The blob complex

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paper: http://tqft.net/blobs

The blob complex

... homotopical topology and TQFT have grown so close that I have started thinking that they are turning into the language of new foundations.

— Yuri Manin, September 2008

What is *the blob complex*?

The blob complex takes an n-manifold $\mathcal{M}$ and an 'n-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$.

$HH_*(\mathcal{C})$ (the Hochschild homology)

$\mathcal{M} = S^1$

$H(\mathcal{B}_*(\mathcal{M};\mathcal{C})) \xrightarrow{*=0} \mathcal{A}(\mathcal{M};\mathcal{C})$ (the usual TQFT Hilbert space)

$\mathcal{C} = k[t]$

$H_*(\Delta^\infty(\mathcal{M}), k)$ (singular homology of the infinite configuration space)

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Motivation: Khovanov homology as a 4d TQFT

Theorem

Khovanov homology gives a 4-category:

3-morphisms *tangles, with the usual 3 operations,*

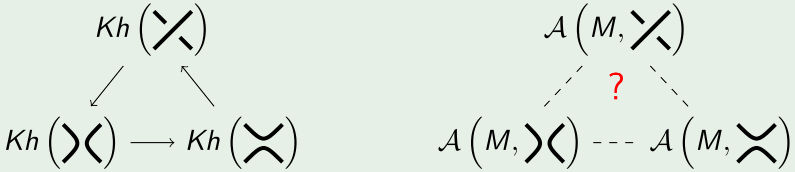
4-morphisms $\mathrm{Hom}_{Kh}(T_1, T_2) = Kh(T_1 \cup \bar{T}_2)$, *composition defined by saddle cobordisms*

There is a corresponding 4-manifold invariant. Given $L \subset \partial W^4$, it associates a doubly-graded vector space $\mathcal{A}(W, L; Kh)$.

$$\mathcal{A}(B^4, L; Kh) \cong Kh(L)$$

Computations are hard

The corresponding 4-manifold invariant is hard to compute, because the TQFT skein module construction breaks the exact triangle for resolving a crossing.



There is a spectral sequence converging to 0 relating the blob homologies for the triangle of resolutions.

Conjecture

It may be possible to compute the skein module by first computing the entire blob homology.

n-categories

Defining n-categories is fraught with difficulties

For now, I'm not going to go into details; I'll draw 2-dimensional pictures, and rely on your intuition for pivotal 2-categories.

Later, I'll explain the notions of 'topological n-categories' and 'A_∞ n-categories'.

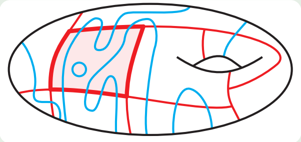
- Defining n-categories: a choice of 'shape' for morphisms.
- We allow all shapes! A vector space for every ball.
- 'Strong duality' is integral in our definition.

Fields and pasting diagrams

Pasting diagrams

Fix an n-category with strong duality $\mathcal{C}$. A *field* on $\mathcal{M}$ is a pasting diagram drawn on $\mathcal{M}$, with cells labelled by morphisms from $\mathcal{C}$.

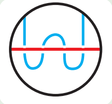
Example ($\mathcal{C} = \mathrm{TL}_d$ the Temperley-Lieb category)

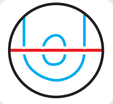


$\in \mathcal{F}\left(T^2\right)$

Given a pasting diagram on a ball, we can evaluate it to a morphism. We call the kernel the *null fields*.

$$\mathrm{ev}\left(\text{[diagram 1]} - \frac{1}{d} \text{[diagram 2]} \right) = 0$$





Background: TQFT invariants

Definition

A decapitated $n+1$-dimensional TQFT associates a vector space $\mathcal{A}(\mathcal{M})$ to each n-manifold $\mathcal{M}$.

('decapitated': no numerical invariants of $n+1$-manifolds.)

If the manifold has boundary, we get a category. Objects are boundary data, $\mathrm{Hom}_{\mathcal{A}(\mathcal{M})}(x, y) = \mathcal{A}(\mathcal{M}; x, y)$.

We want to extend 'all the way down'. The k-category associated to the $n-k$-manifold $\mathcal{Y}$ is $\mathcal{A}(\mathcal{Y} \times B^k)$.

Definition

Given an n-category $\mathcal{C}$, the associated TQFT is

$$\mathcal{A}(\mathcal{M}) = \mathcal{F}(M)/\ker ev,$$

fields modulo fields which evaluate to zero inside some ball.

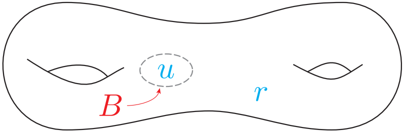
Definition of the blob complex, $k = 0, 1$

Motivation

A *local* construction, such that when $\mathcal{M}$ is a ball, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$ is a resolution of $\mathcal{A}(\mathcal{M},;\mathcal{C})$.

$\mathcal{B}_0(\mathcal{M};\mathcal{C}) = \mathcal{F}(\mathcal{M})$, arbitrary pasting diagrams on $\mathcal{M}$.

$$\mathcal{B}_1(\mathcal{M};\mathcal{C}) = \mathbb{C}\left\{ (B, u, r) \;\middle|\; \begin{array}{c} B \text{ an embedded ball} \\ u \in \mathcal{F}(B) \text{ in the kernel} \\ r \in \mathcal{F}(\mathcal{M} \setminus B) \end{array} \right\}.$$

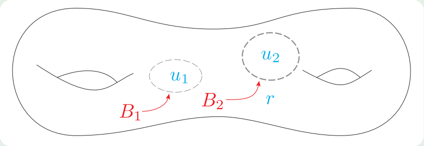


$d_1 : (B, u, r) \mapsto u \circ r$ $\qquad$ $\mathcal{B}_0 / \operatorname{im}(d_1) \cong A(\mathcal{M};\mathcal{C})$

