hom_{\mathcal{C}}(\mathcal{X} \to \mathcal{Y})$, giving it the structure of a chain complex.

Let $\mathcal{Z}$ be another $\mathcal{C}$ module. We define a chain map

$$a : \hom_{\mathcal{C}}(\mathcal{X} \to \mathcal{Y}) \otimes (\mathcal{X} \otimes_{\mathcal{C}} \mathcal{Z}) \to \mathcal{Y} \otimes_{\mathcal{C}} \mathcal{Z}$$

as follows. Recall that the tensor product $\mathcal{X} \otimes_{\mathcal{C}} \mathcal{Z}$ depends on a choice of interval J, labeled by $\mathcal{X}$ on one boundary component and $\mathcal{Z}$ on the other. Because we are using the *local* homotopy colimit, any generator $D \otimes x \otimes \bar{c} \otimes z$ of $\mathcal{X} \otimes_{\mathcal{C}} \mathcal{Z}$ can be written (perhaps non-uniquely) as a gluing $(D' \otimes x \otimes \bar{c}') \bullet (D'' \otimes \bar{c}'' \otimes z)$, for some decomposition $J = L' \cup L''$ and with $D' \otimes x \otimes \bar{c}'$ a generator of $\underrightarrow{\mathcal{X}}(L')$ and $D'' \otimes \bar{c}'' \otimes z$ a generator of $\underrightarrow{\mathcal{Z}}(L'')$. (Such a splitting exists because the blob diagram

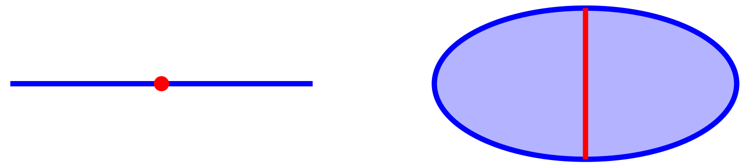


Figure 24: 0-marked 1-ball and 0-marked 2-ball

D can be split into left and right halves, since no blob can include both the leftmost and rightmost intervals in the underlying decomposition. This step would fail if we were using the usual hocolimit instead of the local hocolimit.) We now define

$$a : g \otimes (D \otimes x \otimes \bar{c} \otimes z) \mapsto g(D' \otimes x \otimes \bar{c}') \bullet (D'' \otimes \bar{c}'' \otimes z).$$

This does not depend on the choice of splitting $D = D' \bullet D''$ because g commutes with gluing.

6.7 The $n{+}1$-category of sphere modules

In this subsection we define $n{+}1$-categories $\mathcal{S}$ of "sphere modules" whose objects are n-categories. With future applications in mind, we treat simultaneously the big category of all n-categories and all sphere modules and also subcategories thereof. When $n = 1$ this is closely related to familiar 2-categories consisting of algebras, bimodules and intertwiners (or a subcategory of that).

While it is appropriate to call an S^0 module a bimodule, this is much less true for higher dimensional spheres, so we prefer the term "sphere module" for the general case.

For simplicity, we will assume that n-categories are enriched over $\mathbb{C}$-vector spaces.

The 0- through n-dimensional parts of $\mathcal{S}$ are various sorts of modules, and we describe these first. The $n{+}1$-dimensional part of $\mathcal{S}$ consists of intertwiners of 1-category modules associated to decorated n-balls. We will see below that in order for these $n{+}1$-morphisms to satisfy all of the axioms of an $n{+}1$-category (in particular, duality requirements), we will have to assume that our n-categories and modules have non-degenerate inner products. (In other words, we need to assume some extra duality on the n-categories and modules.)

Our first task is to define an n-category m-sphere module, for $0 \le m \le n-1$. These will be defined in terms of certain classes of marked balls, very similarly to the definition of n-category modules above. (This, in turn, is very similar to our definition of n-category.) Because of this similarity, we only sketch the definitions below.

We start with 0-sphere modules, which also could reasonably be called (categorified) bimodules. (For $n = 1$ they are precisely bimodules in the usual, uncategorified sense.) We prefer the more awkward term "0-sphere module" to emphasize the analogy with the higher sphere modules defined below.

Define a 0-marked k-ball, $1 \le k \le n$, to be a pair (X, M) homeomorphic to the standard (B^k, B^{k-1}). See Figure 24. Another way to say this is that (X, M) is homeomorphic to $B^{k-1} \times ([-1, 1], \{0\})$.

The 0-marked balls can be cut into smaller balls in various ways. We only consider those decompositions in which the smaller balls are either 0-marked (i.e. intersect the 0-marking of the large ball in a disc) or plain (don't intersect the 0-marking of the large ball). We can also take the boundary of a 0-marked ball, which is 0-marked sphere.

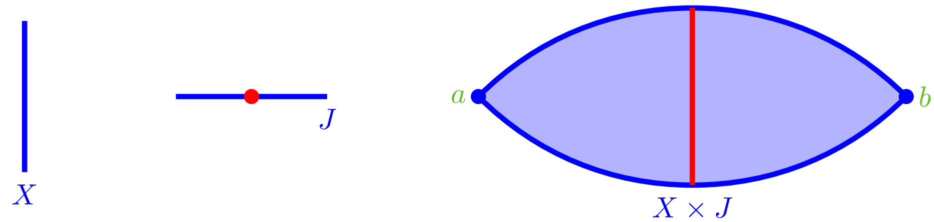


Figure 25: The pinched product $X \times J$

Fix n-categories $\mathcal{A}$ and $\mathcal{B}$. These will label the two halves of a 0-marked k-ball.

An n-category 0-sphere module $\mathcal{M}$ over the n-categories $\mathcal{A}$ and $\mathcal{B}$ is a collection of functors $\mathcal{M}_k$ from the category of 0-marked k-balls, $1 \le k \le n$, (with the two halves labeled by $\mathcal{A}$ and $\mathcal{B}$) to the category of sets. If $k = n$ these sets should be enriched to the extent $\mathcal{A}$ and $\mathcal{B}$ are. Given a decomposition of a 0-marked k-ball X into smaller balls X_i, we have morphism sets $\mathcal{A}_k(X_i)$ (if X_i lies on the $\mathcal{A}$-labeled side) or $\mathcal{B}_k(X_i)$ (if X_i lies on the $\mathcal{B}$-labeled side) or $\mathcal{M}_k(X_i)$ (if X_i intersects the marking and is therefore a smaller 0-marked ball). Corresponding to this decomposition we have a composition (or "gluing") map from the product (fibered over the boundary data) of these various sets into $\mathcal{M}_k(X)$.

Part of the structure of an n-category 0-sphere module $\mathcal{M}$ is captured by saying it is a collection $\mathcal{D}^{ab}$ of $n{-}1$-categories, indexed by pairs (a, b) of objects (0-morphisms) of $\mathcal{A}$ and $\mathcal{B}$. Let J be some standard 0-marked 1-ball (i.e. an interval with a marked point in its interior). Given a j-ball X, $0 \le j \le n-1$, we define

$$\mathcal{D}(X) \stackrel{\text{def}}{=} \mathcal{M}(X \times J).$$

The product is pinched over the boundary of J. The set $\mathcal{D}$ breaks into "blocks" according to the restrictions to the pinched points of $X \times J$ (see Figure 25). These restrictions are 0-morphisms (a, b) of $\mathcal{A}$ and $\mathcal{B}$.

More generally, consider an interval with interior marked points, and with the complements of these points labeled by n-categories $\mathcal{A}_i$ $(0 \le i \le l)$ and the marked points labeled by $\mathcal{A}_i$-$\mathcal{A}_{i+1}$ 0-sphere modules $\mathcal{M}_i$. (See Figure 26.) To this data we can apply the coend construction as in §6.5 above to obtain an $\mathcal{A}_0$-$\mathcal{A}_l$ 0-sphere module and, forgetfully, an $n{-}1$-category. This amounts to a definition of taking tensor products of 0-sphere modules over n-categories.

We could also similarly mark and label a circle, obtaining an $n{-}1$-category associated to the marked and labeled circle. (See Figure 26.) If the circle is divided into two intervals, we can think of this $n{-}1$-category as the 2-sided tensor product of the two 0-sphere modules associated to the two intervals.

Next we define n-category 1-sphere modules. These are just representations of (modules for) $n{-}1$-categories associated to marked and labeled circles (1-spheres) which we just introduced.

Equivalently, we can define 1-sphere modules in terms of 1-marked k-balls, $2 \le k \le n$. Fix a marked (and labeled) circle S. Let $C(S)$ denote the cone of S, a marked 2-ball (Figure 27). A 1-marked k-ball is anything homeomorphic to $B^j \times C(S)$, $0 \le j \le n-2$, where B^j is the standard j-ball. A 1-marked k-ball can be decomposed in various ways into smaller balls, which are either (a) smaller 1-marked k-balls, (b) 0-marked k-balls, or (c) plain k-balls. (See Figure 28.) We now proceed as in the above module definitions.

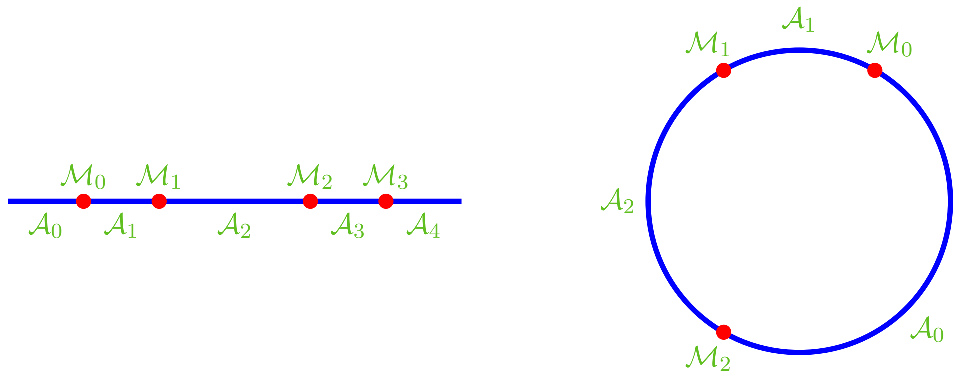


Figure 26: Marked and labeled 1-manifolds

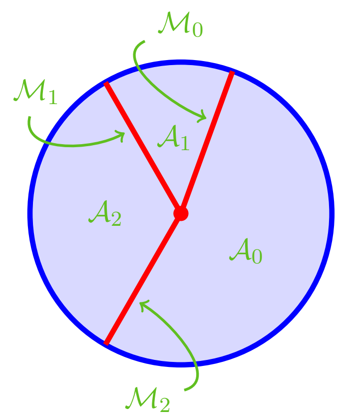


Figure 27: Cone on a marked circle, the prototypical 1-marked ball

A n-category 1-sphere module is, among other things, an $n{-}2$-category $\mathcal{D}$ with

$$\mathcal{D}(X) \stackrel{\text{def}}{=} \mathcal{M}(X \times C(S)).$$

The product is pinched over the boundary of $C(S)$. $\mathcal{D}$ breaks into "blocks" according to the restriction to the image of $\partial C(S) = S$ in $X \times C(S)$.

More generally, consider a 2-manifold Y (e.g. 2-ball or 2-sphere) marked by an embedded 1-complex K. The components of $Y \setminus K$ are labeled by n-categories, the edges of K are labeled by 0-sphere modules, and the 0-cells of K are labeled by 1-sphere modules. We can now apply the coend construction and obtain an $n{-}2$-category. If Y has boundary then this $n{-}2$-category is a module for the $n{-}1$-category associated to the (marked, labeled) boundary of Y. In particular, if ∂Y is a 1-sphere then we get a 1-sphere module as defined above.

It should now be clear how to define n-category m-sphere modules for $0 \le m \le n-1$. For example, there is an $n{-}2$-category associated to a marked, labeled 2-sphere, and a 2-sphere module is a representation of such an $n{-}2$-category.

