A First Definition of the Blob Complex

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0 Background

Recall that given an n-dimensional system of fields and local relations $(\mathcal{F},\mathcal{U})$, we can associated to an n-manifold X its *TQFT invariant*:

$$X \rightsquigarrow A(X) = \mathcal{F}(X)/\mathcal{U}(X).$$

Here $\mathcal{U}(X)$ is the space of local relations in $\mathcal{F}(X)$, i.e. it is generated by fields on X of the form $u \bullet r$, where $u \in \mathcal{U}(B)$ is a local relation on an embedded n-ball $B \subset X$ and $r \in \mathcal{F}(X \backslash B)$. So this is just the vector space of fields on X, up to changing the field in any ball.

The blob complex can be thought of as a derived version of this construction. On a ball B, the blob complex $\mathcal{B}_*(B)$ will just be a free resolution of $H_0(\mathcal{B}_*(B))$. And when we have a short exact sequence of boundary conditions on ∂B, we obtain a long exact sequence in blob homology. In general, we will have $H_0(\mathcal{B}_*(X)) = A(X)$, and the higher blob homology groups will represent syzygies on the local relations not captured by the TQFT invariant itself.

In most of follows, we will assume that X is boundaryless. Otherwise one should fix a boundary condition in $\mathcal{F}(\partial X)$ for once and for all, and then carry out the same constructions under the assumption that everything agrees with that boundary condition.

1 Definition of the blob complex

A blob in X is just a generalization of an embedded n-ball. For this section, we will pretend that this is the definition of a blob. In the next section we will explain the difference, and then the definitions given here will be very nearly correct.

Roughly, the kth level of the blob complex $\mathcal{B}_*(X)$ will be the direct sum, over configurations of k blobs $\{B_1, \ldots, B_k\}$ in X, of the vector spaces of fields splittable over those particular blobs. The boundary map $\partial : \mathcal{B}_k(X) \to \mathcal{B}_{k-1}(X)$ will just be given by (a signed sum of) erasing blobs from the picture, which certainly preserves splittability over the remaining blobs. In this section we unwind exactly what we mean for $k = 0, 1, 2$ and then return to the general case to give a full definition.

1.0 $\mathcal{B}_0(X)$

First, we define $\mathcal{B}_0(X) = \mathcal{F}(X)$.

1.1 $\mathcal{B}_1(X)$

We'll want to quotient $\mathcal{B}_0(X)$ by changing a field $x \in \mathcal{F}(X)$ by a field which splits over a blob and gives a local relation there, i.e. we want $\mathcal{B}_1(X)$ to be fields on X of the form $u \bullet r$, where $u \in \mathcal{U}(B)$ and $r \in \mathcal{F}(X \backslash B)$. Of course, these must give the same boundary condition $c \in \mathcal{F}(\partial B) = \mathcal{F}(\partial(X \backslash B))$. So, for a fixed such boundary condition we get a vector space $\mathcal{U}(B;c) \otimes \mathcal{F}(X \backslash B; c)$. But we want to let the boundary condition vary, so we sum over all c. And we want to let the blob vary too, so we then sum over all $B \subset X$. Thus we define

$$\mathcal{B}_1(X) = \bigoplus_{B \subset X} \left(\bigoplus_{c \in \mathcal{F}(\partial B)} \mathcal{U}(B;c) \otimes \mathcal{F}(X \backslash B; c) \right),$$

with $\partial : \mathcal{B}_1(X) \to \mathcal{B}_0(X)$ given by $\partial(B, u, r) = u \bullet r$. As promised, this gives us

$$A(X) = \mathcal{B}_0(X)/\partial(\mathcal{B}_1(X)) = H_0(\mathcal{B}_*(X)).$$

1.2 $\mathcal{B}_2(X)$

We'll discuss $\mathcal{B}_2(X)$ before moving on to $\mathcal{B}_k(X)$, since this is the first place where we have to start worrying about how our blobs relate to each other. Here we have two blobs $B_1, B_2 \subset X$, and they will either be disjoint or nested. In both cases, we'll want to quotient $\ker(\partial : \mathcal{B}_1(X) \to \mathcal{B}_0(X))$ by fields which can be split over two blobs, identifying any two elements of $\mathcal{B}_1$ that were somehow mutually compatible.

