The blob complex

Scott Morrison
http://tqft.net/
joint work with Kevin Walker

UC Berkeley / Miller Institute for Basic Research

Low-Dimensional Topology and Categorification,
Stony Brook University, June 21-25 2010

slides: http://tqft.net/sunysb-blobs

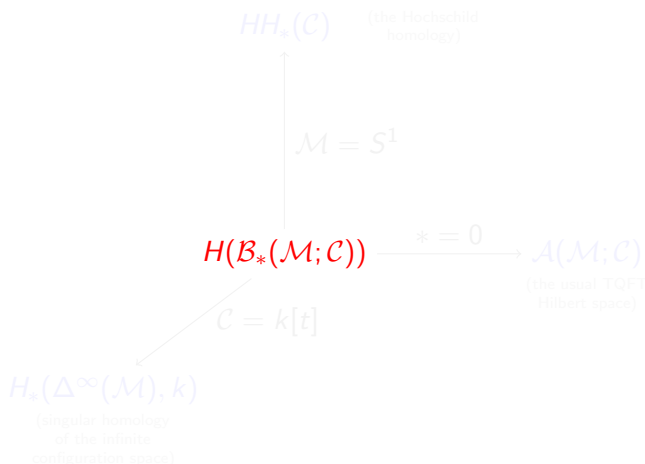paper: http://tqft.net/blobs

The blob complex

... homotopical topology and TQFT have grown so close that I have started thinking that they are turning into the language of new foundations.

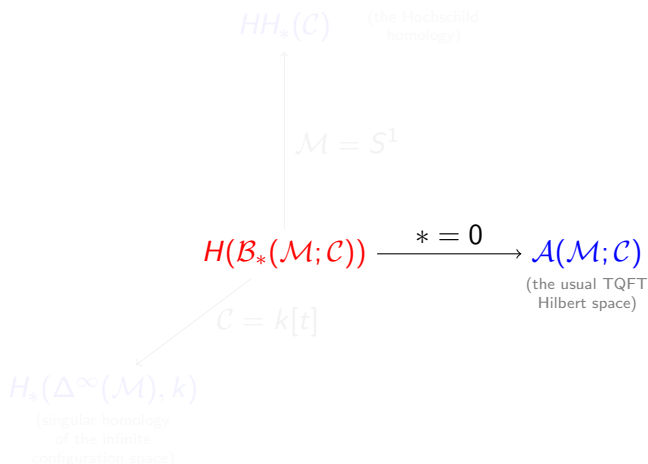— Yuri Manin, September 2008

1 Overview

2 Definition

3 Properties

What is *the blob complex*?

The blob complex takes an n-manifold $\mathcal{M}$ and an 'n-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$.

$HH_*(\mathcal{C})$ (the Hochschild homology)

$\mathcal{M} = S^1$

$H(\mathcal{B}_*(\mathcal{M}; \mathcal{C}))$ $\xrightarrow{* = 0}$ $\mathcal{A}(\mathcal{M}; \mathcal{C})$ (the usual TQFT Hilbert space)

$\mathcal{C} = k[t]$

$H_*(\Delta^\infty(\mathcal{M}), k)$ (singular homology of the infinite configuration space)

What is *the blob complex*?

The blob complex takes an *n*-manifold $\mathcal{M}$ and an '*n*-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$.
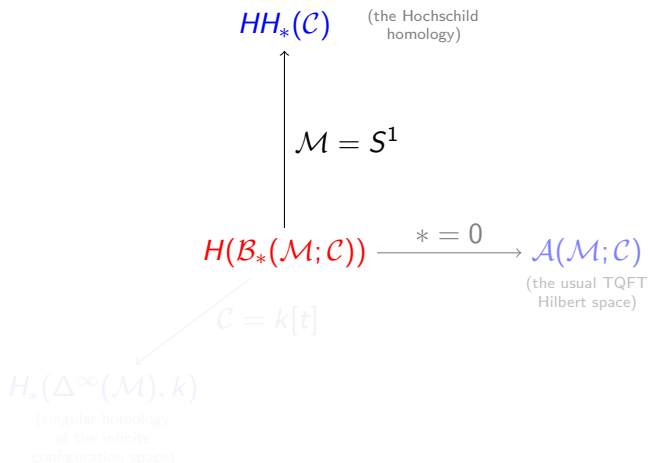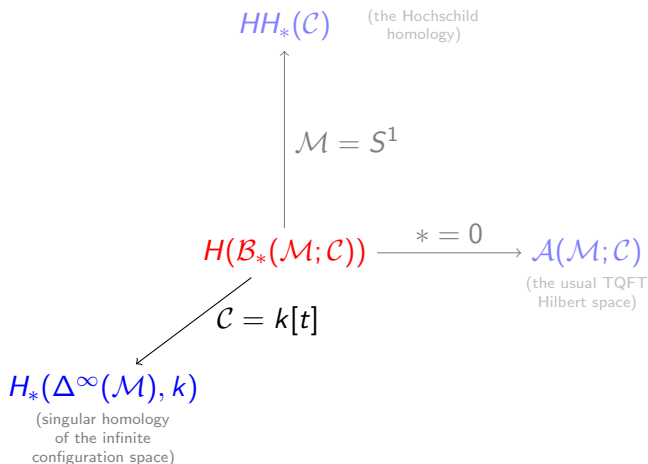
$HH_*(\mathcal{C})$ (the Hochschild homology)