We can now define the n-or-less-dimensional part of our $n{+}1$-category $\mathcal{S}$. Choose some collection of n-categories, then choose some collections of 0-sphere modules between these n-categories, then

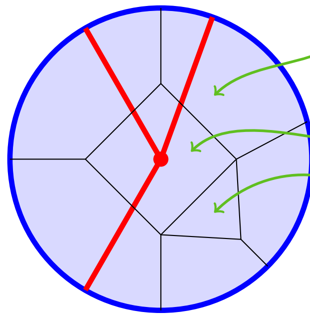


Figure 28: Subdividing a 1-marked ball into plain, 0-marked and 1-marked balls.

choose some collection of 1-sphere modules for the various possible marked 1-spheres labeled by the n-categories and 0-sphere modules, and so on. Let L_i denote the collection of $i{-}1$-sphere modules we have chosen. (For convenience, we declare a (-1)-sphere module to be an n-category.) There is a wide range of possibilities. The set L_0 could contain infinitely many n-categories or just one. For each pair of n-categories in L_0, L_1 could contain no 0-sphere modules at all or it could contain several. The only requirement is that each k-sphere module be a module for a k-sphere $n{-}k$-category constructed out of labels taken from L_j for $j < k$.

We now define $\mathcal{S}(X)$, for X a ball of dimension at most n, to be the set of all cell-complexes K embedded in X, with the codimension-j parts of (X, K) labeled by elements of L_j. As described above, we can think of each decorated k-ball as defining a $k{-}1$-sphere module for the $n{-}k{+}1$-category associated to its decorated boundary. Thus the k-morphisms of $\mathcal{S}$ (for $k \le n$) can be thought of as n-category $k{-}1$-sphere modules (generalizations of bimodules). On the other hand, we can equally well think of the k-morphisms as decorations on k-balls, and from this point of view it is clear that they satisfy all of the axioms of an $n{+}1$-category. (All of the axioms for the less-than-$n{+}1$-dimensional part of an $n{+}1$-category, that is.)

Next we define the $n{+}1$-morphisms of $\mathcal{S}$. The construction of the 0- through n-morphisms was easy and tautological, but the $n{+}1$-morphisms will require some effort and combinatorial topology, as well as additional duality assumptions on the lower morphisms. These are required because we define the spaces of $n{+}1$-morphisms by making arbitrary choices of incoming and outgoing boundaries for each n-ball. The additional duality assumptions are needed to prove independence of our definition form these choices.

Let X be an $n{+}1$-ball, and let c be a decoration of its boundary by a cell complex labeled by 0- through n-morphisms, as above. Choose an $n{-}1$-sphere $E \subset \partial X$ which divides ∂X into "incoming" and "outgoing" boundary $\partial_- X$ and $\partial_+ X$. Let E_c denote E decorated by the restriction of c to E. Recall from above the associated 1-category $\mathcal{S}(E_c)$. We can also have $\mathcal{S}(E_c)$ modules $\mathcal{S}(\partial_- X_c)$ and $\mathcal{S}(\partial_+ X_c)$. Define

$$\mathcal{S}(X; c; E) \stackrel{\text{def}}{=} \hom_{\mathcal{S}(E_c)}(\mathcal{S}(\partial_- X_c), \mathcal{S}(\partial_+ X_c)).$$

We will show that if the sphere modules are equipped with a "compatible family of non-degenerate inner products", then there is a coherent family of isomorphisms $\mathcal{S}(X; c; E) \cong \mathcal{S}(X; c; E')$ for all pairs of choices E and E'. This will allow us to define $\mathcal{S}(X; c)$ independently of the choice of E.

First we must define "inner product", "non-degenerate" and "compatible". Let Y be a decorated n-ball, and $\overline{Y}$ it's mirror image. (We assume we are working in the unoriented category.) Let $Y \cup \overline{Y}$

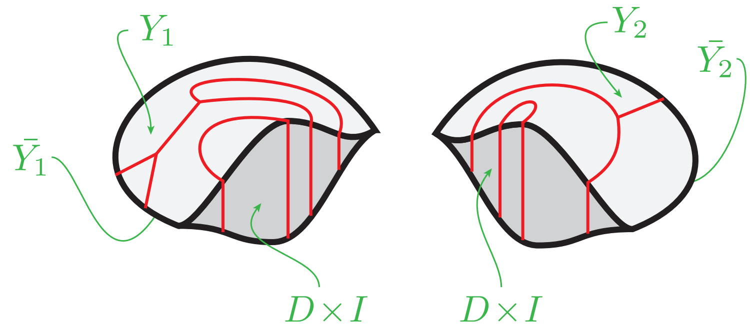


Figure 29: $Y \times I$ sliced open

denote the decorated n-sphere obtained by gluing Y and $\overline{Y}$ along their common boundary. An *inner product* on $\mathcal{S}(Y)$ is a dual vector

$$z_Y : \mathcal{S}(Y \cup \overline{Y}) \to \mathbb{C}.$$

We will also use the notation

$$\langle a, b\rangle \stackrel{\text{def}}{=} z_Y(a \bullet \overline{b}) \in \mathbb{C}.$$

An inner product induces a linear map

$$\begin{aligned} \varphi : \mathcal{S}(Y) &\to \mathcal{S}(Y)^* \\ a &\mapsto \langle a, \cdot\rangle \end{aligned}$$

which satisfies, for all morphisms e of $\mathcal{S}(\partial Y)$,

$$\varphi(ae)(b) = \langle ae, b\rangle = z_Y(a \bullet e \bullet b) = \langle a, eb\rangle = \varphi(a)(eb).$$

In other words, φ is a map of $\mathcal{S}(\partial Y)$ modules. An inner product is *non-degenerate* if φ is an isomorphism. This implies that $\mathcal{S}(Y; c)$ is finite dimensional for all boundary conditions c. (One can think of these inner products as giving some duality in dimension $n+1$; heretofore we have only assumed duality in dimensions 0 through n.)

Next we define compatibility. Let $Y = Y_1 \cup Y_2$ with $D = Y_1 \cap Y_2$. Let X_1 and X_2 be the two components of $Y \times I$ cut along $D \times I$, in both cases using the pinched product. (Here we are overloading notation and letting D denote both a decorated and an undecorated manifold.) We have $\partial X_i = Y_i \cup \overline{Y}_i \cup (D \times I)$ (see Figure 29). Given $a_i \in \mathcal{S}(Y_i)$, $b_i \in \mathcal{S}(\overline{Y}_i)$ and $v \in \mathcal{S}(D \times I)$ which agree on their boundaries, we can evaluate

$$z_{Y_i}(a_i \bullet b_i \bullet v) \in \mathbb{C}.$$

(This requires a choice of homeomorphism $Y_i \cup \overline{Y}_i \cup (D \times I) \cong Y_i \cup \overline{Y}_i$, but the value of z_{Y_i} is independent of this choice.) We can think of z_{Y_i} as giving a function

$$\psi_i : \mathcal{S}(Y_i) \otimes \mathcal{S}(\overline{Y}_i) \to \mathcal{S}(D \times I)^* \xrightarrow{\varphi^{-1}} \mathcal{S}(D \times I).$$

We can now finally define a family of inner products to be *compatible* if for all decompositions $Y = Y_1 \cup Y_2$ as above and all $a_i \in \mathcal{S}(Y_i)$, $b_i \in \mathcal{S}(\overline{Y}_i)$ we have

$$z_Y(a_1 \bullet a_2 \bullet b_1 \bullet b_2) = z_{D\times I}(\psi_1(a_1 \otimes b_1) \bullet \psi_2(a_2 \otimes b_2)).$$

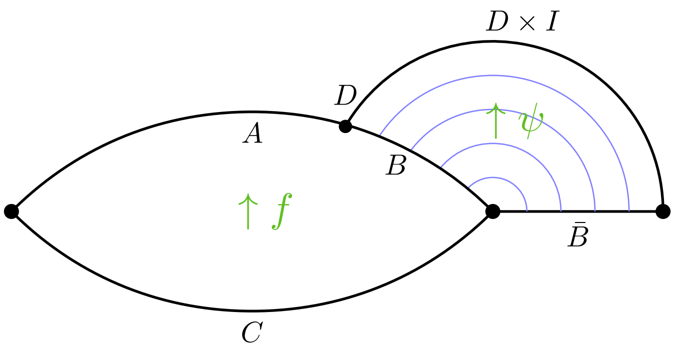


Figure 30: Moving B from top to bottom

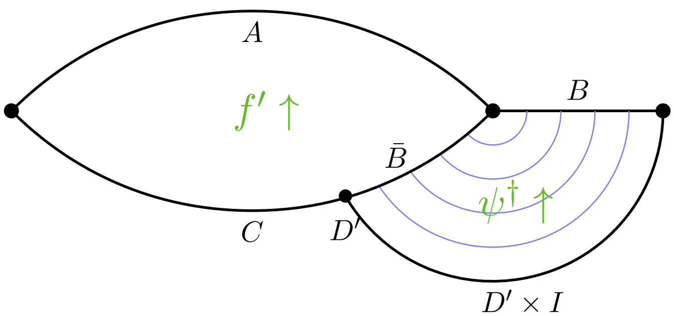


Figure 31: Moving B from bottom to top

In other words, the inner product on Y is determined by the inner products on Y_1, Y_2 and $D \times I$.

Now we show how to unambiguously identify $\mathcal{S}(X; c; E)$ and $\mathcal{S}(X; c; E')$ for any two choices of E and E'. Consider first the case where ∂X is decomposed as three n-balls A, B and C, with $E = \partial(A \cup B)$ and $E' = \partial A$. We must provide an isomorphism between $\mathcal{S}(X; c; E) = \hom(\mathcal{S}(C), \mathcal{S}(A \cup B))$ and $\mathcal{S}(X; c; E') = \hom(\mathcal{S}(C \cup \overline{B}), \mathcal{S}(A))$. Let $D = B \cap A$. Then as above we can construct a map

$$\psi : \mathcal{S}(B) \otimes \mathcal{S}(\overline{B}) \to \mathcal{S}(D \times I).$$

Given $f \in \hom(\mathcal{S}(C), \mathcal{S}(A \cup B))$ we define $f' \in \hom(\mathcal{S}(C \cup \overline{B}), \mathcal{S}(A))$ to be the composition

$$\mathcal{S}(C \cup \overline{B}) \xrightarrow{f \otimes \mathbf{1}} \mathcal{S}(A \cup B \cup \overline{B}) \xrightarrow{\mathbf{1} \otimes \psi} \mathcal{S}(A \cup (D \times I)) \xrightarrow{\cong} \mathcal{S}(A).$$

(See Figure 30.) Let $D' = B \cap C$. Using the inner products there is an adjoint map

$$\psi^\dagger : \mathcal{S}(D' \times I) \to \mathcal{S}(\overline{B}) \otimes \mathcal{S}(B).$$

Given $f' \in \hom(\mathcal{S}(C \cup \overline{B}), \mathcal{S}(A))$ we define $f \in \hom(\mathcal{S}(C), \mathcal{S}(A \cup B))$ to be the composition

$$\mathcal{S}(C) \xrightarrow{\cong} \mathcal{S}(C \cup (D' \times I)) \xrightarrow{\mathbf{1} \otimes \psi^\dagger} \mathcal{S}(C \cup \overline{B} \cup B) \xrightarrow{f' \otimes \mathbf{1}} \mathcal{S}(A \cup B).$$

(See Figure 31.) Let $D' = B \cap C$. It is not hard too show that the above two maps are mutually inverse.

