

Blob homology, part II

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slides, part II: <http://tqft.net/UCR-blobs1>

slides, part III: <http://tqft.net/UCR-blobs2>

draft: <http://tqft.net/blobs>

Blob homology

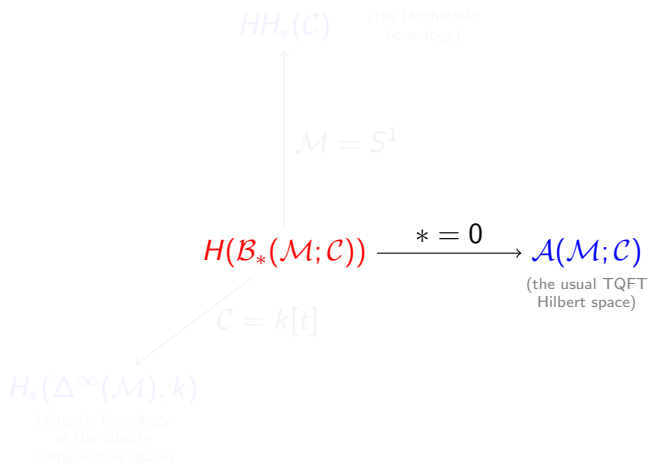
... homotopical topology and TQFT have grown so close that I have started thinking that they are turning into the language of new foundations.

— Yuri Manin, September 2008

- 1 Overview
- 2 Definition
- 3 Properties

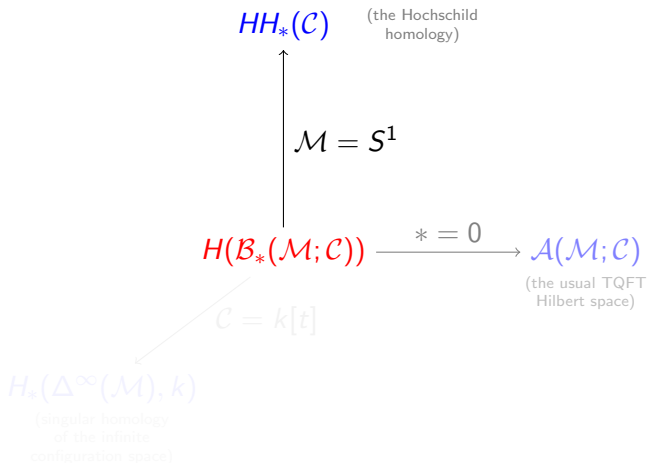
What is *blob homology*?

The blob complex takes an n -manifold \mathcal{M} and an ' n -category with strong duality' \mathcal{C} and produces a chain complex, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$.



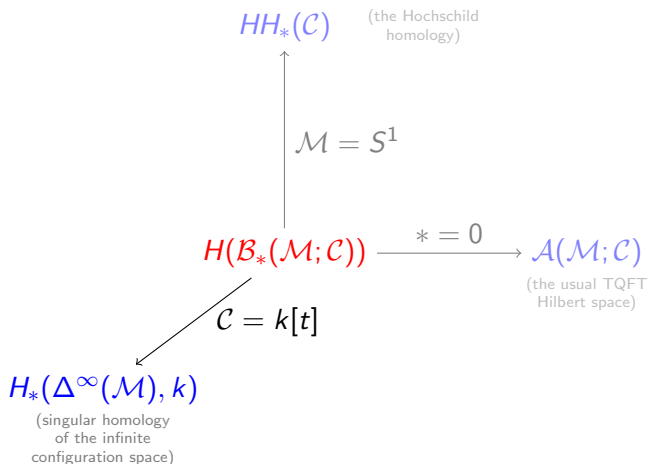
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n -categories

Defining n -categories is fraught with difficulties

I'm not going to go into details; I'll draw 2-dimensional pictures, and rely on your intuition for pivotal 2-categories.

Kevin's talk (part III) will explain the notions of 'topological n -categories' and ' A_∞ n -categories'.

