The blob complex arXiv: 1009.5025 & arXiv: 1108.5386

Topological Quantum Field Theories are some of the nicest invariants of manifolds.

An $(n+1)$-dimensional TQFT associates

$$n\text{-manifolds} \rightsquigarrow \text{vector spaces}$$
$$\Sigma \longmapsto Z(\Sigma)$$

$$(n+1)\text{-manifolds} \rightsquigarrow \text{linear maps}$$
$$M: \Sigma_1 \to \Sigma_2 \longmapsto Z(M): Z(\Sigma_1) \to Z(\Sigma_2)$$

(Sometimes not all $n+1$ manifolds are allowed; when M must be a mapping cylinder we say we have a $(n+\varepsilon)$-dimensional TQFT.)

The nicest TQFTs are the <u>fully extended</u> ones, which we can compute by decomposing our manifolds into smaller pieces.

We associate to

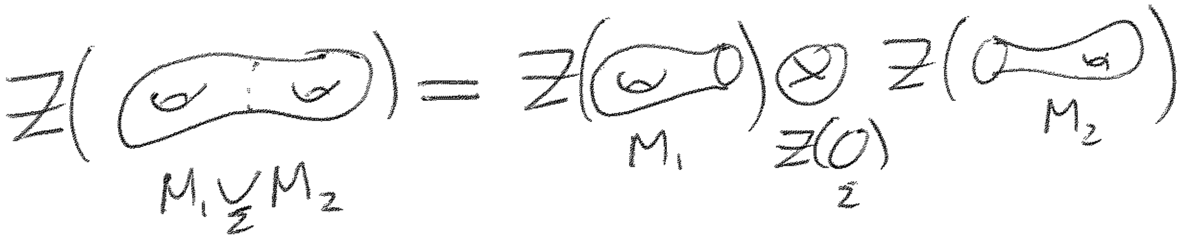
(n-1)-manifolds $\rightsquigarrow$ categories

(n-2)-manifolds $\rightsquigarrow$ 2-categories

$\vdots$

0-manifolds $\rightsquigarrow$ n-categories

and at each level have a formula



which translates gluing of manifolds into an algebraic operation.

In fact, the Cobordism Hypothesis of Baez and Dolan roughly says:

$$\left(\text{fully extended } n+\varepsilon\text{-TQFTs}\right) \longleftrightarrow \left(n\text{-categories with duals}\right)$$

This has since been made precise, and proved, by Lurie.

Today, I want to show you an explicit construction of a fully extended TQFT from an n-category, and show that we can get more: the vector spaces $Z(M^n)$ for n-manifolds are actually the 0-th homology of a natural chain complex.

A <u>disklike n-category</u> consists of

- functors $\mathcal{C}_k : \{k\text{-balls}\} \longrightarrow \mathrm{Set}$ for $0 \le k \le n$

  $\circlearrowleft$

  $\{\text{homeomorphisms}\}$

  "$\mathcal{C}_k(X)$ is "the set of k-morphisms of shape X"

- restriction maps $\mathcal{C}_k(X) \to \mathcal{C}_{k-1}(Y)$ when $Y \subset \partial X$.

- gluing maps $\mathcal{C}_k(X_1) \underset{\mathcal{C}_{k-1}(Y)}{\times} \mathcal{C}_k(X_2) \longrightarrow \mathcal{C}_k(X_1 \underset{Y}{\cup} X_2)$

such that

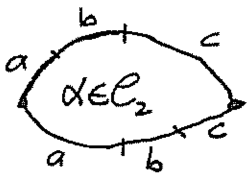
- at level n, isotopic homeomorphisms act identically
- the gluing maps are strictly associative.

Although gluing is strictly associative, disklike categories are 'weak'.

- To obtain a composition on $\mathcal{C}_1(I)$, you need a reparametrization $I \cup I \to I$

$$\mathcal{C}_1(I) \times \mathcal{C}_1(I) \to \mathcal{C}_1(I \cup I) \to \mathcal{C}_1(I).$$

- $(a \bullet b) \bullet c \neq a \bullet (b \bullet c)$, but the (omitted) identity axiom ensures there exists

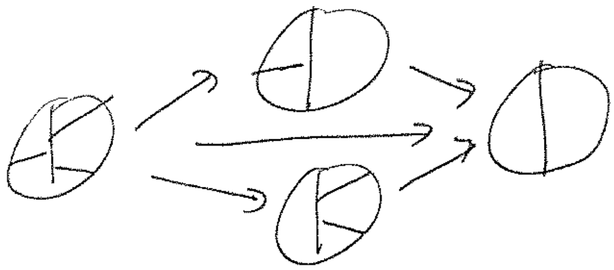


an invertible (up-to higher morphisms) morphism between them.