For two disjoint blobs $B_1, B_2 \subset X$ (actually they only need disjoint interiors), this is just a field $x \in \mathcal{F}(X)$ from which we can simultaneously split off local relations on B_1 and B_2, i.e. $x = u_1 \bullet u_2 \bullet r$. Again, these must agree on the boundary conditions $c_1 \in \mathcal{F}(\partial B_1)$ and $c_2 \in \mathcal{F}(\partial B_2)$, so for a fixed pair of such boundary conditions we get a vector space $\mathcal{U}(B_1; c_1) \otimes \mathcal{U}(B_2; c_2) \otimes \mathcal{F}(X \backslash (B_1 \cup B_2); c_1, c_2)$. We make the identification $(B_1, B_2, u_1, u_2, r) = -(B_2, B_1, u_2, u_1, r)$, and we define $\partial(B_1, B_2, u_1, u_2, r) = (B_2, u_2, u_1 \bullet r) - (B_1, u_1, u_2 \bullet r)$. Hold on to that for a second.

For two nested (possibly equal) blobs $B_1 \subset B_2 \subset X$, we will want to kill a field $x \in \mathcal{F}(X)$ from which not only can we split off local relations on B_1 and B_2, but we can do so compatibly. Note, however, that if a field $x \in \mathcal{F}(X)$ restricts down to a local relation $u_1 \in \mathcal{U}(B_1)$ and splits over B_2, then it must restrict to a local relation $u_2 \in \mathcal{U}(B_2) \subset \mathcal{F}(B_2)$ as well since local relations are contagious, so we really just need to look for compatible fields $r' \in \mathcal{F}(B_2 \backslash B_1; c_1, c_2)$. So for a fixed pair of boundary conditions $c_1 \in \mathcal{F}(\partial B_1)$ and $c_2 \in \mathcal{F}(\partial B_2)$, we get a vector space $\mathcal{U}(B_1; c_1) \otimes \mathcal{F}(B_2 \backslash B_1; c_1, c_2) \otimes \mathcal{F}(X \backslash B_2; c_2)$. We define $\partial(B_1, B_2, u, r', r) = (B_2, u \bullet r', r) - (B_1, u, r' \bullet r)$.

Now that we have investigated both types of configurations of two blobs, we make the definition

$$\begin{aligned}
\mathcal{B}_2(X) \quad &= \quad \left(\bigoplus_{\substack{B_1, B_2 \subset X \\ B_1^\circ \cap B_2^\circ = \emptyset}} \bigoplus_{\substack{c_1 \in \mathcal{F}(\partial B_1) \\ c_2 \in \mathcal{F}(\partial B_2)}} \mathcal{U}(B_1; c_1) \otimes \mathcal{U}(B_2; c_2) \otimes \mathcal{F}(X \backslash (B_1 \cup B_2); c_1, c_2) \right) \\
&\qquad \bigoplus \left(\bigoplus_{B_1 \subset B_2 \subset X} \bigoplus_{\substack{c_1 \in \mathcal{F}(\partial B_1) \\ c_2 \in \mathcal{F}(\partial B_2)}} \mathcal{U}(B_1; c_1) \otimes \mathcal{F}(B_2 \backslash B_1; c_1, c_2) \otimes \mathcal{F}(X \backslash B_2; c_2) \right).
\end{aligned}$$

1.3 $\mathcal{B}_k(X)$

We define $\mathcal{B}_k(X)$ to be generated by fields on x along with configurations of k blobs that are all disjoint or nested and over which x splits, such that if a blob B_i does not strictly contain any other blob (this is called a *twig*) then x must restrict to a local relation in $\mathcal{U}(B_i)$. (Again, this last condition ensures that x restricts to a local relation on any blob, by the condition that local relations form an ideal.) In symbols, if we let