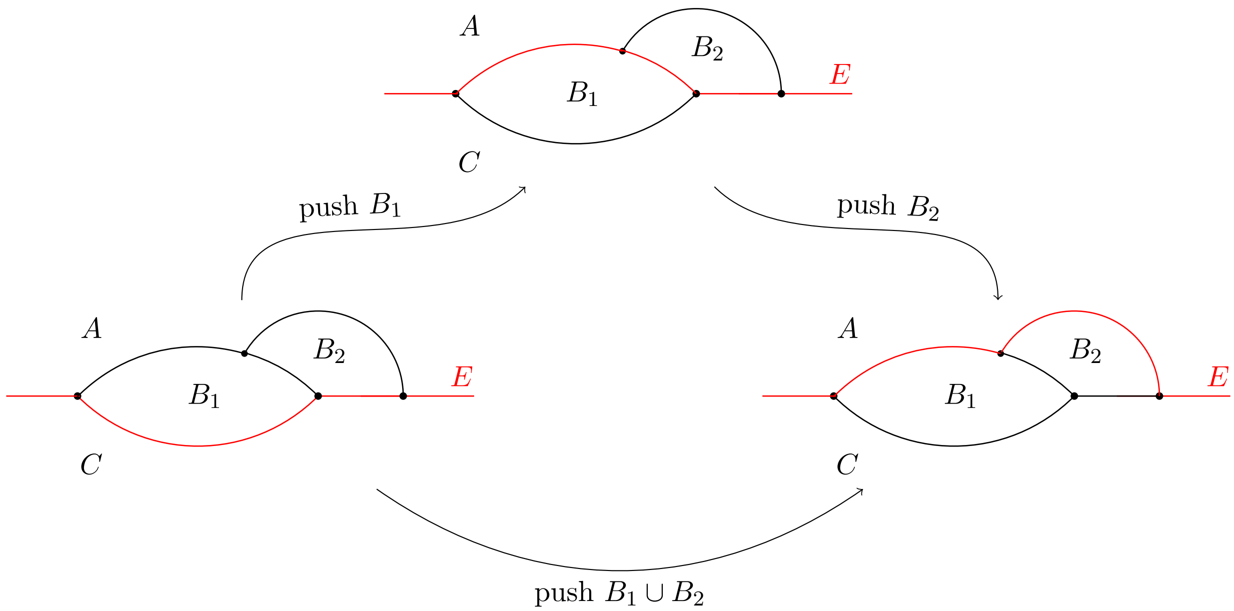


Figure 32: A movie move

Lemma 6.7.1. *Any two choices of E and E' are related by a series of modifications as above.*

Proof. (Sketch) E and E' are isotopic, and any isotopy is homotopic to a composition of small isotopies which are either (a) supported away from E, or (b) modify E in the simple manner described above. □

It follows from the lemma that we can construct an isomorphism between $\mathcal{S}(X; c; E)$ and $\mathcal{S}(X; c; E')$ for any pair E, E'. This construction involves on a choice of simple "moves" (as above) to transform E to E'. We must now show that the isomorphism does not depend on this choice. We will show below that it suffice to check two "movie moves".

The first movie move is to push E across an n-ball B as above, then push it back. The result is equivalent to doing nothing. As we remarked above, the isomorphisms corresponding to these two pushes are mutually inverse, so we have invariance under this movie move.

The second movie move replaces two successive pushes in the same direction, across B_1 and B_2, say, with a single push across $B_1 \cup B_2$. (See Figure 32.) Invariance under this movie move follows from the compatibility of the inner product for $B_1 \cup B_2$ with the inner products for B_1 and B_2.

If $n \geq 2$, these two movie move suffice:

Lemma 6.7.2. *Assume $n \geq 2$ and fix E and E' as above. Then any two sequences of elementary moves connecting E to E' are related by a sequence of the two movie moves defined above.*

Proof. (Sketch) Consider a two parameter family of diffeomorphisms (one parameter family of isotopies) of ∂X. Up to homotopy, such a family is homotopic to a family which can be decomposed into small families which are either (a) supported away from E, (b) have boundaries corresponding to the two movie moves above. Finally, observe that the space of E's is simply connected. (This fails for $n = 1$.) □

For $n = 1$ we have to check an additional "global" relations corresponding to rotating the 0-sphere E around the 1-sphere ∂X. But if $n = 1$, then we are in the case of ordinary algebroids and bimodules, and this is just the well-known "Frobenius reciprocity" result for bimodules [1].

We have now defined $\mathcal{S}(X; c)$ for any $n{+}1$-ball X with boundary decoration c. We must also define, for any homeomorphism $X \to X'$, an action $f : \mathcal{S}(X; c) \to \mathcal{S}(X', f(c))$. Choosing an equator $E \subset \partial X$ we have

$$\mathcal{S}(X; c) \cong \mathcal{S}(X; c; E) \stackrel{\text{def}}{=} \hom_{\mathcal{S}(E_c)}(\mathcal{S}(\partial_- X_c), \mathcal{S}(\partial_+ X_c)).$$

We define $f : \mathcal{S}(X; c) \to \mathcal{S}(X', f(c))$ to be the tautological map

$$f : \mathcal{S}(X; c; E) \to \mathcal{S}(X'; f(c); f(E)).$$

It is easy to show that this is independent of the choice of E. Note also that this map depends only on the restriction of f to ∂X. In particular, if $F : X \to X$ is the identity on ∂X then f acts trivially, as required by Axiom 6.1.10.

We define product $n{+}1$-morphisms to be identity maps of modules.

To define (binary) composition of $n{+}1$-morphisms, choose the obvious common equator then compose the module maps. The proof that this composition rule is associative is similar to the proof of Lemma 6.7.1.

7 The blob complex for A_∞ n-categories

Given an A_∞ n-category $\mathcal{C}$ and an n-manifold M, we make the anticlimactically tautological definition of the blob complex $\mathcal{B}_*(M; \mathcal{C})$ to be the homotopy colimit $\underrightarrow{\mathcal{C}}(M)$ of §6.3.

We will show below in Corollary 7.1.3 that when $\mathcal{C}$ is obtained from a system of fields $\mathcal{D}$ as the blob complex of an n-ball (see Example 6.2.8), $\underrightarrow{\mathcal{C}}(M)$ is homotopy equivalent to our original definition of the blob complex $\mathcal{B}_*(M; \mathcal{D})$.

7.1 A product formula

Given an n-dimensional system of fields $\mathcal{E}$ and a $n{-}k$-manifold F, recall from Example 6.2.8 that there is an A_∞ k-category $\mathcal{C}_F$ defined by $\mathcal{C}_F(X) = \mathcal{E}(X \times F)$ if $\dim(X) < k$ and $\mathcal{C}_F(X) = \mathcal{B}_*(X \times F; \mathcal{E})$ if $\dim(X) = k$.

Theorem 7.1.1. *Let Y be a k-manifold. Then there is a homotopy equivalence between "old-fashioned" (blob diagrams) and "new-fangled" (hocolimit) blob complexes*

$$\mathcal{B}_*(Y \times F) \simeq \underrightarrow{\mathcal{C}_F}(Y).$$

Proof. We will use the concrete description of the homotopy colimit from §6.3.

First we define a map

$$\psi : \underrightarrow{\mathcal{C}_F}(Y) \to \mathcal{B}_*(Y \times F; C).$$

On 0-simplices of the hocolimit we just glue together the various blob diagrams on $X_i \times F$ (where X_i is a component of a permissible decomposition of Y) to get a blob diagram on $Y \times F$. For simplices of dimension 1 and higher we define the map to be zero. It is easy to check that this is a chain map.

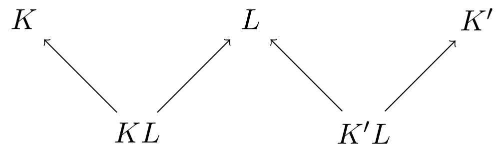


Figure 33: Connecting K and K' via L

In the other direction, we will define a subcomplex $G_* \subset \mathcal{B}_*(Y \times F; C)$ and a map

$$\phi : G_* \to \mathcal{C}_{\underrightarrow{F}}(Y).$$

Given a decomposition K of Y into k-balls X_i, let $K \times F$ denote the corresponding decomposition of $Y \times F$ into the pieces $X_i \times F$.

Let $G_* \subset \mathcal{B}_*(Y \times F; C)$ be the subcomplex generated by blob diagrams a such that there exists a decomposition K of Y such that a splits along $K \times F$. It follows from Lemma 5.1.1 that $\mathcal{B}_*(Y \times F; C)$ is homotopic to a subcomplex of G_*. (If the blobs of a are small with respect to a sufficiently fine cover then their projections to Y are contained in some disjoint union of balls.) Note that the image of ψ is equal to G_*.

We will define $\phi : G_* \to \mathcal{C}_{\underrightarrow{F}}(Y)$ using the method of acyclic models. Let a be a generator of G_*. Let $D(a)$ denote the subcomplex of $\mathcal{C}_{\underrightarrow{F}}(Y)$ generated by all $(b, \overline{K})$ such that a splits along $K_0 \times F$ and b is a generator appearing in an iterated boundary of a (this includes a itself). (Recall that $\overline{K} = (K_0, \ldots, K_l)$ denotes a chain of decompositions; see §6.3.) By $(b, \overline{K})$ we really mean $(b^\sharp, \overline{K})$, where $b^\sharp$ is b split according to $K_0 \times F$. To simplify notation we will just write plain b instead of $b^\sharp$. Roughly speaking, $D(a)$ consists of 0-simplices which glue up to give a (or one of its iterated boundaries), 1-simplices which connect all the 0-simplices, 2-simplices which kill the homology created by the 1-simplices, and so on. More formally,

Lemma 7.1.2. *$D(a)$ is acyclic.*

Proof. We will prove acyclicity in the first couple of degrees, and leave the general case to the reader.

Let K and K' be two decompositions (0-simplices) of Y compatible with a. We want to find 1-simplices which connect K and K'. We might hope that K and K' have a common refinement, but this is not necessarily the case. (Consider the x-axis and the graph of $y = x^2 \sin(1/x)$ in $\mathbb{R}^2$.) However, we *can* find another decomposition L such that L shares common refinements with both K and K'. Let KL and $K'L$ denote these two refinements. Then 1-simplices associated to the four anti-refinements $KL \to K$, $KL \to L$, $K'L \to L$ and $K'L \to K'$ give the desired chain connecting (a, K) and (a, K') (see Figure 33).

Consider a different choice of decomposition L' in place of L above. This leads to a cycle of 1-simplices. We want to find 2-simplices which fill in this cycle. Choose a decomposition M which has common refinements with each of K, KL, L, $K'L$, K', $K'L'$, L' and KL'. (We also also require that KLM antirefines to KM, etc.) Then we have 2-simplices, as shown in Figure 34, which do the trick. (Each small triangle in Figure 34 can be filled with a 2-simplex.)

Continuing in this way we see that $D(a)$ is acyclic. □

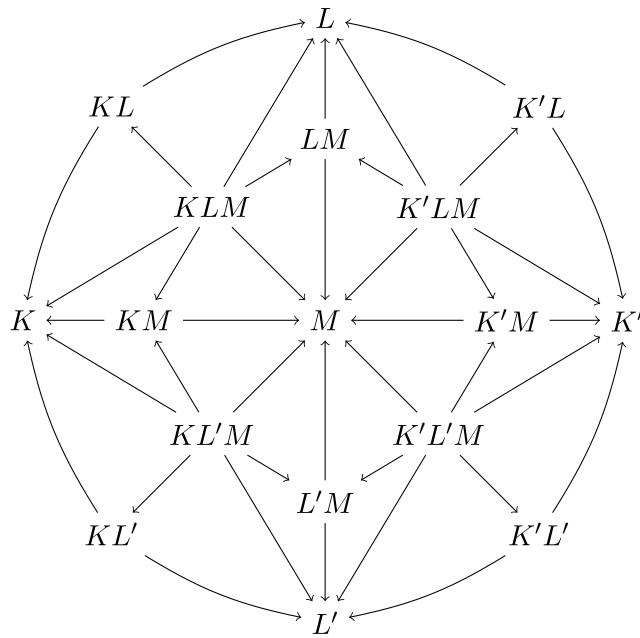


Figure 34: Filling in K-KL-L-$K'L$-K'-$K'L'$-L'-KL'-K

We are now in a position to apply the method of acyclic models to get a map $\phi : G_* \to \underrightarrow{\mathcal{C}_F}(Y)$. We may assume that $\phi(a)$ has the form $(a, K) + r$, where (a, K) is a 0-simplex and r is a sum of simplices of dimension 1 or higher.

We now show that $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic to the identity.

First, $\psi \circ \phi$ is the identity on the nose:

$$\psi(\phi(a)) = \psi((a, K)) + \psi(r) = a + 0.$$

Roughly speaking, (a, K) is just a chopped up into little pieces, and ψ glues those pieces back together, yielding a. We have $\psi(r) = 0$ since ψ is zero on (≥ 1)-simplices.

Second, $\phi \circ \psi$ is the identity up to homotopy by another argument based on the method of acyclic models. To each generator $(b, \overline{K})$ of G_* we associate the acyclic subcomplex $D(b)$ defined above. Both the identity map and $\phi \circ \psi$ are compatible with this collection of acyclic subcomplexes, so by the usual method of acyclic models argument these two maps are homotopic.