- At level n equations hold on the nose, by the isotopy axiom.

From a disklike n-category we immediately obtain invariants of n-manifolds.

Given M an n-manifold, consider $\mathbb{D}(M)$, the poset of ball decompositions



The n-category $\mathcal{C}$ gives a functor $\mathbb{D}(M) \to \mathcal{S}et$ and the TQFT skein module is

$$A(M; \mathcal{C}) = \operatorname*{colim}_{\mathbb{D}(M)} \mathcal{C}$$

Examples

① $\mathcal{C}_{k<n}(X^k) = \mathrm{Maps}(X \to T)$

$\mathcal{C}_n(X^n) = [X \to T]_{\mathrm{rel}\,\partial}$

and then $A(M;\mathcal{C}) = [M \to T]_{\mathrm{rel}\,\partial}$

② $\mathcal{C}$ a fusion category

$A(\Sigma^2;\mathcal{C})$ is the Turaev-Viro vector space for a surface

③ $\mathcal{C}$ a modular tensor category

$A(M^3;\mathcal{C})$ is the Reshetikhin-Turaev vector space for ∂M.

④ We can build a 4-category from Khovanov homology (mod 2).

What are the invariants??

What is the blob complex?

We replace colimits by homotopy colimits.

$$B_*(M; \mathcal{C}) = \operatorname*{hocolim}_{D(M)} \mathcal{C}.$$

Let's make this much more explicit.

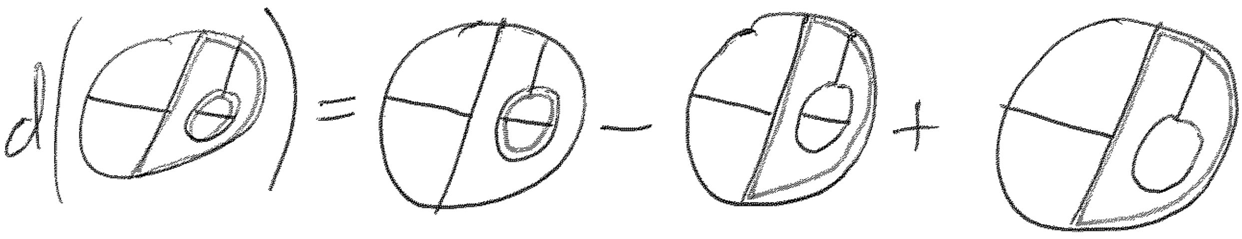
A k-chain of the blob complex consists of

- k balls ("blobs") in W, pairwise nested or disjoint.
- a compatible decomposition of W into balls,
- a labelling of the balls by n-morphisms from $\mathcal{C}$.

The differential is a (signed) sum over

a) ways to forget a blob

b) ways to forget an innermost blob,
gluing up its contents



It is easy to see that H_0 recovers the original colimit.

Theorem (Families of diffeomorphisms act)

There are chain maps

$$C_*(\mathrm{Diff}(M)) \otimes B_*(M;\mathcal{C}) \longrightarrow B_*(M;\mathcal{C})$$

- so $C_0\,\mathrm{Diff}(M) \otimes B_*(M;\mathcal{C}) \to B_*(M;\mathcal{C})$ is the obvious action
- compatible with gluing (up to homotopy)
- and in fact uniquely (up-to-homotopy) determined by these conditions.

Examples

- $B_*(S^1;\mathcal{C})$ is the Hochschild complex; rotation around S^1 gives the cyclic differential
- rotation along rational slopes on $\mathbb{T}^2$ giving a degree-raising map $HB_*(\mathbb{T}^2;\mathcal{C}) \to HB_{*+1}(\mathbb{T}^2;\mathcal{C})$

Sketch define BT_*, total complex of $BT_{ij} = C_j(i\text{-blob diagrams})$

BT_* has an obvious action of $C_*\mathrm{Diff}$

the inclusion $B_* = BT_{*0} \subset BT_*$ is a homotopy equivalence

Sketch $B_* = B_*^{\mathcal{U}}$ (blobs smaller than an open cover $\mathcal{U}$)

$BT_* = BT_*^{\mathcal{U}}$

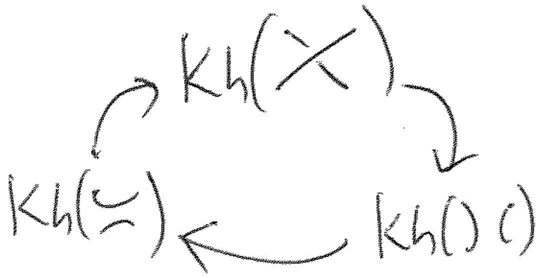
$BT_*(B^n)$ is contractible (acyclic in positive degrees)

The blob complex reflects triangulated structure in the n-category $\mathcal{C}$ which the TQFT invariant misses.