- $T \subseteq \{1, \ldots, k\}$ be the indices of the twig blobs (for some particular configuration of blobs)
- $B_{\varphi(i)}$ be a largest blob contained in B_i (for $i \notin T$)
- $X' = X \backslash (B_1 \cup \ldots \cup B_k)$
- $c \in \mathcal{F}(\partial X')$ be the sum of the boundary conditions $c_i \in \mathcal{F}(\partial B_i)$

then

$$\mathcal{B}_k(X) = \left(\bigoplus_{\substack{B_1, \ldots, B_k \subset X \\ B_i, B_j \text{ disjoint} \\ \text{or nested}}} \bigoplus_{c \in \mathcal{F}(\partial X')} \left(\bigotimes_{i \in T} \mathcal{U}(B_i; c_i) \right) \bigotimes \left(\bigotimes_{i \notin T} \mathcal{F}(B_i \backslash B_{\varphi(i)}; c_i, c_{\varphi(i)}) \right) \bigotimes \mathcal{F}(X'; c) \right) \Big/ \begin{array}{l}\text{permutations} \\ \text{of blobs} \\ \text{with signs.}\end{array}$$

The boundary map $\partial : \mathcal{B}_k(X) \to \mathcal{B}_{k-1}(X)$, as stated before, is the alternating sum of erasing one of the balls.

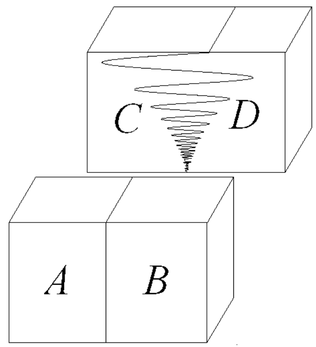
2 Blobs

The following definitions are motivated by the fact that we would like the following two operations on blob configurations to yield blob configurations:

- For any (possibly empty) blob configuration on an n-ball B, we can add B itself as an outermost blob.

- If we obtain X_{gl} from X by gluing, then any blob configuration on X gives a blob configuration on X_{gl}.

However, allowing these operations gives blob configurations whose complements are not manifolds. For example, suppose we have two $1 \times 1 \times 2$ blocks that have each been decomposed into two blobs as in the picture, and suppose we are planning to glue these two blocks together in the evident way.



Certainly we must allow $\{A\}$ as a blob configuration in $A \cup B$ and $\{D\}$ as a blob configuration in $C \cup D$. But then we also must allow $\{A, D\}$ as a blob configuration in $(A \cup B) \cup_{\text{face}} (C \cup D)$, whose complement is not a manifold.

Therefore, we define a *gluing decomposition* of a manifold X to be a sequence of manifolds $M_0 \to M_1 \to \ldots \to M_m = X$ such that each M_k is obtained from M_{k-1} by gluing together a disjoint pair of homeomorphic $(n-1)$-manifolds in the boundary of M_{k-1}. So all the points of X are already contained in M_0, and we just need to find a sequence of gluings such that at every stage we still have a manifold. If we have a gluing decomposition that begins as a disjoint union of balls, we call it a *ball decomposition.* Thus, the final $1 \times 2 \times 2$ block in the above example can be realized as the ball decomposition

$$A \sqcup B \sqcup C \sqcup D \to (A \cup B) \sqcup (C \cup D) \to A \cup B \cup C \cup D.$$

Given a gluing decomposition $M_0 \to \ldots \to M_m = X$, we say that a field on X is *splittable* if it is the image of a field on M_0.

Now we can give a precise definition: a *blob configuration* in X is an ordered collection of k subsets $\{B_1, \ldots, B_k\}$ of X such that there exists a gluing decomposition $M_0 \to \ldots \to M_m = X$ for which each B_i is the image in X of some connected component M'_j of some M_j, where M'_j must be a ball. We say that the gluing decomposition is *compatible* with the configuration.

Observe the following:

- Any two blobs must be nested or have disjoint interiors.
- Nested blobs may have boundaries that overlap (or even coincide).
- Blobs may meet ∂X.
- Through the sequence of gluings, M'_j may have been glued to itself, and so blobs need not actually be embedded balls.
- Complements of blob configurations need not be manifolds.