This concludes the proof of Theorem 7.1.1. □

If Y has dimension $k - m$, then we have an m-category $\mathcal{C}_{Y \times F}$ whose value at a j-ball X is either $\mathcal{E}(X \times Y \times F)$ (if $j < m$) or $\mathcal{B}_*(X \times Y \times F)$ (if $j = m$). (See Example 6.2.8.) Similarly we have an m-category whose value at X is $\underrightarrow{\mathcal{C}_F}(X \times Y)$. These two categories are equivalent, but since we do not define functors between topological n-categories in this paper we are unable to say precisely what "equivalent" means in this context. We hope to include this stronger result in a future paper.

Taking F in Theorem 7.1.1 to be a point, we obtain the following corollary.

Corollary 7.1.3. *Let $\mathcal{E}$ be a system of fields (with local relations) and let $\mathcal{C}_\mathcal{E}$ be the A_∞ n-category obtained from $\mathcal{E}$ by taking the blob complex of balls. Then for all n-manifolds Y the old-fashioned and new-fangled blob complexes are homotopy equivalent:*

$$\mathcal{B}_*^\mathcal{E}(Y) \simeq \underrightarrow{\mathcal{C}_\mathcal{E}}(Y).$$

Theorem 7.1.1 extends to the case of general fiber bundles

$$F \to E \to Y,$$

an indeed even to the case of general maps

$$M \to Y.$$

We outline two approaches to these generalizations. The first is somewhat tautological, while the second is more amenable to calculation.

We can generalize the definition of a k-category by replacing the categories of j-balls ($j \le k$) with categories of j-balls D equipped with a map $p : D \to Y$ (c.f. [10]). Call this a k-category over Y. A fiber bundle $F \to E \to Y$ gives an example of a k-category over Y: assign to $p : D \to Y$ the blob complex $\mathcal{B}_*(p^*(E))$, if $\dim(D) = k$, or the fields $\mathcal{E}(p^*(E))$, if $\dim(D) < k$. ($p^*(E)$ denotes the pull-back bundle over D.) Let $\mathcal{F}_E$ denote this k-category over Y. We can adapt the homotopy colimit construction (based decompositions of Y into balls) to get a chain complex $\underrightarrow{\mathcal{F}_E}(Y)$. The proof of Theorem 7.1.1 goes through essentially unchanged to show that

Theorem 7.1.4. *Let $F \to E \to Y$ be a fiber bundle and let $\mathcal{F}_E$ be the k-category over Y defined above. Then*

$$\mathcal{B}_*(E) \simeq \underrightarrow{\mathcal{F}_E}(Y).$$

□

We can generalize this result still further by noting that it is not really necessary for the definition of $\mathcal{F}_E$ that $E \to Y$ be a fiber bundle. Let $M \to Y$ be a map, with $\dim(M) = n$ and $\dim(Y) = k$. Call a map $D^j \to Y$ "good" with respect to M if the fibered product $D \widetilde{\times} M$ is a manifold of dimension $n - k + j$ with a collar structure along the boundary of D. (If $D \to Y$ is an embedding then $D \widetilde{\times} M$ is just the part of M lying above D.) We can define a k-category $\mathcal{F}_M$ based on maps of balls into Y which a re good with respect to M. We can again adapt the homotopy colimit construction to get a chain complex $\underrightarrow{\mathcal{F}_M}(Y)$. The proof of Theorem 7.1.1 again goes through essentially unchanged to show that

Theorem 7.1.5. *Let $M \to Y$ be a map of manifolds and let $\mathcal{F}_M$ be the k-category over Y defined above. Then*

$$\mathcal{B}_*(M) \simeq \underrightarrow{\mathcal{F}_M}(Y).$$

□

In the second approach we use a decorated colimit (as in §6.7) and various sphere modules based on $F \to E \to Y$ or $M \to Y$, instead of an undecorated colimit with fancier k-categories over Y.

Information about the specific map to Y has been taken out of the categories and put into sphere modules and decorations.

Let $F \to E \to Y$ be a fiber bundle as above. Choose a decomposition $Y = \cup X_i$ such that the restriction of E to X_i is homeomorphic to a product $F \times X_i$, and choose trivializations of these products as well.

Let $\mathcal{F}$ be the k-category associated to F. To each codimension-1 face $X_i \cap X_j$ we have a bimodule (S^0-module) for $\mathcal{F}$. More generally, to each codimension-m face we have an S^{m-1}-module for a $(k-m+1)$-category associated to the (decorated) link of that face. We can decorate the strata of the decomposition of Y with these sphere modules and form a colimit as in §6.7. This colimit computes $\mathcal{B}_*(E)$.

There is a similar construction for general maps $M \to Y$.

7.2 A gluing theorem

Next we prove a gluing theorem. Let X be a closed k-manifold with a splitting $X = X'_1 \cup_Y X'_2$. We will need an explicit collar on Y, so rewrite this as $X = X_1 \cup (Y \times J) \cup X_2$. Given this data we have:

- An A_∞ $n{-}k$-category $\mathcal{B}(X)$, which assigns to an m-ball D fields on $D \times X$ (for $m + k < n$) or the blob complex $\mathcal{B}_*(D \times X; c)$ (for $m + k = n$). (See Example 6.2.8.)
- An A_∞ $n{-}k{+}1$-category $\mathcal{B}(Y)$, defined similarly.
- Two $\mathcal{B}(Y)$ modules $\mathcal{B}(X_1)$ and $\mathcal{B}(X_2)$, which assign to a marked m-ball (D, H) either fields on $(D \times Y) \cup (H \times X_i)$ (if $m+k < n$) or the blob complex $\mathcal{B}_*((D \times Y) \cup (H \times X_i))$ (if $m+k = n$). (See Example 6.4.13.)
- The tensor product $\mathcal{B}(X_1) \otimes_{\mathcal{B}(Y),J} \mathcal{B}(X_2)$, which is an A_∞ $n{-}k$-category. (See §6.5.)

It is the case that the $n{-}k$-categories $\mathcal{B}(X)$ and $\mathcal{B}(X_1) \otimes_{\mathcal{B}(Y),J} \mathcal{B}(X_2)$ are equivalent for all k, but since we do not develop a definition of functor between n-categories in this paper, we cannot state this precisely. (It will appear in a future paper.) So we content ourselves with

Theorem 7.2.1. *When $k = n$ above, $\mathcal{B}(X)$ is homotopy equivalent to $\mathcal{B}(X_1) \otimes_{\mathcal{B}(Y),J} \mathcal{B}(X_2)$.*

Proof. The proof is similar to that of Theorem 7.1.1. We give a short sketch with emphasis on the differences from the proof of Theorem 7.1.1.

Let $\mathcal{T}$ denote the chain complex $\mathcal{B}(X_1) \otimes_{\mathcal{B}(Y),J} \mathcal{B}(X_2)$. Recall that this is a homotopy colimit based on decompositions of the interval J.

We define a map $\psi : \mathcal{T} \to \mathcal{B}_*(X)$. On 0-simplices it is given by gluing the pieces together to get a blob diagram on X. On simplices of dimension 1 and greater ψ is zero.

The image of ψ is the subcomplex $G_* \subset \mathcal{B}(X)$ generated by blob diagrams which split over some decomposition of J. It follows from Lemma 5.1.1 that $\mathcal{B}_*(X)$ is homotopic to a subcomplex of G_*.

Next we define a map $\phi : G_* \to \mathcal{T}$ using the method of acyclic models. As in the proof of Theorem 7.1.1, we assign to a generator a of G_* an acyclic subcomplex which is (roughly) $\psi^{-1}(a)$. The proof of acyclicity is easier in this case since any pair of decompositions of J have a common refinement.

The proof that these two maps are inverse to each other is the same as in Theorem 7.1.1. □

7.3 Reconstructing mapping spaces

The next theorem shows how to reconstruct a mapping space from local data. Let T be a topological space, let M be an n-manifold, and recall the A_∞ n-category $\pi^\infty_{\leq n}(T)$ of Example 6.2.7. Think of $\pi^\infty_{\leq n}(T)$ as encoding everything you would ever want to know about spaces of maps of k-balls into T ($k \leq n$). To simplify notation, let $\mathcal{T} = \pi^\infty_{\leq n}(T)$.

Theorem 7.3.1. *The blob complex for M with coefficients in the fundamental A_∞ n-category for T is quasi-isomorphic to singular chains on maps from M to T.*

$$\mathcal{B}^{\mathcal{T}}(M) \simeq C_*(\mathrm{Maps}(M \to T)).$$

Remark. Lurie has shown in [6, Theorem 3.8.6] that the topological chiral homology of an n-manifold M with coefficients in a certain E_n algebra constructed from T recovers the same space of singular chains on maps from M to T, with the additional hypothesis that T is $n-1$-connected. This extra hypothesis is not surprising, in view of the idea described in Example 6.2.10 that an E_n algebra is roughly equivalent data to an A_∞ n-category which is trivial at levels 0 through $n-1$. Ricardo Andrade also told us about a similar result.

Proof. The proof is again similar to that of Theorem 7.1.1.

We begin by constructing chain map $\psi : \mathcal{B}^{\mathcal{T}}(M) \to C_*(\mathrm{Maps}(M \to T))$.

Recall that the 0-simplices of the homotopy colimit $\mathcal{B}^{\mathcal{T}}(M)$ are a direct sum of chain complexes with the summands indexed by decompositions of M which have their $n-1$-skeletons labeled by $n-1$-morphisms of $\mathcal{T}$. Since $\mathcal{T} = \pi^\infty_{\leq n}(T)$, this means that the summands are indexed by pairs (K, φ), where K is a decomposition of M and φ is a continuous map from the $n-1$-skeleton of K to T. The summand indexed by (K, φ) is

$$\bigotimes_b D_*(b, \varphi),$$

where b runs through the n-cells of K and $D_*(b, \varphi)$ denotes chains of maps from b to T compatible with φ. We can take the product of these chains of maps to get chains of maps from all of M to K. This defines ψ on 0-simplices.

We define ψ to be zero on (≥ 1)-simplices. It is not hard to see that this defines a chain map from $\mathcal{B}^{\mathcal{T}}(M)$ to $C_*(\mathrm{Maps}(M \to T))$.

The image of ψ is the subcomplex $G_* \subset C_*(\mathrm{Maps}(M \to T))$ generated by families of maps whose support is contained in a disjoint union of balls. It follows from Lemma B.0.5 that $C_*(\mathrm{Maps}(M \to T))$ is homotopic to a subcomplex of G_*.

We will define a map $\phi : G_* \to \mathcal{B}^{\mathcal{T}}(M)$ via acyclic models. Let a be a generator of G_*. Define $D(a)$ to be the subcomplex of $\mathcal{B}^{\mathcal{T}}(M)$ generated by all pairs $(b, \overline{K})$, where b is a generator appearing in an iterated boundary of a and $\overline{K}$ is an index of the homotopy colimit $\mathcal{B}^{\mathcal{T}}(M)$. (See the proof of Theorem 7.1.1 for more details.) The same proof as of Lemma 7.1.2 shows that $D(a)$ is acyclic. By the usual acyclic models nonsense, there is a (unique up to homotopy) map $\phi : G_* \to \mathcal{B}^{\mathcal{T}}(M)$ such that $\phi(a) \in D(a)$. Furthermore, we may choose ϕ such that for all a

$$\phi(a) = (a, K) + r$$

where (a, K) is a 0-simplex and r is a sum of simplices of dimension 1 and greater.

It is now easy to see that $\psi \circ \phi$ is the identity on the nose. Another acyclic models argument shows that $\phi \circ \psi$ is homotopic to the identity. (See the proof of Theorem 7.1.1 for more details.) □

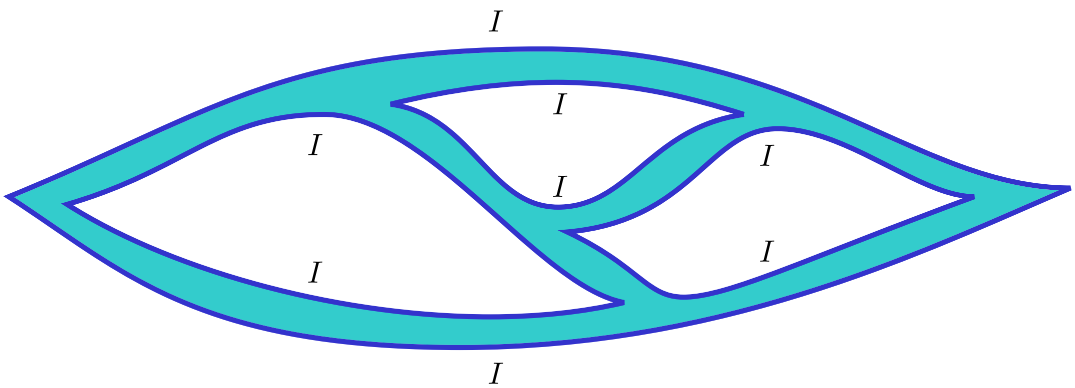


Figure 35: Little bigons, though of as encoding surgeries

8 Higher-dimensional Deligne conjecture

In this section we prove a higher dimensional version of the Deligne conjecture about the action of the little disks operad on Hochschild cochains. The first several paragraphs lead up to a precise statement of the result (Theorem 8.0.2 below). Then we give the proof.