Definition, $k = 2$

$$\mathcal{B}_2 = \mathcal{B}_2^{\text{disjoint}} \oplus \mathcal{B}_2^{\text{nested}}$$

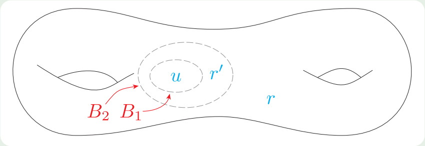
$\mathcal{B}_2^{\text{disjoint}} = \mathbb{C} \Bigg\{$



$\Bigg| \; \mathrm{ev}_{B_i}(u_i) = 0 \Bigg\}$

$$d_2 : (B_1, B_2, u_1, u_2, r) \mapsto (B_2, u_2, r \circ u_1) - (B_1, u_1, r \circ u_2)$$

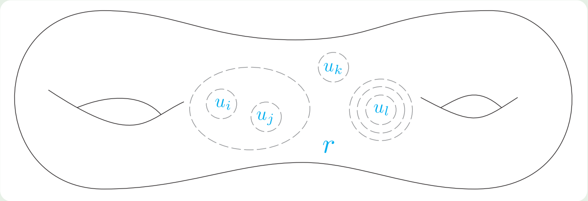
$\mathcal{B}_2^{\text{nested}} = \mathbb{C} \Bigg\{$



$\Bigg| \; \mathrm{ev}_{B_1}(u) = 0 \Bigg\}$

$$d_2 : (B_1, B_2, u, r', r) \mapsto (B_2, u \circ r', r) - (B_1, u, r \circ r')$$

Definition, general case

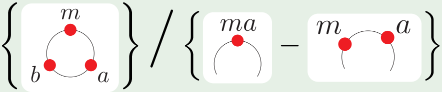
$\mathcal{B}_k = \mathbb{C} \Bigg\{$  $\Bigg\}$

k blobs, properly nested or disjoint, with "innermost" blobs labelled by pasting diagrams that evaluate to zero.

$$d_k : \mathcal{B}_k \to \mathcal{B}_{k-1} = \sum_i (-1)^i (\text{erase blob } i)$$

Hochschild homology

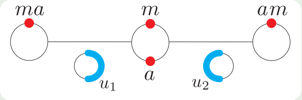
TQFT on S^1 is 'coinvariants'

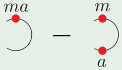
$\mathcal{A}(S^1, A) = \mathbb{C}$  $= A/(ab - ba)$

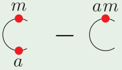
The Hochschild complex is 'coinvariants of the bar resolution'

$$\cdots \to A \otimes A \otimes A \to A \otimes A \xrightarrow{m \otimes a \mapsto ma - am} A$$

Theorem ($\mathrm{Hoch}_*(A) \cong \mathcal{B}_*(S^1; A)$)

$m \otimes a \mapsto$ 

$u_1 =$ 

$u_2 =$ 

An action of $C_*(\mathrm{Homeo}(\mathcal{M}))$

Theorem

There's a chain map

$$C_*(\mathrm{Homeo}(\mathcal{M})) \otimes \mathcal{B}_*(\mathcal{M}) \to \mathcal{B}_*(\mathcal{M}).$$

which is associative up to homotopy, and compatible with gluing.

Taking H_0, this is the mapping class group acting on a TQFT skein module.

Gluing

$\mathcal{B}_*(Y \times [0,1])$ is naturally an A_∞ category

m_2: gluing $[0,1] \simeq [0,1] \cup [0,1]$

m_k: reparametrising $[0,1]$

If $Y \subset \partial X$ then $\mathcal{B}_*(X)$ is an A_∞ module over $\mathcal{B}_*(Y)$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{B}_*\Big(X \bigcup_Y \circlearrowleft\Big) \cong \mathcal{B}_*(X) \bigotimes_{\mathcal{B}_*(Y)}^{A_\infty} \circlearrowleft.$$

In principle, we can compute blob homology from a handle decomposition, by iterated Hochschild homology.

Higher Deligne conjecture

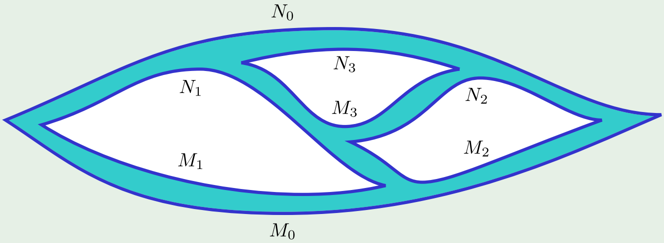
Deligne conjecture

Chains on the little discs operad acts on Hochschild cohomology.

Call $\mathrm{Hom}_{\mathcal{B}_*(\partial M)}(\mathcal{B}_*(\mathcal{M}), \mathcal{B}_*(\mathcal{M}))$ 'blob cochains on $\mathcal{M}$'.

Theorem (Higher Deligne conjecture)

Chains on the n-dimensional fat graph operad acts on blob cochains.



Maps to a space

Fix a target space T. There is an A_∞ n-category $\pi^\infty_{\leq n}(T)$ defined by

$$\pi^\infty_{\leq n}(T)(B) = C_*(\mathrm{Maps}(B \to T)).$$

Theorem

The blob complex recovers mapping spaces:

$$\mathcal{B}_*(M; \pi^\infty_{\leq n}(T)) \cong C_*(\mathrm{Maps}(M \to T))$$

This generalizes a result of Lurie: if T is $n-1$ connected, $\pi^\infty_{\leq n}(T)$ is an E_n-algebra and the blob complex is the same as his topological chiral homology.