REFEREE REPORT FOR "THE BLOB COMPLEX" BY SCOTT MORRISON AND KEVIN WALKER

The paper "The Blob Complex" by Scott Morrison and Kevin Walker defines a chain complex, the so called "blob complex" to the initial data of an n-category. Here, the authors define their own version of an n-category, which, in the text, is compared to more traditional definitions of n-categories, and moreover is closely related and motivated by the notion of TQFTs coming from system of fields and local relations. One of the main theorems of the paper states that in the case of a 1-category, the blob complex associated to this 1-category is homotopy equivalent to its Hochschild chain complex. Furthermore, it is shown how to recover the chains on the mapping space of a space as the blob complex of a certain n-category associated to the base space. Various other properties of the blob complex are studied. In particular an action of the singular chains of the space of homeomorphisms between n-manifolds on the blob complex is given. As an application this is used later to describe a higher dimensional version of the Deligne conjecture.

I think the paper is both substantial and significant. It touches on a variety of subjects both in pure and physically motivated mathematics and relates them in a non-trivial way. The topics include a wide range from n-categories to TQFTs to the Deligne conjecture, with, as the authors state in the introduction, possible future applications to the Khovanov homology of a link. Moreover, the definitions in the paper are broad enough to allow for an additional variation of the setup, placing it certainly in the range of papers published in Geometry & Topology. To further improve the paper, I have attached a list of specific comments and other minor typos below, which should be addressed before publication.

Overall, this is a very interesting paper, and I believe that Geometry & Topology would be a suitable journal for it.

Mathematical points and recommendations on the presentation of the paper:

1. ~~page 3, lines 8-9: Remove the comment "(Don't worry...)"~~
2. ~~p.4, Figure 1: This is a nice overview of the notions used in the paper. In fact, I recommend to further expand on the references of the arrows in the paper: the first diagonal is from Appendix C with inverse arrow in Section 2.2. Other references for the arrows are Definition 2.4.1 ("$\mathcal{F}(M)/U$"), Section 3.1 ("blob complex for M"), Example 6.2.8 ("blob complex for balls"),~~ Lemma from p.48 ("restrict to balls", see item 13 below). ~~The top right vertex should read "$A(M)$" instead of "$\underrightarrow{\mathcal{C}}(M)$".~~
3. ~~p.6, Property 1.3.3: The way it is stated would also be satisfied by the trivial map $0 : \mathcal{B}_*(X) \to \mathcal{B}_*(X_{gl})$. State that this map is non trivial, e.g. by identifying it via the explicit formula for $\mathcal{B}_*(X_{gl})$ from Theorem 7.2.1.~~

   I just added "injective"; do we need/want more?

4. p.9, Section 1.7, and p.12, Section 2.2: Provide some references that contain precise definitions for the type of traditional n-categories you are considering in the paper (including pivotal 2-categories, and *-1-categories).

   SM?

5. ~~Section 1: The use of the word n-category both for disk-like n-categories and traditional n-categories is sometimes confusing in the introduction. To~~

minimize confusion, add the corresponding qualifier throughout Sections 1.5-1.7. Also in Figure 1, change "topological n-category" to "disk-like n-category".