The usual Deligne conjecture (proved variously in [4, 11, 3, 13] gives a map

$$C_*(LD_k) \otimes \overbrace{Hoch^*(C,C) \otimes \cdots \otimes Hoch^*(C,C)}^{k \text{ copies}} \to Hoch^*(C,C).$$

Here LD_k is the k-th space of the little disks operad and $Hoch^*(C,C)$ denotes Hochschild cochains.

We now reinterpret $C_*(LD_k)$ and $Hoch^*(C,C)$ in such a way as to make the generalization to higher dimensions clear.

The little disks operad is homotopy equivalent to configurations of little bigons inside a big bigon, as shown in Figure 35. We can think of such a configuration as encoding a sequence of surgeries, starting at the bottommost interval of Figure 35 and ending at the topmost interval. The surgeries correspond to the k bigon-shaped "holes". We remove the bottom interval of each little bigon and replace it with the top interval. To convert this topological operation to an algebraic one, we need, for each hole, an element of $\hom(\mathcal{B}_*^C(I_{\text{bottom}}), \mathcal{B}_*^C(I_{\text{top}}))$, which is homotopy equivalent to $Hoch^*(C,C)$. So for each fixed configuration we have a map

$$\hom(\mathcal{B}_*^C(I), \mathcal{B}_*^C(I)) \otimes \cdots \otimes \hom(\mathcal{B}_*^C(I), \mathcal{B}_*^C(I)) \to \hom(\mathcal{B}_*^C(I), \mathcal{B}_*^C(I)).$$

If we deform the configuration, corresponding to a 1-chain in $C_*(LD_k)$, we get a homotopy between the maps associated to the endpoints of the 1-chain. Similarly, higher-dimensional chains in $C_*(LD_k)$ give rise to higher homotopies.

We emphasize that in $\hom(\mathcal{B}_*^C(I), \mathcal{B}_*^C(I))$ we are thinking of $\mathcal{B}_*^C(I)$ as a module for the A_∞ 1-category associated to ∂I, and hom means the morphisms of such modules as defined in §6.6.

It should now be clear how to generalize this to higher dimensions. In the sequence-of-surgeries description above, we never used the fact that the manifolds involved were 1-dimensional. So we

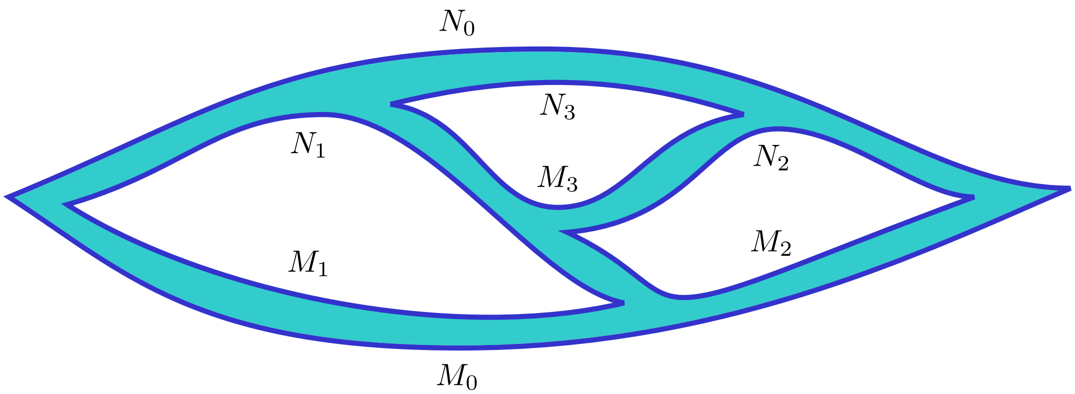


Figure 36: An n-dimensional surgery cylinder

will define, below, the operad of n-dimensional surgery cylinders, analogous to mapping cylinders of homeomorphisms (Figure 36). (Note that n is the dimension of the manifolds we are doing surgery on; the surgery cylinders are $n+1$-dimensional.)

An n-dimensional surgery cylinder (n-SC for short) consists of:

- "Lower" n-manifolds $M_0, \ldots, M_k$ and "upper" n-manifolds $N_0, \ldots, N_k$, with $\partial M_i = \partial N_i = E_i$ for all i. We call M_0 and N_0 the outer boundary and the remaining M_i's and N_i's the inner boundaries.
- Additional manifolds $R_1, \ldots, R_k$, with $\partial R_i = E_0 \cup \partial M_i = E_0 \cup \partial N_i$.
- Homeomorphisms

$$\begin{aligned} f_0 : M_0 &\to R_1 \cup M_1 \\ f_i : R_i \cup N_i &\to R_{i+1} \cup M_{i+1} \text{ for } 1 \le i \le k-1 \\ f_k : R_k \cup N_k &\to N_0. \end{aligned}$$

  Each f_i should be the identity restricted to E_0.

We can think of the above data as encoding the union of the mapping cylinders $C(f_0), \ldots, C(f_k)$, with $C(f_i)$ glued to $C(f_{i+1})$ along R_{i+1} (see Figure 37). We regard two such surgery cylinders as the same if there is a homeomorphism between them which is the identity on the boundary and which preserves the 1-dimensional fibers coming from the mapping cylinders. More specifically, we impose the following two equivalence relations:

- If $g : R_i \to R_i'$ is a homeomorphism, we can replace

$$\begin{aligned} (\ldots, R_{i-1}, R_i, R_{i+1}, \ldots) &\to (\ldots, R_{i-1}, R_i', R_{i+1}, \ldots) \\ (\ldots, f_{i-1}, f_i, \ldots) &\to (\ldots, g \circ f_{i-1}, f_i \circ g^{-1}, \ldots), \end{aligned}$$

  leaving the M_i and N_i fixed. (Keep in mind the case $R_i' = R_i$.) (See Figure 38.)

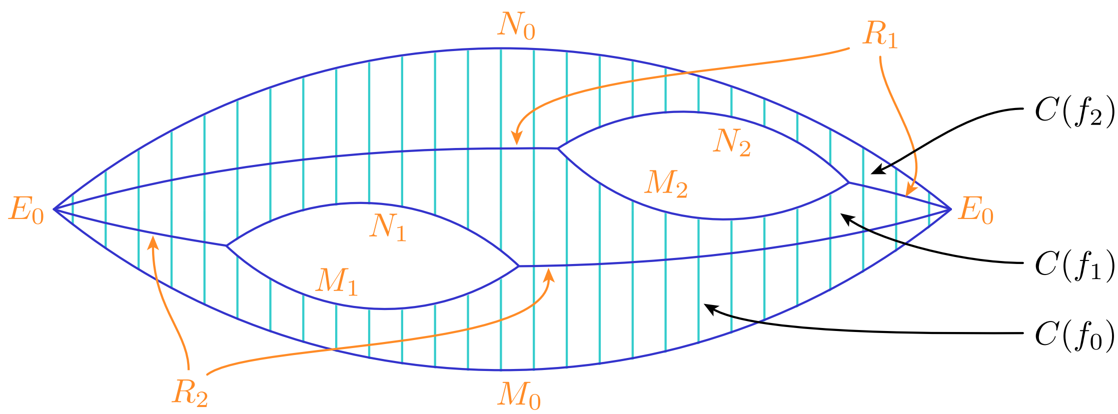


Figure 37: An n-dimensional surgery cylinder constructed from mapping cylinders

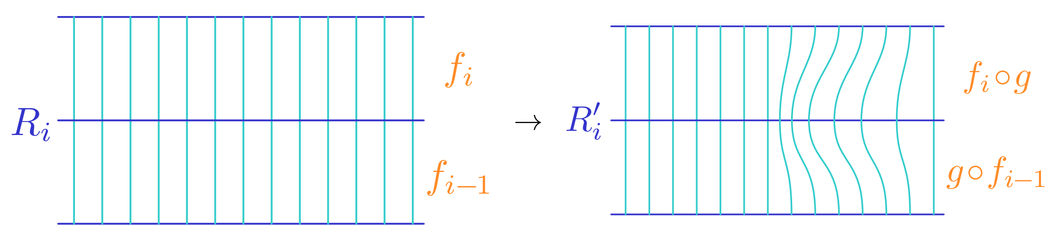


Figure 38: Conjugating by a homeomorphism.

- If $M_i = M'_i \sqcup M''_i$ and $N_i = N'_i \sqcup N''_i$ (and there is a compatible disjoint union of $\partial M = \partial N$), we can replace

$$
\begin{aligned}
(\ldots, M_{i-1}, M_i, M_{i+1}, \ldots) &\to (\ldots, M_{i-1}, M'_i, M''_i, M_{i+1}, \ldots) \\
(\ldots, N_{i-1}, N_i, N_{i+1}, \ldots) &\to (\ldots, N_{i-1}, N'_i, N''_i, N_{i+1}, \ldots) \\
(\ldots, R_{i-1}, R_i, R_{i+1}, \ldots) &\to (\ldots, R_{i-1}, R_i \cup M''_i, R_i \cup N'_i, R_{i+1}, \ldots) \\
(\ldots, f_{i-1}, f_i, \ldots) &\to (\ldots, f_{i-1}, \mathrm{id}, \mathrm{f_i}, \ldots).
\end{aligned}
$$

(See Figure 39.)

Note that the second equivalence increases the number of holes (or arity) by 1. We can make a similar identification with the roles of M'_i and M''_i reversed. In terms of the "sequence of surgeries" picture, this says that if two successive surgeries do not overlap, we can perform them in reverse order or simultaneously.

There is an operad structure on n-dimensional surgery cylinders, given by gluing the outer boundary of one cylinder into one of the inner boundaries of another cylinder. We leave it to the reader to work out a more precise statement in terms of M_i's, f_i's etc.

For fixed $\overline{M} = (M_0, \ldots, M_k)$ and $\overline{N} = (N_0, \ldots, N_k)$, we let $SC^n_{\overline{MN}}$ denote the topological space of all n-dimensional surgery cylinders as above. (Note that in different parts of $SC^n_{\overline{MN}}$ the M_i's and

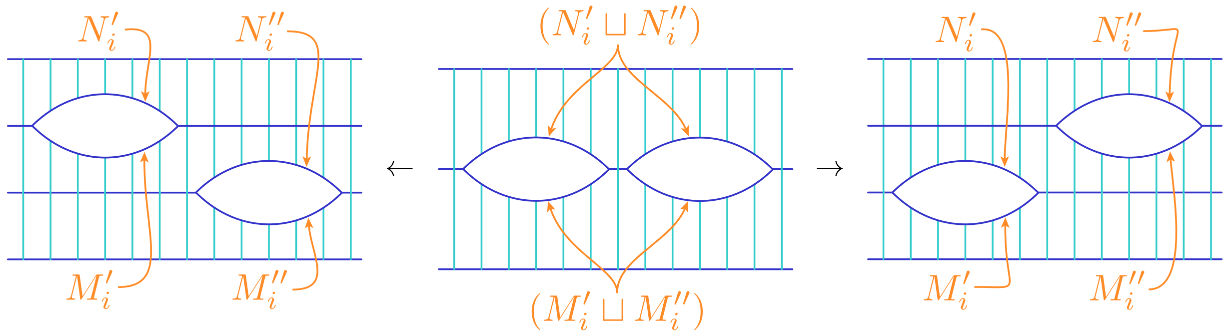


Figure 39: Changing the order of a surgery.