$\mathcal{M} = S^1$

$$H(\mathcal{B}_*(\mathcal{M}; \mathcal{C})) \xrightarrow{\quad * = 0 \quad} \mathcal{A}(\mathcal{M}; \mathcal{C})$$

(the usual TQFT Hilbert space)

$\mathcal{C} = k[t]$

$H_*(\Delta^\infty(\mathcal{M}), k)$

(singular homology of the infinite configuration space)

What is *the blob complex*?

The blob complex takes an *n*-manifold $\mathcal{M}$ and an '*n*-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$.

$HH_*(\mathcal{C})$ (the Hochschild homology)

$\mathcal{M} = S^1$

$H(\mathcal{B}_*(\mathcal{M}; \mathcal{C})) \xrightarrow{\ *\, =\, 0\ } \mathcal{A}(\mathcal{M}; \mathcal{C})$

(the usual TQFT Hilbert space)

$\mathcal{C} = k[t]$

$H_*(\Delta^\infty(\mathcal{M}), k)$

(singular homology of the infinite configuration space)

What is *the blob complex*?

The blob complex takes an *n*-manifold $\mathcal{M}$ and an '*n*-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$.

$HH_*(\mathcal{C})$ (the Hochschild homology)

$\mathcal{M} = S^1$

$H(\mathcal{B}_*(\mathcal{M}; \mathcal{C})) \xrightarrow{\quad * = 0 \quad} \mathcal{A}(\mathcal{M}; \mathcal{C})$

(the usual TQFT Hilbert space)

$\mathcal{C} = k[t]$

$H_*(\Delta^\infty(\mathcal{M}), k)$

(singular homology of the infinite configuration space)

n-categories

There are many definitions of n-categories!

For most of what follows, I'll draw 2-dimensional pictures and rely on your intuition for pivotal 2-categories.

We have another definition!

Many axioms; geometric examples are easy, algebraic ones hard.

- A vector space $\mathcal{C}(B^n)$ for every n-ball B.
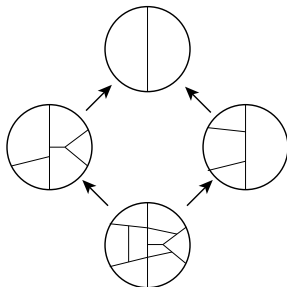- An associative gluing map: with $B = \bigcup_i B_i$, balls glued together to form a ball,

$$\bigotimes \mathcal{C}(B_i) \to \mathcal{C}(B)$$

(the $\otimes$ is fibered over 'boundary restriction' maps).

- ...

Cellulations of manifolds

Consider cell(M), the category of cellulations of a manifold M, with morphisms 'antirefinements'.

An n-category $\mathcal{C}$ gives a functor from cell(M) to vector spaces.

objects send a cellulation to the product of $\mathcal{C}$ on each top-cell, restricting to the subset where boundaries agree

morphisms send an antirefinement to the appropriate gluing map.

Fields

A field on $\mathcal{M}^n$ is a choice of cellulation and a choice of n-morphism for each top-cell.

Example ($\mathcal{C} = \mathsf{TL}_d$ the Temperley-Lieb category)

 $\in \mathcal{F}\left(T^2\right)$
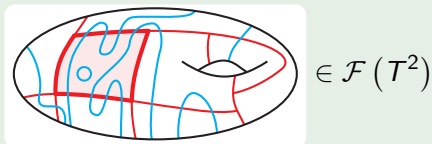
Given a field on a ball, we can evaluate it to a morphism using the gluing map. We call the kernel the *null fields*.

$$\mathrm{ev}\left(\vcenter{\hbox{}} - \frac{1}{d} \vcenter{\hbox{}} \right) = 0$$

Background: TQFT invariants

Definition

We associate to an n-manifold $\mathcal{M}$ the skein module

$$\mathcal{A}(\mathcal{M}) = \mathcal{F}(\mathcal{M})/\ker ev,$$

fields modulo fields which evaluate to zero inside some ball.

Equivalently, $\mathcal{A}(\mathcal{M})$ is the colimit of $\mathcal{C}$ along cell(M).