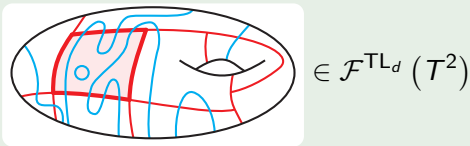
- Defining n -categories: a choice of 'shape' for morphisms.
- We allow all shapes! A vector space for every ball.
- 'Strong duality' is integral in our definition.

Fields and pasting diagrams

Pasting diagrams

Fix an n -category with strong duality \mathcal{C} . A *field* on \mathcal{M} is a pasting diagram drawn on \mathcal{M} , with cells labelled by morphisms from \mathcal{C} .

Example ($\mathcal{C} = \text{TL}_d$ the Temperley-Lieb category)



Given a pasting diagram on a ball, we can evaluate it to a morphism. We call the kernel the *null fields*.

$$\text{ev} \left(\left(\text{Diagram 1} - \frac{1}{d} \text{Diagram 2} \right) \right) = 0$$

Background: TQFT invariants

Definition

A decapitated $n + 1$ -dimensional TQFT associates a vector space $\mathcal{A}(\mathcal{M})$ to each n -manifold \mathcal{M} .

(‘decapitated’: no numerical invariants of $n + 1$ -manifolds.)

If the manifold has boundary, we get a category. Objects are boundary data, $\text{Hom}_{\mathcal{A}(\mathcal{M})}(x, y) = \mathcal{A}(\mathcal{M}; x, y)$.

We want to extend ‘all the way down’. The k -category associated to the $n - k$ -manifold \mathcal{Y} is $\mathcal{A}(\mathcal{Y} \times B^k)$.

Definition

Given an n -category \mathcal{C} , the associated TQFT is

$$\mathcal{A}(\mathcal{M}) = \mathcal{F}(\mathcal{M}) / \ker \text{ev},$$

fields modulo fields which evaluate to zero inside some ball.

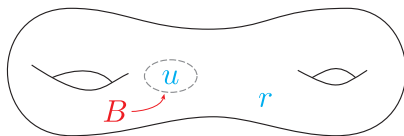
Definition of the blob complex, $k = 0, 1$

Motivation

A *local* construction, such that when \mathcal{M} is a ball, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$ is a resolution of $\mathcal{A}(\mathcal{M}, ; \mathcal{C})$.

$\mathcal{B}_0(\mathcal{M}; \mathcal{C}) = \mathcal{F}(\mathcal{M})$, arbitrary pasting diagrams on \mathcal{M} .

$$\mathcal{B}_1(\mathcal{M}; \mathcal{C}) = \left\{ (B, u, r) \mid \begin{array}{l} B \text{ an embedded ball} \\ u \in \mathcal{F}(B) \text{ in the kernel} \\ r \in \mathcal{F}(\mathcal{M} \setminus B) \end{array} \right\}.$$



$$d_1 : (B, u, r) \mapsto u \circ r \quad \mathcal{B}_0 / \text{im}(d_1) \cong \mathcal{A}(\mathcal{M}; \mathcal{C})$$

Definition, $k = 2$

$$\mathcal{B}_2 = \mathcal{B}_2^{\text{disjoint}} \oplus \mathcal{B}_2^{\text{nested}}$$

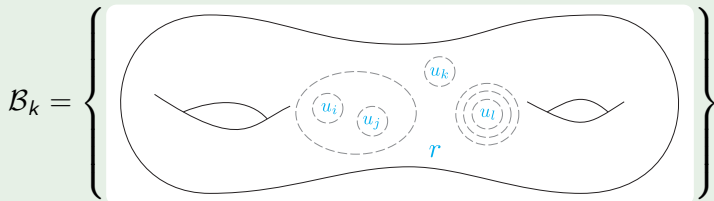
$$\mathcal{B}_2^{\text{disjoint}} = \left\{ \begin{array}{c} \text{Diagram of a genus-2 surface with two disjoint dashed circles } u_1 \text{ and } u_2. \text{ Red arrows } B_1 \text{ and } B_2 \text{ point to } u_1 \text{ and } u_2 \text{ respectively. A red arrow } r \text{ points to } u_2. \end{array} \middle| \text{ev}_{B_i}(u_i) = 0 \right\}$$