N_i's are ordered differently.) The topology comes from the spaces

$$\mathrm{Homeo}(M_0 \to R_1 \cup M_1) \times \mathrm{Homeo}(R_1 \cup N_1 \to R_2 \cup M_2) \times \cdots \times \mathrm{Homeo}(R_k \cup N_k \to N_0)$$

and the above equivalence relations. We will denote the typical element of $SC^n_{\overline{MN}}$ by $\overline{f} = (f_0, \ldots, f_k)$.

The n-SC operad contains the little n+1-balls operad. Roughly speaking, given a configuration of k little n+1-balls in the standard n+1-ball, we fiber the complement of the balls by vertical intervals and let M_i $[N_i]$ be the southern [northern] hemisphere of the i-th ball. More precisely, let $x_1, \ldots, x_{n+1}$ be the coordinates of $\mathbb{R}^{n+1}$. Let z be a point of the k-th space of the little n+1-balls operad, with little balls $D_1, \ldots, D_k$ inside the standard n+1-ball. We assume the D_i's are ordered according to the x_{n+1} coordinate of their centers. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection corresponding to x_{n+1}. Let $B \subset \mathbb{R}^n$ be the standard n-ball. Let M_i and N_i be B for all i. Identify $\pi(D_i)$ with B (a.k.a. M_i or N_i) via translations and dilations (no rotations). Let $R_i = B \setminus \pi(D_i)$. Let $f_i = \mathrm{id}$ for all i. We have now defined a map from the little n+1-balls operad to the n-SC operad, with contractible fibers. (The fibers correspond to moving the D_i's in the x_{n+1} direction without changing their ordering.)

Another familiar subspace of the n-SC operad is $\mathrm{Homeo}(M_0 \to N_0)$, which corresponds to case $k = 0$ (no holes). In this case the surgery cylinder is just a single mapping cylinder.

Let $\overline{f} \in SC^n_{\overline{MN}}$. Let $\hom(\mathcal{B}_*(M_i), \mathcal{B}_*(N_i))$ denote the morphisms from $\mathcal{B}_*(M_i)$ to $\mathcal{B}_*(N_i)$, as modules of the A_∞ 1-category $\mathcal{B}_*(E_i)$. We define a map

$$p(\overline{f}) : \hom(\mathcal{B}_*(M_1), \mathcal{B}_*(N_1)) \otimes \cdots \otimes \hom(\mathcal{B}_*(M_k), \mathcal{B}_*(N_k)) \to \hom(\mathcal{B}_*(M_0), \mathcal{B}_*(N_0)).$$

Given $\alpha_i \in \hom(\mathcal{B}_*(M_i), \mathcal{B}_*(N_i))$, we define $p(\overline{f})$ to be the composition

$$\mathcal{B}_*(M_0) \xrightarrow{f_0} \mathcal{B}_*(R_1 \cup M_1) \xrightarrow{\mathbf{1}\otimes\alpha_1} \mathcal{B}_*(R_1 \cup N_1) \xrightarrow{f_1} \mathcal{B}_*(R_2 \cup M_2) \xrightarrow{\mathbf{1}\otimes\alpha_2} \cdots \xrightarrow{\mathbf{1}\otimes\alpha_k} \mathcal{B}_*(R_k \cup N_k) \xrightarrow{f_k} \mathcal{B}_*(N_0)$$

(Recall that the maps $\mathbf{1} \otimes \alpha_i$ were defined in §6.6.) It is easy to check that the above definition is compatible with the equivalence relations and also the operad structure. We can reinterpret the above as a chain map

$$p : C_0(SC^n_{\overline{MN}}) \otimes \hom(\mathcal{B}_*(M_1), \mathcal{B}_*(N_1)) \otimes \cdots \otimes \hom(\mathcal{B}_*(M_k), \mathcal{B}_*(N_k)) \to \hom(\mathcal{B}_*(M_0), \mathcal{B}_*(N_0)).$$

The main result of this section is that this chain map extends to the full singular chain complex $C_*(SC^n_{\overline{M}\overline{N}})$.

Theorem 8.0.2. *There is a collection of chain maps*

$$C_*(SC^n_{\overline{M},\overline{N}}) \otimes \hom(\mathcal{B}_*(M_1), \mathcal{B}_*(N_1)) \otimes \cdots \otimes \hom(\mathcal{B}_*(M_k), \mathcal{B}_*(N_k)) \to \hom(\mathcal{B}_*(M_0), \mathcal{B}_*(N_0))$$

which satisfy the operad compatibility conditions. On $C_0(SC^n_{\overline{M}\overline{N}})$ this agrees with the chain map p defined above. When $k = 0$, this coincides with the $C_*(\mathrm{Homeo}(M_0 \to N_0))$ action of §5.

If, in analogy to Hochschild cochains, we define elements of $\hom(M, N)$ to be "blob cochains", we can summarize the above proposition by saying that the n-SC operad acts on blob cochains. As noted above, the n-SC operad contains the little n+1-balls operad, so this constitutes a higher dimensional version of the Deligne conjecture for Hochschild cochains and the little 2-disks operad.

Proof. As described above, $SC^n_{\overline{M},\overline{N}}$ is equal to the disjoint union of products of homeomorphism spaces, modulo some relations. By Theorem 5.2.1 and the Eilenberg-Zilber theorem, we have for each such product P a chain map

$$C_*(P) \otimes \hom(\mathcal{B}_*(M_1), \mathcal{B}_*(N_1)) \otimes \cdots \otimes \hom(\mathcal{B}_*(M_k), \mathcal{B}_*(N_k)) \to \hom(\mathcal{B}_*(M_0), \mathcal{B}_*(N_0)).$$

It suffices to show that the above maps are compatible with the relations whereby $SC^n_{\overline{M},\overline{N}}$ is constructed from the various P's. This in turn follows easily from the fact that the actions of $C_*(\mathrm{Homeo}(\cdot \to \cdot))$ are local (compatible with gluing) and associative. □

We note that even when $n = 1$, the above theorem goes beyond an action of the little disks operad. M_i could be a disjoint union of intervals, and N_i could connect the end points of the intervals in a different pattern from M_i. The genus of the surface associated to the surgery cylinder could be greater than zero.

A The method of acyclic models

Let F_* and G_* be chain complexes. Assume F_k has a basis $\{x_{kj}\}$ (that is, F_* is free and we have specified a basis). (In our applications, $\{x_{kj}\}$ will typically be singular k-simplices or k-blob diagrams.) For each basis element x_{kj} assume we have specified a "target" $D^{kj}_* \subset G_*$.

We say that a chain map $f : F_* \to G_*$ is *compatible* with the above data (basis and targets) if $f(x_{kj}) \in D^{kj}_*$ for all k and j. Let $\mathrm{Compat}(D^\bullet_*)$ denote the subcomplex of maps from F_* to G_* such that the image of each higher homotopy applied to x_{kj} lies in D^{kj}_*.

Theorem A.0.3 (Acyclic models)**.** *Suppose*

- $D^{k-1,l}_* \subset D^{kj}_*$ *whenever $x_{k-1,l}$ occurs in ∂x_{kj} with non-zero coefficient;*
- D^{0j}_0 *is non-empty for all j; and*
- D^{kj}_* *is $(k-1)$-acyclic (i.e. $H_{k-1}(D^{kj}_*) = 0$) for all k, j .*

Then $\mathrm{Compat}(D^\bullet_*)$ *is non-empty. If, in addition,*

- D_*^{kj} *is m-acyclic for* $k \le m \le k+i$ *and for all* k, j,

then $\mathrm{Compat}(D_*^\bullet)$ *is i-connected.*

Proof. (Sketch) This is a standard result; see, for example, [9, Chapter 4].

We will build a chain map $f \in \mathrm{Compat}(D_*^\bullet)$ inductively. Choose $f(x_{0j}) \in D_0^{0j}$ for all j (possible since D_0^{0j} is non-empty). Choose $f(x_{1j}) \in D_1^{1j}$ such that $\partial f(x_{1j}) = f(\partial x_{1j})$ (possible since $D_*^{0l} \subset D_*^{1j}$ for each x_{0l} in ∂x_{1j} and D_*^{1j} is 0-acyclic). Continue in this way, choosing $f(x_{kj}) \in D_k^{kj}$ such that $\partial f(x_{kj}) = f(\partial x_{kj})$ We have now constructed $f \in \mathrm{Compat}(D_*^\bullet)$, proving the first claim of the theorem.

Now suppose that D_*^{kj} is k-acyclic for all k and j. Let f and f' be two chain maps (0-chains) in $\mathrm{Compat}(D_*^\bullet)$. Using a technique similar to above we can construct a homotopy (1-chain) in $\mathrm{Compat}(D_*^\bullet)$ between f and f'. Thus $\mathrm{Compat}(D_*^\bullet)$ is 0-connected. Similarly, if D_*^{kj} is $(k+i)$-acyclic then we can show that $\mathrm{Compat}(D_*^\bullet)$ is i-connected. □

B Adapting families of maps to open covers

Let X and T be topological spaces, with X compact. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X which affords a partition of unity $\{r_\alpha\}$. (That is, $r_\alpha : X \to [0,1]$; $r_\alpha(x) = 0$ if $x \notin U_\alpha$; for fixed x, $r_\alpha(x) \neq 0$ for only finitely many α; and $\sum_\alpha r_\alpha = 1$.) Since X is compact, we will further assume that $r_\alpha = 0$ (globally) for all but finitely many α.

Consider $C_*(\mathrm{Maps}(X \to T))$, the singular chains on the space of continuous maps from X to T. $C_k(\mathrm{Maps}(X \to T))$ is generated by continuous maps

$$f : P \times X \to T,$$

where P is some convex linear polyhedron in $\mathbb{R}^k$. Recall that f is *supported* on $S \subset X$ if $f(p, x)$ does not depend on p when $x \notin S$, and that f is *adapted* to $\mathcal{U}$ if f is supported on the union of at most k of the U_α's. A chain $c \in C_*(\mathrm{Maps}(X \to T))$ is adapted to $\mathcal{U}$ if it is a linear combination of generators which are adapted.

Lemma B.0.4. *Let* $f : P \times X \to T$, *as above. Then there exists*

$$F : I \times P \times X \to T$$

such that

1. $F(0, \cdot, \cdot) = f$.

2. *We can decompose* $P = \cup_i D_i$ *so that the restrictions* $F(1, \cdot, \cdot) : D_i \times X \to T$ *are all adapted to* $\mathcal{U}$.

3. *If* f *has support* $S \subset X$, *then* $F : (I \times P) \times X \to T$ *(a* $k{+}1$*-parameter family of maps) also has support* S. *Furthermore, if* Q *is a convex linear subpolyhedron of* ∂P *and* f *restricted to* Q *has support* $S' \subset X$, *then* $F : (I \times Q) \times X \to T$ *also has support* S'.

4. *Suppose both X and T are smooth manifolds, metric spaces, or PL manifolds, and let $\mathcal{X}$ denote the subspace of* $\mathrm{Maps}(X \to T)$ *consisting of immersions or of diffeomorphisms (in the smooth case), bi-Lipschitz homeomorphisms (in the metric case), or PL homeomorphisms (in the PL case). If f is smooth, Lipschitz or PL, as appropriate, and $f(p,\cdot) : X \to T$ is in $\mathcal{X}$ for all $p \in P$ then $F(t,p,\cdot)$ is also in $\mathcal{X}$ for all $t \in I$ and $p \in P$.*

Proof. Our homotopy will have the form

$$\begin{aligned} F : I \times P \times X &\to X \\ (t,p,x) &\mapsto f(u(t,p,x),x) \end{aligned}$$

for some function

$$u : I \times P \times X \to P.$$

First we describe u, then we argue that it makes the conclusions of the lemma true.

For each cover index α choose a cell decomposition K_α of P such that the various K_α are in general position with respect to each other. If we are in one of the cases of item 4 of the lemma, also choose K_α sufficiently fine as described below.

Let L be a common refinement of all the K_α's. Let $\tilde{L}$ denote the handle decomposition of P corresponding to L. Each i-handle C of $\tilde{L}$ has an i-dimensional tangential coordinate and, more importantly for our purposes, a $k{-}i$-dimensional normal coordinate. We will typically use the same notation for i-cells of L and the corresponding i-handles of $\tilde{L}$.