<u>Example</u>

Khovanov homology associates a vector space to a link in the boundary of the 4-ball.

Resolutions of a crossing are related by an exact triangle

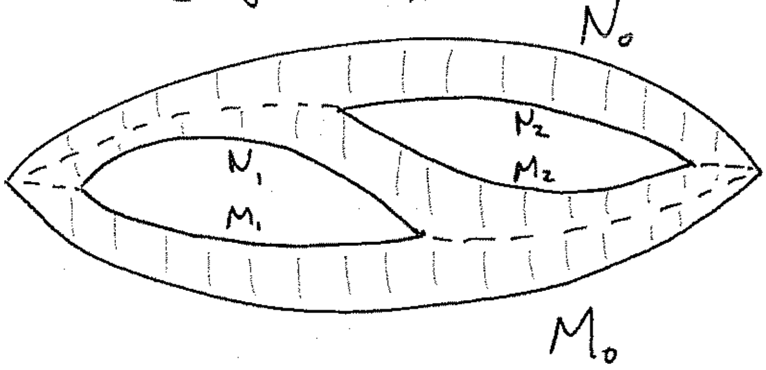


However in a general 4-manifold this structure disappears.

It survives on the blob complex, and gives a spectral sequence perhaps allowing us to compute examples.

Generalized Deligne conjecture

The 'surgery cylinder operad': $SC_{M,N}$



- $\partial M_i = \partial N_i = E_i$
- mapping cylinders between.
- $SC_{M,N}$ has a natural topology.

Write $\hom_i = \mathrm{Hom}_{B_*(E_i)}\big(B_*(M_i) \to B_*(N_i)\big)$.

Theorem There are a collection of ~~map~~ maps

$$C_*(SC_{M,N}) \otimes \bigotimes_i \hom_i \longrightarrow \hom_0$$

giving an action of the operad up to coherent homotopy.

Specialising to $n=1$, $N_i = M_i = I$, this gives the Deligne conjecture: the little discs operad acts on Hochschild cochains.

(Even at $n=1$ there more here:

$(S^1 \to$ ⓐⓑ $\to S^1)$ gives a map

$$\mathrm{Hom}_{B_*(\cdots\cdots)}\big(B_*(\smile\smile\smile\smile) \to B_*(\frown\frown)\big)$$
$$\downarrow$$
$$\mathrm{End}(HC_*) \qquad\qquad)$$

To state the next theorems, we first need the notion of 'A$_\infty$ disklike n-categories'

as before, but

- $\mathcal{C}_n(X;c)$ is a chain complex, not a vector space
- the action of diffeomorphisms of balls lifts to

$$\mathcal{C}_*(\mathrm{Diff}(X^n)) \otimes \mathcal{C}_n(X) \to \mathcal{C}_n(X)$$

<u>Theorem</u> With M a k-manifold, $\mathcal{C}$ a disklike n-category, the association $X^{n-k} \longmapsto B_*(M \times X; \mathcal{C})$ defines an A$_\infty$ disklike $(n-k)$ category, which we call $B_*(M)$

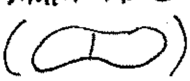
(or $\underset{\to}{\mathcal{C}}(M)$, depending on what we want to emphasize)

<u>Theorem</u> If $M^{n-1} \subset \partial N^n$, $B_*(N)$ is a module over $B_*(M)$.

<u>Theorem</u> If $N = N_1 \underset{M}{\cup} N_2$

$$B_*(N) \underset{q.i.}{=} B_*(N_1) \overset{A_\infty}{\underset{B_*(M)}{\otimes}} B_*(N_2)$$

Sketch: prove a much more general fibre product formula

$B_*($  $)$

$\shortparallel$

$B_*($  $)$

k-cells labelled by order $(n-k)$ modules

Prove this using 'small blobs', acyclic models, and a somewhat technical argument!

<u>Theorem</u> Define the fundamental (∞,n)-groupoid of T: $\pi_\infty^{\leq n}(T)(X^n) = C_*(\mathrm{Maps}(X \to T))$

Then $$B_*(S; \pi_\infty^{\leq n}(T)) \underset{q.i.}{=} C_*\mathrm{Maps}(S \to T).$$

<u>Corollary</u> $\mathrm{Hoch}_*(C_*\Omega T) \simeq \mathrm{Hoch}_*(\pi_{\leq 1}^{\infty}(T)) \simeq B_*(S^1; \pi_{\leq}^{\infty}(T)) \simeq C_*(LT)$