6. p.15, lines 8-21: Add references to later sections where these constructions (of assigning a k-category $A(Y)$ to the $n-k$-manifold Y, and a representation of $A(\partial X)$ on $A(X)$) are carried out in a more general setting (e.g. Example 6.2.8, Section 6.4, etc.).
7. p.21, Proof of Property 1.3.2: In degree zero it is $\mathcal{B}_0(X)\otimes\mathcal{B}_0(Y)=\mathcal{F}(X)\otimes\mathcal{F}(Y)$ and $\mathcal{B}_0(X\sqcup Y)=\mathcal{F}(X\sqcup Y)\cong\mathcal{F}(X)\times\mathcal{F}(Y)$ (by p.11 first two lines), while the tensor product is not the coproduct (nor is it the product, or else how do you project $\mathcal{C}(X_1\sqcup X_2)\to\mathcal{C}(X_i)$ on the third line of p.11?). Please comment on this point.
8. Section 4.1: It is my understanding from Section 2.2 that the notation for a traditional 1-category is C whereas its associated system of fields is denoted by $\mathcal{C}$. Unfortunately, this distinction is not carried through Section 4.1, making it difficult to follow the exact statements of this section. I recommend to keep the distinction throughout Section 4.1. (Also, the same comment applies to the statement of Theorem 4.1.1 on p. 6 and to p. 77.) Moreover, since the proof of Proposition 4.1.3 relies on abstract properties for modules, some further comments on the relationship between modules over the traditional category C and the concept of modules over $\mathcal{C}$ from Section 6.4 should be added.
9. p.33, and Section 5.2: The notation $CH_*(X,Y)$ for singular chains on the space of homeomorphisms between X and Y with fixed boundary map $\partial X\to\partial Y$ is confusing, since CH_* is usually used as a standard notation for the Hochschild chain complex. I recommend to stick with the notation $C_*(\mathrm{Homeo}(X\to Y))$ throughout the paper (or even better: $C_*(\mathrm{Homeo}_\partial(X\to Y)))$, which has also been used on p. 7 and on p. 47.
10. p.33, l.20: Instead of "define the support ... to be" say "recall from p.20 that the support ... is".
11. p.42, l.1, l.5, and 2nd line from bottom: Describe the restriction maps ($\underrightarrow{\mathcal{C}}(S)_E\to\mathcal{C}(B_i)$, $\mathcal{C}(X)_E\to\mathcal{C}(B_i)$, and $\mathcal{C}(X)_E\to\mathcal{C}(B_i)_E$) in more detail as compositions of given data. For example $\underrightarrow{\mathcal{C}}(S)_E\xrightarrow{gl_E^{-1}}\mathcal{C}(B_1)\times_E\mathcal{C}(B_2)\xrightarrow{pr_i}\mathcal{C}(B_i)$, etc.
12. p.47, 5th line from bottom: Since there are two alternative constructions of the blob complex in Section 5.1, specify "the alternative construction $\mathcal{BT}_*(X)$ of the blob complex".
13. p.48, l.10-15: The passage from fields and local relations to n-categories is important and should be stated (at least) as a Lemma.
14. Lower half of p.48: One major difficulty in a first reading of the paper is multiple referrals to constructions appearing in later sections. In Section 2, details of systems of fields are referred to Section 6. Now in Section 6.1, the definition of an n-category relies in Lemma 6.1.2 on the colimit construction in Section 6.3. Furthermore, the colimit construction is implicitly used in all the following axioms 6.1.3, 6.1.5, 6.1.8, and 6.1.11, while at the same time the colimit construction relies on axioms 6.1.3 and 6.1.5. Thus, to make the overview on p. 48 concerning the data of an n-category even

more useful, and to describe the whole data in one place, I recommend to repeat and expand on the comment given right before Lemma 6.1.2, that describes the inductive nature of the definition by its dimension.