For each (top-dimensional) k-cell C of each K_α, choose a point $p(C) \in C \subset P$. If C meets a subpolyhedron Q of ∂P, we require that $p(C) \in Q$. (It follows that if C meets both Q and Q', then $p(C) \in Q \cap Q'$. Ensuring this is possible corresponds to some mild constraints on the choice of the K_α.)

Let D be a k-handle of $\tilde{L}$. For each α let $C(D,\alpha)$ be the k-cell of K_α which contains D and let $p(D,\alpha) = p(C(D,\alpha))$.

For $p \in D$ we define

$$u(t,p,x) = (1-t)p + t\sum_\alpha r_\alpha(x)p(D,\alpha).$$

(Recall that P is a convex linear polyhedron, so the weighted average of points of P makes sense.)

Thus far we have defined $u(t,p,x)$ when p lies in a k-handle of $\tilde{L}$. We will now extend u inductively to handles of index less than k.

Let E be a $k{-}1$-handle. E is homeomorphic to $B^{k-1} \times [0,1]$, and meets the k-handles at $B^{k-1} \times \{0\}$ and $B^{k-1} \times \{1\}$. Let $\eta : E \to [0,1]$, $\eta(x,s) = s$ be the normal coordinate of E. Let D_0 and D_1 be the two k-handles of $\tilde{L}$ adjacent to E. There is at most one index β such that $C(D_0,\beta) \neq C(D_1,\beta)$. (If there is no such index, choose β arbitrarily.) For $p \in E$, define

$$u(t,p,x) = (1-t)p + t\left(\sum_{\alpha\neq\beta} r_\alpha(x)p(D_0,\alpha) + r_\beta(x)(\eta(p)p(D_0,\beta) + (1-\eta(p))p(D_1,\beta))\right).$$

Now for the general case. Let E be a $k{-}j$-handle. Let $D_0,\ldots,D_a$ be the k-handles adjacent to E. There is a subset of cover indices $\mathcal{N}$, of cardinality j, such that if $\alpha \notin \mathcal{N}$ then $p(D_u,\alpha) = p(D_v,\alpha)$ for all $0 \le u,v \le a$. For fixed $\beta \in \mathcal{N}$ let $\{q_{\beta i}\}$ be the set of values of $p(D_u,\beta)$ for $0 \le u \le a$. Recall the product structure $E = B^{k-j} \times B^j$. Inductively, we have defined functions $\eta_{\beta i} : \partial B^j \to [0,1]$ such that $\sum_i \eta_{\beta i} = 1$ for all $\beta \in \mathcal{N}$. Choose extensions of $\eta_{\beta i}$ to all of B^j. Via the projection $E \to B^j$, regard $\eta_{\beta i}$ as a function on E. Now define, for $p \in E$,

$$u(t,p,x) = (1-t)p + t\left(\sum_{\alpha\notin\mathcal{N}} r_\alpha(x)p(D_0,\alpha) + \sum_{\beta\in\mathcal{N}} r_\beta(x)\left(\sum_i \eta_{\beta i}(p)\cdot q_{\beta i}\right)\right). \tag{B.1}$$

This completes the definition of $u : I \times P \times X \to P$. The formulas above are consistent: for p at the boundary between a $k-j$-handle and a $k-(j+1)$-handle the corresponding expressions in Equation (B.1) agree, since one of the normal coordinates becomes 0 or 1. Note that if $Q \subset \partial P$ is a convex linear subpolyhedron, then $u(I \times Q \times X) \subset Q$.

Next we verify that u affords F the properties claimed in the statement of the lemma.

Since $u(0,p,x) = p$ for all $p \in P$ and $x \in X$, $F(0,p,x) = f(p,x)$ for all p and x. Therefore F is a homotopy from f to something.

Next we show that for each handle D of J, $F(1,\cdot,\cdot) : D \times X \to X$ is a singular cell adapted to $\mathcal{U}$. Let $k-j$ be the index of D. Referring to Equation (B.1), we see that $F(1,p,x)$ depends on p only if $r_\beta(x) \neq 0$ for some $\beta \in \mathcal{N}$, i.e. only if $x \in \bigcup_{\beta\in\mathcal{N}} U_\beta$. Since the cardinality of $\mathcal{N}$ is j which is less than or equal to k, this shows that $F(1,\cdot,\cdot) : D \times X \to X$ is adapted to $\mathcal{U}$.

Next we show that F does not increase supports. If $f(p,x) = f(p',x)$ for all $p,p' \in P$, then

$$F(t,p,x) = f(u(t,p,x),x) = f(u(t',p',x),x) = F(t',p',x)$$

for all (t,p) and (t',p') in $I \times P$. Similarly, if $f(q,x) = f(q',x)$ for all $q,q' \in Q \subset \partial P$, then

$$F(t,q,x) = f(u(t,q,x),x) = f(u(t',q',x),x) = F(t',q',x)$$

for all (t,q) and (t',q') in $I \times Q$. (Recall that we arranged above that $u(I \times Q \times X) \subset Q$.)

Now for claim 4 of the lemma. Assume that X and T are smooth manifolds and that f is a smooth family of diffeomorphisms. We must show that we can choose the K_α's and u so that $F(t,p,\cdot)$ is a diffeomorphism for all t and p. It suffices to show that the derivative $\frac{\partial F}{\partial x}(t,p,x)$ is non-singular for all (t,p,x). We have

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial u}{\partial x}.$$

Since f is a family of diffeomorphisms and X and P are compact, $\frac{\partial f}{\partial x}$ is non-singular and bounded away from zero. Also, since f is smooth $\frac{\partial f}{\partial p}$ is bounded. Thus if we can insure that $\frac{\partial u}{\partial x}$ is sufficiently small, we are done. It follows from Equation (B.1) above that $\frac{\partial u}{\partial x}$ depends on $\frac{\partial r_\alpha}{\partial x}$ (which is bounded) and the differences amongst the various $p(D_0,\alpha)$'s and $q_{\beta i}$'s. These differences are small if the cell decompositions K_α are sufficiently fine. This completes the proof that F is a homotopy through diffeomorphisms.

If we replace "diffeomorphism" with "immersion" in the above paragraph, the argument goes through essentially unchanged.

Next we consider the case where f is a family of bi-Lipschitz homeomorphisms. Recall that we assume that f is Lipschitz in the P direction as well. The argument in this case is similar to the one above for diffeomorphisms, with bounded partial derivatives replaced by Lipschitz constants. Since X and P are compact, there is a universal bi-Lipschitz constant that works for $f(p,\cdot)$ for all p. By choosing the cell decompositions K_α sufficiently fine, we can insure that u has a small Lipschitz constant in the X direction. This allows us to show that $F(t,p,\cdot)$ has a bi-Lipschitz constant close to the universal bi-Lipschitz constant for f.

Since PL homeomorphisms are bi-Lipschitz, we have established this last remaining case of claim 4 of the lemma as well. □

Lemma B.0.5. *Let $\mathcal{X}_*$ be any of $C_*(\mathrm{Maps}(X \to T))$ or singular chains on the subspace of $\mathrm{Maps}(X \to T)$ consisting of immersions, diffeomorphisms, bi-Lipschitz homeomorphisms or PL homeomorphisms. Let $G_* \subset \mathcal{X}_*$ denote the chains adapted to an open cover $\mathcal{U}$ of X. Then G_* is a strong deformation retract of $\mathcal{X}_*$.*

Proof. It suffices to show that given a generator $f : P \times X \to T$ of $\mathcal{X}_k$ with $\partial f \in G_{k-1}$ there exists $h \in \mathcal{X}_{k+1}$ with $\partial h = f + g$ and $g \in G_k$. This is exactly what Lemma B.0.4 gives us. More specifically, let $\partial P = \sum Q_i$, with each $Q_i \in G_{k-1}$. Let $F : I \times P \times X \to T$ be the homotopy constructed in Lemma B.0.4. Then ∂F is equal to f plus $F(1,\cdot,\cdot)$ plus the restrictions of F to $I \times Q_i$. Part 2 of Lemma B.0.4 says that $F(1,\cdot,\cdot) \in G_k$, while part 3 of Lemma B.0.4 says that the restrictions to $I \times Q_i$ are in G_k. □

C Comparing n-category definitions

In §2.2 we showed how to construct a topological n-category from a traditional n-category; the morphisms of the topological n-category are string diagrams labeled by the traditional n-category. In this appendix we sketch how to go the other direction, for $n = 1$ and 2. The basic recipe, given a topological n-category $\mathcal{C}$, is to define the k-morphisms of the corresponding traditional n-category to be $\mathcal{C}(B^k)$, where B^k is the *standard* k-ball. One must then show that the axioms of §6.1 imply the traditional n-category axioms. One should also show that composing the two arrows (between traditional and topological n-categories) yields the appropriate sort of equivalence on each side. Since we haven't given a definition for functors between topological n-categories (the paper is already too long!), we do not pursue this here.

We emphasize that we are just sketching some of the main ideas in this appendix — it falls well short of proving the definitions are equivalent.

C.1 1-categories over Set or Vect

Given a topological 1-category $\mathcal{X}$ we construct a 1-category in the conventional sense, $c(\mathcal{X})$. This construction is quite straightforward, but we include the details for the sake of completeness, because it illustrates the role of structures (e.g. orientations, spin structures, etc) on the underlying manifolds, and to shed some light on the $n = 2$ case, which we describe in §C.2.

Let B^k denote the *standard* k-ball. Let the objects of $c(\mathcal{X})$ be $c(\mathcal{X})^0 = \mathcal{X}(B^0)$ and the morphisms of $c(\mathcal{X})$ be $c(\mathcal{X})^1 = \mathcal{X}(B^1)$. The boundary and restriction maps of $\mathcal{X}$ give domain and range maps from $c(\mathcal{X})^1$ to $c(\mathcal{X})^0$.

Choose a homeomorphism $B^1 \cup_{pt} B^1 \to B^1$. Define composition in $c(\mathcal{X})$ to be the induced map $c(\mathcal{X})^1 \times c(\mathcal{X})^1 \to c(\mathcal{X})^1$ (defined only when range and domain agree). By isotopy invariance in $\mathcal{X}$, any other choice of homeomorphism gives the same composition rule. Also by isotopy invariance, composition is strictly associative.

Given $a \in c(\mathcal{X})^0$, define $\mathbf{1}_a \stackrel{\text{def}}{=} a \times B^1$. By extended isotopy invariance in $\mathcal{X}$, this has the expected properties of an identity morphism.

If the underlying manifolds for $\mathcal{X}$ have further geometric structure, then we obtain certain functors. The base case is for oriented manifolds, where we obtain no extra algebraic data.

For 1-categories based on unoriented manifolds, there is a map $* : c(\mathcal{X})^1 \to c(\mathcal{X})^1$ coming from $\mathcal{X}$ applied to an orientation-reversing homeomorphism (unique up to isotopy) from B^1 to itself. Topological properties of this homeomorphism imply that $a^{**} = a$ (* is order 2), * reverses domain and range, and $(ab)^* = b^*a^*$ (* is an anti-automorphism).

For 1-categories based on Spin manifolds, the the nontrivial spin homeomorphism from B^1 to itself which covers the identity gives an order 2 automorphism of $c(\mathcal{X})^1$.

For 1-categories based on Pin_- manifolds, we have an order 4 antiautomorphism of $c(\mathcal{X})^1$. For 1-categories based on Pin_+ manifolds, we have an order 2 antiautomorphism and also an order 2 automorphism of $c(\mathcal{X})^1$, and these two maps commute with each other.

Similar arguments show that modules for topological 1-categories are essentially the same thing as traditional modules for traditional 1-categories.

C.2 Pivotal 2-categories

Let $\mathcal{C}$ be a topological 2-category. We will construct from $\mathcal{C}$ a traditional pivotal 2-category. (The "pivotal" corresponds to our assumption of strong duality for $\mathcal{C}$.)

We will try to describe the construction in such a way the the generalization to $n > 2$ is clear, though this will make the $n = 2$ case a little more complicated than necessary.

Before proceeding, we must decide whether the 2-morphisms of our pivotal 2-category are shaped like rectangles or bigons. Each approach has advantages and disadvantages. For better or worse, we choose bigons here.