$\mathcal{A}(Y \times [0, 1])$ is a 1-category, and when $Y \subset \partial X$, $\mathcal{A}(X)$ is a module over $\mathcal{A}(Y \times [0, 1])$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{A}(X \bigcup_Y \circlearrowleft) \cong \mathcal{A}(X) \bigotimes_{\mathcal{A}(Y \times [0,1])} \circlearrowleft.$$

Motivation: Khovanov homology as a 4d TQFT

Theorem

Khovanov homology gives a 4-category:

3-morphisms *tangles, with the usual 3 operations,*

4-morphisms $\mathrm{Hom}_{Kh}(T_1, T_2) = Kh(T_1 \cup \bar{T}_2)$, *composition defined by saddle cobordisms*

There is a corresponding 4-manifold invariant. Given $L \subset \partial W^4$, it associates a doubly-graded vector space $\mathcal{A}(W, L; Kh)$.

$$\mathcal{A}(B^4, L; Kh) \cong Kh(L)$$

Computations are hard

This invariant is hard to compute, because the TQFT skein module construction breaks the exact triangle for resolving a crossing.

$$Kh\left(\diagup\!\!\!\!\diagdown\right)$$

$$Kh\left(\,\rangle\langle\,\right) \longrightarrow Kh\left(\asymp\right)$$

$$\mathcal{A}\left(M, \diagup\!\!\!\!\diagdown\right)$$

$$\mathcal{A}\left(M, \,\rangle\langle\,\right) \;\text{---}\; \mathcal{A}\left(M, \asymp\right)$$

?

There is a spectral sequence converging to 0 relating the blob homologies for the triangle of resolutions.

Conjecture

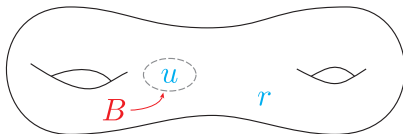*It may be possible to compute the skein module by first computing the entire blob homology.*

Definition of the blob complex, $k = 0, 1$

Motivation

A *local* construction, such that when $\mathcal{M}$ is a ball, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$ is a resolution of $\mathcal{A}(\mathcal{M}; \mathcal{C})$.

$$\mathcal{B}_0(\mathcal{M}; \mathcal{C}) = \mathcal{F}(\mathcal{M}), \text{ arbitrary fields on } \mathcal{M}.$$

$$\mathcal{B}_1(\mathcal{M}; \mathcal{C}) = \mathbb{C} \left\{ (B, u, r) \; \middle| \; \begin{array}{c} B \text{ an embedded ball} \\ u \in \mathcal{F}(B) \text{ in the kernel} \\ r \in \mathcal{F}(\mathcal{M} \setminus B) \end{array} \right\}.$$

$$d_1 : (B, u, r) \mapsto u \circ r \qquad \mathcal{B}_0 / \operatorname{im}(d_1) \cong A(\mathcal{M}; \mathcal{C})$$

Definition, $k = 2$

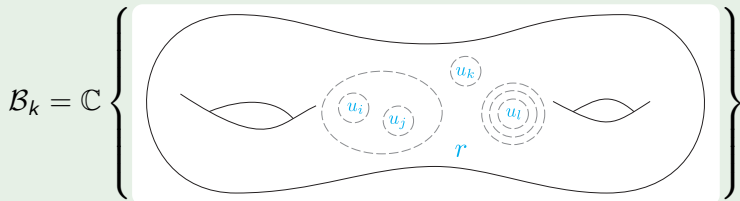$$\mathcal{B}_2 = \mathcal{B}_2^{\mathsf{disjoint}} \oplus \mathcal{B}_2^{\mathsf{nested}}$$

$$\mathcal{B}_2^{\mathsf{disjoint}} = \mathbb{C} \left\{ \quad \middle| \quad \mathsf{ev}_{B_i}(u_i) = 0 \right\}$$

$$d_2 : (B_1, B_2, u_1, u_2, r) \mapsto (B_2, u_2, r \circ u_1) - (B_1, u_1, r \circ u_2)$$

$$\mathcal{B}_2^{\mathsf{nested}} = \mathbb{C} \left\{ \quad \middle| \quad \mathsf{ev}_{B_1}(u) = 0 \right\}$$

$$d_2 : (B_1, B_2, u, r', r) \mapsto (B_2, u \circ r', r) - (B_1, u, r \circ r')$$

Definition, general case

$$\mathcal{B}_k = \mathbb{C}\left\{ \begin{array}{c} \end{array} \right\}$$

k blobs, properly nested or disjoint, with "innermost" blobs labelled by fields that evaluate to zero.

$$d_k : \mathcal{B}_k \to \mathcal{B}_{k-1} = \sum_i (-1)^i (\text{erase blob } i)$$
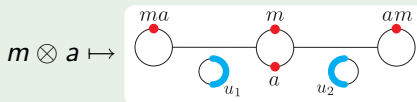
Hochschild homology