$$d_2 : (B_1, B_2, u_1, u_2, r) \mapsto (B_2, u_2, r \circ u_1) - (B_1, u_1, r \circ u_2)$$

$$\mathcal{B}_2^{\text{nested}} = \left\{ \begin{array}{c} \text{Diagram of a genus-2 surface with a nested dashed circle } u \text{ (inner) and } r' \text{ (outer). Red arrows } B_1 \text{ and } B_2 \text{ point to } u \text{ and } r' \text{ respectively. A blue arrow } r \text{ points to } r'. \end{array} \middle| \text{ev}_{B_1}(u) = 0 \right\}$$

$$d_2 : (B_1, B_2, u, r', r) \mapsto (B_2, u \circ r', r) - (B_1, u, r \circ r')$$

Definition, general case



k blobs, properly nested or disjoint, with “innermost” blobs labelled by pasting diagrams that evaluate to zero.

$$d_k : \mathcal{B}_k \rightarrow \mathcal{B}_{k-1} = \sum_i (-1)^i (\text{erase blob } i)$$

Hochschild homology

TQFT on S^1 is 'coinvariants'

$$\mathcal{A}(S^1, A) = \mathbb{C} \left\{ \begin{array}{c} m \\ \circlearrowleft \\ b \quad a \end{array} \right\} / \left\{ \begin{array}{c} ma \\ \circlearrowleft \\ \bullet \end{array} \right\} - \left\{ \begin{array}{c} m \quad a \\ \circlearrowleft \\ \bullet \quad \bullet \end{array} \right\} = A / (ab - ba)$$

The Hochschild complex is 'coinvariants of the bar resolution'

$$\dots \rightarrow A \otimes A \otimes A \rightarrow A \otimes A \xrightarrow{m \otimes a \mapsto ma - am} A$$

Theorem ($\text{Hoch}_*(A) \cong \mathcal{B}_*(S^1; A)$)

$$m \otimes a \mapsto \begin{array}{c} ma \quad m \quad am \\ \circlearrowleft \quad \circlearrowleft \quad \circlearrowleft \\ \bullet \quad \bullet \quad \bullet \\ \circlearrowleft \quad \circlearrowleft \\ u_1 \quad u_2 \end{array}$$

$$u_1 = \begin{array}{c} ma \\ \circlearrowleft \\ \bullet \end{array} - \begin{array}{c} m \\ \circlearrowleft \\ \bullet \\ a \end{array}$$

$$u_2 = \begin{array}{c} m \\ \circlearrowleft \\ \bullet \\ a \end{array} - \begin{array}{c} am \\ \circlearrowleft \\ \bullet \end{array}$$

An action of $C_*(\text{Homeo}(\mathcal{M}))$

Theorem

There's a chain map

$$C_*(\text{Homeo}(\mathcal{M})) \otimes \mathcal{B}_*(\mathcal{M}) \rightarrow \mathcal{B}_*(\mathcal{M}).$$

which is associative up to homotopy, and compatible with gluing.

Taking H_0 , this is the mapping class group acting on a TQFT skein module.

Gluing

$\mathcal{B}_*(Y \times [0, 1])$ is naturally an A_∞ category

m_2 : gluing $[0, 1] \simeq [0, 1] \cup [0, 1]$

m_k : reparametrising $[0, 1]$

If $Y \subset \partial X$ then $\mathcal{B}_*(X)$ is an A_∞ module over $\mathcal{B}_*(Y)$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{B}_*\left(X \bigcup_Y \circlearrowleft\right) \cong \mathcal{B}_*(X) \underset{\mathcal{B}_*(Y)}{\overset{A_\infty}{\otimes}} \circlearrowleft.$$

In principle, we can compute blob homology from a handle decomposition, by iterated Hochschild homology.