Define the k-morphisms C^k of C to be $\mathcal{C}(B^k)_E$, where B^k denotes the standard k-ball, which we also think of as the standard bihedron (a.k.a. globe). (For $k = 1$ this is an interval, and for $k = 2$ it is a bigon.) Since we are thinking of B^k as a bihedron, we have a standard decomposition of the ∂B^k into two copies of B^{k-1} which intersect along the "equator" $E \cong S^{k-2}$. Recall that the subscript in $\mathcal{C}(B^k)_E$ means that we consider the subset of $\mathcal{C}(B^k)$ whose boundary is splittable along E. This allows us to define the domain and range of morphisms of C using boundary and restriction maps of $\mathcal{C}$.

Choosing a homeomorphism $B^1 \cup B^1 \to B^1$ defines a composition map on C^1. This is not associative, but we will see later that it is weakly associative.

Choosing a homeomorphism $B^2 \cup B^2 \to B^2$ defines a "vertical" composition map on C^2 (Figure 40). Isotopy invariance implies that this is associative. We will define a "horizontal" composition later.

Given $a \in C^1$, define $\mathbf{1}_a = a \times I \in C^2$ (pinched boundary). Extended isotopy invariance for $\mathcal{C}$ shows that this morphism is an identity for vertical composition.

Given $x \in C^0$, define $\mathbf{1}_x = x \times B^1 \in C^1$. We will show that this 1-morphism is a weak identity. This would be easier if our 2-morphisms were shaped like rectangles rather than bigons.

In showing that identity 1-morphisms have the desired properties, we will rely heavily on the extended isotopy invariance of 2-morphisms in $\mathcal{C}$. This means we are free to add or delete product

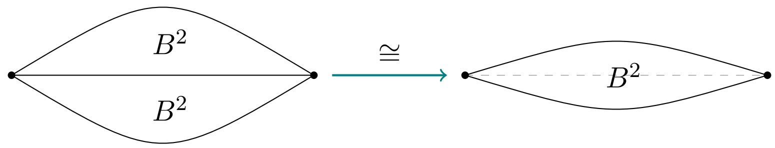


Figure 40: Vertical composition of 2-morphisms

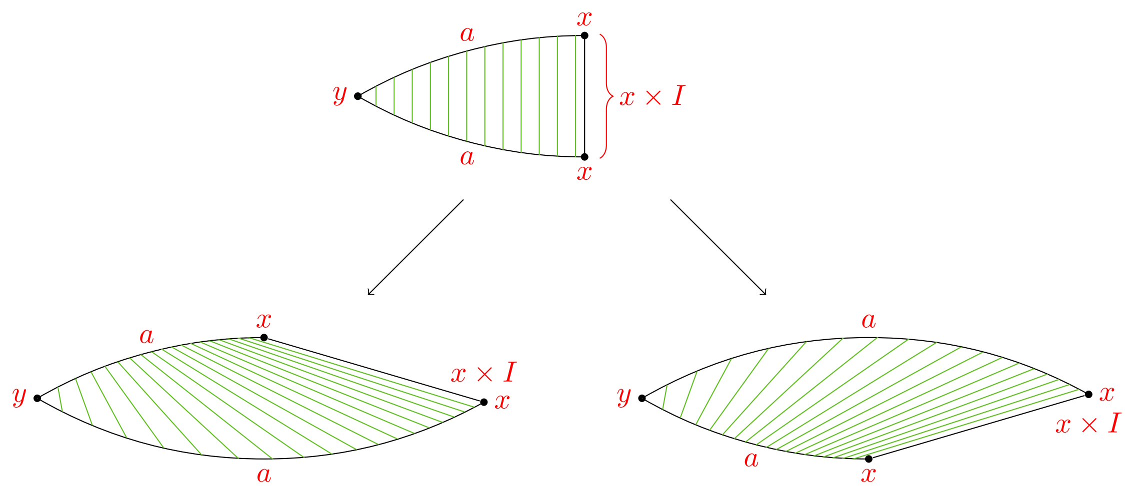


Figure 41: Producing weak identities from half pinched products

regions from 2-morphisms.

Let $a : y \to x$ be a 1-morphism. Define 2-morphsims $a \to a \bullet \mathbf{1}_x$ and $a \bullet \mathbf{1}_x \to a$ as shown in Figure 41. As suggested by the figure, these are two different reparameterizations of a half-pinched version of $a \times I$. We must show that the two compositions of these two maps give the identity 2-morphisms on a and $a \bullet \mathbf{1}_x$, as defined above. Figure 42 shows one case. In the first step we have inserted a copy of $(x \times I) \times I$. Figure 43 shows the other case. We identify a product region and remove it.

We define horizontal composition $f *_h g$ of 2-morphisms f and g as shown in Figure 44. It is not hard to show that this is independent of the arbitrary (left/right) choice made in the definition, and that it is associative.

C.3 A_∞ 1-categories

In this section, we make contact between the usual definition of an A_∞ category and our definition of a topological A_∞ 1-category, from §6.1.

Given a topological A_∞ 1-category $\mathcal{C}$, we define an "m_k-style" A_∞ 1-category A as follows. The objects of A are $\mathcal{C}(pt)$. The morphisms of A, from x to y, are $\mathcal{C}(I; x, y)$ ($\mathcal{C}$ applied to the standard interval with boundary labeled by x and y). For simplicity we will now assume there is only one object and suppress it from the notation.

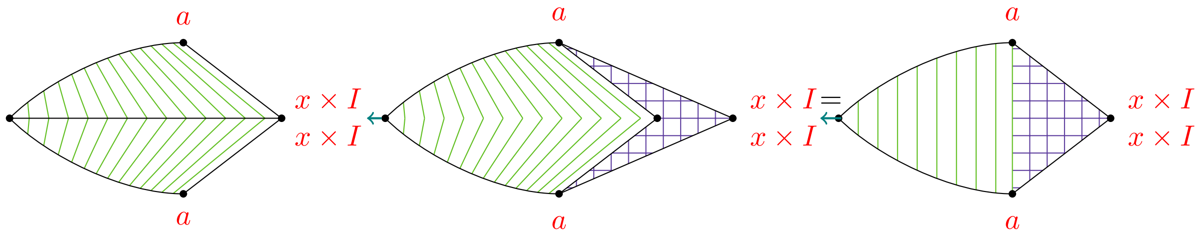


Figure 42: Composition of weak identities, 1

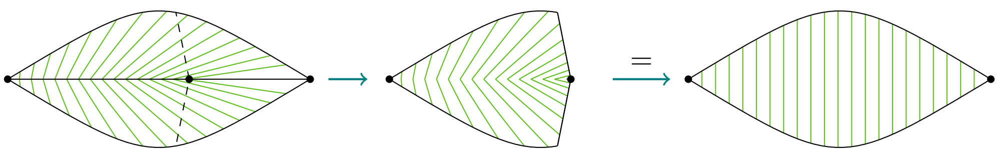


Figure 43: Composition of weak identities, 2

A choice of homeomorphism $I \cup I \to I$ induces a chain map $m_2 : A \times A \to A$. We now have two different homeomorphisms $I \cup I \cup I \to I$, but they are isotopic. Choose a specific 1-parameter family of homeomorphisms connecting them; this induces a degree 1 chain homotopy $m_3 : A \otimes A \otimes A \to A$. Proceeding in this way we define the rest of the m_i's. It is straightforward to verify that they satisfy the necessary identities.

In the other direction, we start with an alternative conventional definition of an A_∞ algebra: an algebra A for the A_∞ operad. (For simplicity, we are assuming our A_∞ 1-category has only one object.) We are free to choose any operad with contractible spaces, so we choose the operad whose k-th space is the space of decompositions of the standard interval I into k parameterized copies of I. Note in particular that when $k = 1$ this implies a $C_*(\mathrm{Homeo}(I))$ action on A. (Compare with Example 6.2.10 and the discussion which precedes it.) Given a non-standard interval J, we define $\mathcal{C}(J)$ to be $(\mathrm{Homeo}(I \to J) \times A)/\mathrm{Homeo}(I \to I)$, where $\beta \in \mathrm{Homeo}(I \to I)$ acts via $(f, a) \mapsto (f \circ \beta^{-1}, \beta_*(a))$. Note that $\mathcal{C}(J) \cong A$ (non-canonically) for all intervals J. We define a $\mathrm{Homeo}(J)$ action on $\mathcal{C}(J)$ via $g_*(f, a) = (g \circ f, a)$. The $C_*(\mathrm{Homeo}(J))$ action is defined similarly.

Let J_1 and J_2 be intervals. We must define a map $\mathcal{C}(J_1) \otimes \mathcal{C}(J_2) \to \mathcal{C}(J_1 \cup J_2)$. Choose a homeomorphism $g : I \to J_1 \cup J_2$. Let $(f_i, a_i) \in \mathcal{C}(J_i)$. We have a parameterized decomposition of I into two intervals given by $g^{-1} \circ f_i$, $i = 1, 2$. Corresponding to this decomposition the operad action gives a map $\mu : A \otimes A \to A$. Define the gluing map to send $(f_1, a_1) \otimes (f_2, a_2)$ to $(g, \mu(a_1 \otimes a_2))$. Operad associativity for A implies that this gluing map is independent of the choice of g and the choice of representative (f_i, a_i).

It is straightforward to verify the remaining axioms for a topological A_∞ 1-category.

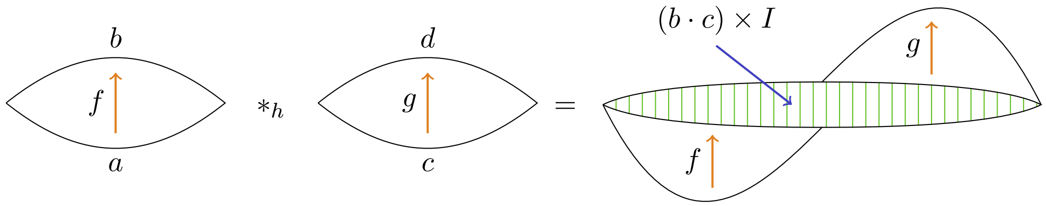


Figure 44: Horizontal composition of 2-morphisms

References

[1] Dietmar Bisch. Bimodules, higher relative commutants and the fusion algebra associated to a subfactor. In *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, volume 13 of *Fields Inst. Commun.*, pages 13–63. Amer. Math. Soc., Providence, RI, 1997. MR1424954 (preview at google books).

[2] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer-Verlag, Berlin, 1996. Translated from the 1988 Russian original MR1438306 (preview at google books).

[3] E. Getzler and J. D. S. Jones. Operads, homotopy algebra, and iterated integrals for double loop spaces, 1994. arXiv:hep-th/9403055.

[4] Maxim Kontsevich and Yan Soibelman. Deformations of algebras over operads and the Deligne conjecture. In *Conférence Moshé Flato 1999, Vol. I (Dijon)*, volume 21 of *Math. Phys. Stud.*, pages 255–307. Kluwer Acad. Publ., Dordrecht, 2000. MR1805894 arXiv:math.QA/0001151.

[5] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004. MR2094071.

[6] Jacob Lurie. Derived Algebraic Geometry VI: E_k Algebras. arXiv:0911.0018.

[7] Roger Penrose. Applications of negative dimensional tensors. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 221–244. Academic Press, London, 1971.

[8] Roger Penrose and Wolfgang Rindler. *Spinors and space-time. Vol. 1*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1984. MR776784.

[9] Edwin H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York, 1966. MR0210112 (preview at google books).

[10] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004. MR2079378 DOI:10.1017/CBO9780511526398.013 http://math.berkeley.edu/~teichner/Papers/Oxford.pdf.

[11] Dmitry E. Tamarkin. Formality of chain operad of little discs. *Lett. Math. Phys.*, 66(1-2):65–72, 2003. MR2064592 DOI:10.1023/B:MATH.0000017651.12703.a1.

[12] Alexander A. Voronov. The Swiss-cheese operad. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 365–373. Amer. Math. Soc., Providence, RI, 1999. MR1718089 arXiv:math.QA/9807037.

[13] Alexander A. Voronov. Homotopy Gerstenhaber algebras. In *Conférence Moshé Flato 1999, Vol. II (Dijon)*, volume 22 of *Math. Phys. Stud.*, pages 307–331. Kluwer Acad. Publ., Dordrecht, 2000. MR1805923 arXiv:math.QA/9908040.

[14] Kevin Walker. Topological quantum field theories. Available at http://canyon23.net/math/.

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