TQFT on S^1 is 'coinvariants'

$$\mathcal{A}(S^1, A) = \mathbb{C}\left\{ \raisebox{-1em}{} \right\} \Big/ \left\{ \raisebox{-0.5em}{} - \raisebox{-0.5em}{} \right\} = A/(ab - ba)$$

The Hochschild complex is 'coinvariants of the bar resolution'

$$\cdots \to A \otimes A \otimes A \to A \otimes A \xrightarrow{m \otimes a \mapsto ma - am} A$$

Theorem (Hoch$_*(A) \cong \mathcal{B}_*(S^1; A)$)

$$m \otimes a \mapsto \raisebox{-2em}{}$$

$$u_1 = \raisebox{-0.5em}{} - \raisebox{-0.5em}{} \qquad u_2 = \raisebox{-0.5em}{} - \raisebox{-0.5em}{}$$

An action of $C_*(\text{Homeo}(\mathcal{M}))$

Theorem

There's a chain map

$$C_*(\text{Homeo}(\mathcal{M})) \otimes \mathcal{B}_*(\mathcal{M}) \to \mathcal{B}_*(\mathcal{M}).$$

which is associative up to homotopy, and compatible with gluing.

Taking H_0, this is the mapping class group acting on a TQFT skein module.

An action of $C_*(\mathrm{Homeo}(\mathcal{M}))$

Proof.

Step 1 If $\mathcal{M} = B^n$ or a union of balls, there's a unique chain map, since $\mathcal{B}_*(B^n; \mathcal{C}) \simeq \mathcal{C}$ is concentrated in homological degree 0.

Step 2 Fix an open cover $\mathcal{U}$ of balls.
A family of homeomorphisms $P^k \to \mathrm{Homeo}(\mathcal{M})$ can be broken up in into pieces, each of which is supported in at most k open sets from $\mathcal{U}$. $\qquad \square$

Gluing

> $\mathcal{B}_*(Y \times [0,1])$ is naturally an A_∞ category
>
> multiplication (m_2): gluing $[0,1] \simeq [0,1] \cup [0,1]$
> associativity up to homotopy (m_k): reparametrising $[0,1]$ using the
> action of $C_*(\text{Homeo}([0,1]))$.

> If $Y \subset \partial X$ then $\mathcal{B}_*(X)$ is an A_∞ module over $\mathcal{B}_*(Y)$.

Theorem (Gluing formula)

> When $Y \sqcup Y^{op} \subset \partial X$,
> $$\mathcal{B}_*(X \bigcup_Y \bigcirc\!\!\curvearrowleft) \cong \mathcal{B}_*(X) \overset{A_\infty}{\underset{\mathcal{B}_*(Y)}{\bigotimes}} \bigcirc\!\!\curvearrowleft.$$

In principle, we can compute blob homology from a handle
decomposition, by iterated Hochschild homology.
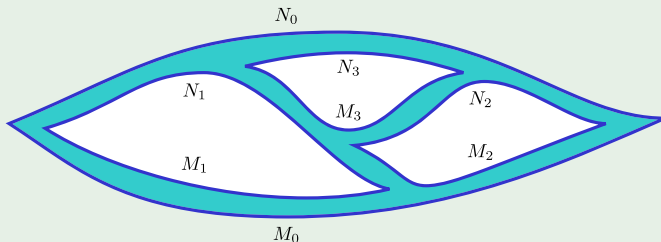
Higher Deligne conjecture

Deligne conjecture

Chains on the little discs operad acts on Hochschild cohomology.

Call $\text{Hom}_{\mathcal{B}_*(\partial M)}(\mathcal{B}_*(\mathcal{M}), \mathcal{B}_*(\mathcal{M}))$ 'blob cochains on $\mathcal{M}$'.

Theorem (Higher Deligne conjecture)

Chains on the *n*-dimensional fat graph operad acts on blob cochains.

Maps to a space

Fix a target space $\mathcal{T}$. There is an A_∞ n-category $\pi_{\leq n}^\infty(\mathcal{T})$ defined by

$$\pi_{\leq n}^\infty(\mathcal{T})(B) = C_*(\mathrm{Maps}(B \to \mathcal{T})).$$

Theorem

The blob complex recovers mapping spaces:

$$\mathcal{B}_*(\mathcal{M}; \pi_{\leq n}^\infty(\mathcal{T})) \cong C_*(\mathrm{Maps}(\mathcal{M} \to \mathcal{T}))$$

This generalizes a result of Lurie: if $\mathcal{T}$ is $n - 1$ connected, $\pi_{\leq n}^\infty(\mathcal{T})$ is an E_n-algebra and the blob complex is the same as his topological chiral homology.