15. p.50, Example 6.2.7: There is a mixup with a generalization of Example 6.2.1 or 6.2.2. Include or remove F consistently.
16. p.50: After Example 6.2.8, also recall the example of n-categories from fields and local relations from p.48, i.e. see item 13 above.
17. p.51, l.12-13: Give some details on how to go from (certain) disk like A_∞ n-category to $\mathcal{EB}_n$-algebras.
18. p.52, 5th line from bottom: When saying "as described above", add a reference to the multi-compositions from p.43.
19. p.56, Lemma 6.4.4: The figure to have in mind is slightly different than that of Figure 12 on p.41. I suggest to provide an extra figure here.
20. XX p.59, Module Axiom 6.4.9: Provide a figure of a pinched product, similar to Figure 16 from p.45.
21. p.65, 9th and 5th line from bottom: Again a picture analogous to Figure 23 would be helpful here.
22. p.67, l.11: Add a sentence here that the $n+1$ category $\mathcal{S} = \mathcal{S}_{\{L_i\},\{z_Y\}}$ depends on the choice of the collection L_i and inner products z_Y described below.
23. p.67, l.26-28: State if you are assuming any compatibility (such as tranversality) between the decoration c and the $n-1$-sphere E.
24. p.67, bottom line. Remove the bar from $\bar{b}$, since this notation is not used on the following page.
25. p.71, l.17-18: Make the definition of the blob complex of an A_∞ n-category into an actual definition to underscore its importance.
26. p.72, l.22: Since you gave an example for K and K' (the x-axis and $y = x^2\sin(1/x)$) also give an example for L (e.g. $y = x^2+1$).
27. p.74: Theorems 7.1.4 and 7.1.5 really require a more detailed discussion of their definitions and proofs, which are only indicated on p.74. I recommend to remove the theorems and make this whole page (up to Section 7.2) into a remark instead.
28. p.75, l.6: State the exact assumptions you are making for Section 7.2 (e.g. that you are assuming an n-dimensional system of fields as in Example 6.2.8).
29. p.76, Remark after Theorem 7.3.1: The remark explains how Theorem 7.3.1 differs from [Lur09, Theorem 3.8.6], where the chains on the mapping space are recovered only under suitable connectivity coniditions. However, the classical result $HH_*(C_*(\Omega X)) \cong H_*(LX)$ by [Goodwillie, *Cyclic homology, derivation, and the free loopspace*, Topology 24, no.2] and [Burghelea, Fiedorowicz, *Cyclic homology and algebraic K-theory of spaces II*, Topology 25, no.3] does not require such assumptions. It would be a great confirmation of Theorem 7.3.1 if it would be possible to recover this result via the identification of the Hochschild homology given in Theorem 4.1.1. Please comment on this.
30. p.79, 5th line from bottom: Say there is "a colored operad structure" instead of "an operad structure", since only homeomorphic manifolds can be composed.

31. p.80, l.13-14: The statement that the fibers correspond to moving D_i's without changing their ordering is not completely accurate, since by the second relation (i.e. Figure 39) the ordering may change when $\pi(D_i)$ and $\pi(D_{i+1})$ are disjoint.
32. p.80, l.17-18: State the assumptions again (e.g. $\mathcal{C}$ is an A_∞ n-category), and add a reference to Section 6.6 for the notion of morphisms of modules. XX
33. p.81, End of proof of Theorem 8.0.2: The compatibility of the action of $C_*(Homeo(\cdot \to \cdot))$ with gluing and associativity are only true up to homotopy (see Theorems 5.2.1 and 5.2.2). How is this reconciled with the operad compatibility that should not be up to homotopy but on the nose?
34. p.81: In the beginning of Appendix A, state where this appendix will be used (Lemma 3.2.3, etc.)
35. p.82: In the beginning of Appendix B, state where this appendix will be used (Lemma 5.1.5, Theorem 7.3.1, etc.)
36. p.85, l.26-27: Remove the comment "(the paper is already too long!)"
37. p.87, last line: Say that A now also denotes the space of morphisms of the A_∞ 1-category A.
38. p.88, 7th line from bottom: Add that J_1 and J_2 are "connected intervals which intersect non-trivially" in the first sentence.

Typos and other minor issues: (all done)

1. p.10, 3rd line from bottom: State what "**Kom**" is.
2. p.13, l.7, and p.14, l.7: Make "**Vect**" bold.
3. p.14, l.10: Fix the numeration, which starts with "Example 2.3.-1". Similarly on p.81f, the first Theorem is 8.0.2, but there is no 8.0.1. Appendix A starts with A.0.3. Appendix B starts with B.0.4.
4. p.14, l.15: Add "of" to get "a collection of subspaces".
5. p.14, Definition 2.3.1 item 2: The property states that extended isotopy implies local relations, and not the other way around.
6. p.14, Definition 2.3.1 item 3: Replace "$c \in \mathcal{C}(B'')$" by "$r \in \mathcal{C}(B'')$".
7. p.15, Definition 2.4.1: To stay consistent with the rest of the paper, the local relations should be denoted by U instead of $\mathcal{U}$.
8. p.18, l.13: Add "we" to get "Thus we will need".
9. p.19, l.6 and l.26: Add a comma for "$\{B_1, \dots, B_k\}$".
10. p.22, l.4: Replace B' by b', i.e. "$T(b') \subset T(b)$".
11. p.26, l.15: Fix parenthesis to "$\dots -1 \otimes \left(\sum_i a_i g(\tilde{q}_i) b_i\right) \otimes 1$"
12. p.27, l.8: Remove "$\otimes$" to get "$\dots\ker(\bigoplus_{k,k'} C^{\otimes k} \otimes M \otimes C^{\otimes k'} \to M)$,".
13. p.27, l.10-12: Equation $\mathrm{ev}(\partial y) = \dots = \dots = 0$ is only true after taking π.
14. p.34, l.17: Replace twice "is" by "in" ("contained in").
15. p.37, l.8: Remove "s" for "if there exist $a' \in$".
16. p.38, l.2: Write "...$= s(\partial b) \subset \mathcal{B}_1(X)$" instead of "...$= s(\partial b) \subset \mathcal{B}_2(X)$".
17. p.48, l.7: Add "the" to get "Second, in the category definition".
18. p.51, l.6: Replace "who" by "whose".
19. p.52, 6th line from bottom: Add "the" for "is the value".
20. p.55, l.9: Replace "there" with "exist" to get "There exist decompositions".
21. p.56, Figure 20: Rename "M" by "N" to fit with the text above the figure.
22. p.63, l.10: Change "$\underrightarrow{\mathcal{C}(K)}$" to "$\underrightarrow{\mathcal{C}}(K)$".

23. p.64, 8th line from bottom: Add "a" to get "which is a 0-marked sphere".
24. p.69, 3rd line from bottom: Delete "Let $D' = B \cap C$", since it is a copy error from 4 lines above this.
25. p.70, l.5: Remove "on" for "involves a choice".
26. p.70, l.7: Add "s" to get "it suffices to show".
27. p.70, l.14: Add "s" to get "two movie moves suffice".
28. p.70, last line: Remove "s" for "an additional "global" relation".
29. p.72, 8th line from bottom: Remove one of the "also"s in "We also require".
30. p.74, l.3: Add "d" for "and indeed".
31. p.79, l.6: Make "f_1" italic.
32. p.80, 6th line from bottom: Instead of "we define $p(\overline{f})$ to be" write "we define $p(\overline{f})(\alpha_1 \otimes \cdots \otimes \alpha_k)$ to be".
33. p.81, l.4, l.11, and l.15: Change "$SC^n_{\bar{M},\bar{N}}$" to "$SC^n_{\overline{MN}}$", which is how the space is denoted on the previous pages.
34. p.86, l.11: Remove one "the" in "the the nontrivial".
35. p.86, l.21: Change "the" to "that" to get "in such a way that the generalization".
36. p.88, l.1: Write "$m_2 : A \otimes A \to A$" instead of "$m_2 : A \times A \to A$".
37. "a" versus "an": p.10, l.23: "An n-dimensional", p.16, l.15: "A closed", p.22, l.24: "an n-category", p.47, 2nd line from bottom: "an ordinary", p.50 Example 6.2.8: "an m-ball" and "a k-ball", p.51, l.23: "an ordinary"
38. p. 90, reference [VG95]: Correct the name "Gerstenhaber".