Remember that our old definition involved choosing fields on the complements of blob configurations. But we can only choose fields on manifolds. Thus, in light of the final observation, we make a new, final definition: a *k-blob diagram* on X is a configuration of k blobs $\{B_1, \ldots, B_k\}$ on X and a field $r \in \mathcal{F}(X)$ which is splittable along some gluing decomposition compatible with the configuration, such that the restriction to each twig B_i is a local relation, or more precisely that the restriction to the associated ball M'_j is a local relation.

When we write $\{B_1, \ldots, B_k\}$ we mean a configuration of k blobs, and when we write $(\{B_1, \ldots, B_k\}, r)$ we mean a k-blob diagram. Now we can now make the succinct definitions

$$\begin{aligned}
\mathcal{B}_k(X) &= \left(\bigoplus_{\{B_1,\ldots,B_k\}} \{(\{B_1, \ldots, B_k\}, r)\} \right) \Big/ \begin{array}{l}\text{permutations}\\ \text{of blobs}\\ \text{with signs}\end{array} \\
\partial(\{B_1, \ldots, B_k\}, r) &= \sum_{i=1}^{k} (-1)^{i+1} (\{B_1, \ldots, \hat{B}_i, \ldots, B_k\}, r).
\end{aligned}$$

It is immediate that ∂ really does take $\mathcal{B}_k(X)$ into $\mathcal{B}_{k-1}(X)$.

3 Basic properties

Throughout Morrison & Walker's original paper, results are mostly proved using the following properties rather than the actual definition of blob homology.

1. *The blob complex is functorial with respect to homeomorphisms.*

2. *There is a natural isomorphism $\mathcal{B}_*(X \sqcup Y) \cong B_*(X) \otimes B_*(Y)$.*

   Proof. We can combine any pair of blob diagrams on X and Y to a blob diagram on $X \sqcup Y$ by listing first the blobs on X, then the blobs on Y. Up to sign, every blob diagram on $X \sqcup Y$ arises in this way. ☐

3. *Let $c \in \mathcal{F}(\partial B)$ be any boundary condition. If the natural quotient map $p : \mathcal{B}_*(B;c) \to H_0(\mathcal{B}_*(B;c))$ has a splitting $s : H_0(\mathcal{B}_*(B;c)) \to \mathcal{B}_0(B;c)$, then these two maps induce a chain homotopy equivalence between $\mathcal{B}_*(B;c)$ and the complex $H_0 = \cdots \to 0 \to H_0(\mathcal{B}_*(B;c)) \to 0 \to \cdots$.*

   Proof. By assumption $ps = \mathrm{id}_{H_0}$, so we just need a collection of maps $h : \mathcal{B}_k(B;c) \to \mathcal{B}_{k+1}(B;c)$ such that $\partial h + h\partial = \mathrm{id}_{\mathcal{B}_*(B;c)} - sp$. For $k \geq 1$ we define $h_k((\{B_1, \ldots, B_k\}, r)) = (\{B_1, \ldots, B_k, B\}, r)$, and we define $h_0(r) = (\{B\}, r - s(p(r)))$. This gives the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{B}_3(B;c) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{B}_2(B;c) \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, "h_2"'] & \mathcal{B}_1(B;c) \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, "h_1"'] & \mathcal{B}_0(B;c) \arrow[r] \arrow[d, "\mathrm{id}-sp"] \arrow[dl, "h_0"'] & 0 \\
\cdots \arrow[r] & \mathcal{B}_3(B;c) \arrow[r] & \mathcal{B}_2(B;c) \arrow[r] & \mathcal{B}_1(B;c) \arrow[r] & \mathcal{B}_0(B;c) \arrow[r] & 0.
\end{tikzcd}$$

   It is obvious that $\partial h + h\partial = \mathrm{id} - sp$ on $\mathcal{B}_k(B;c)$ for all $k \neq 1$. At $k = 1$ we have

$$\begin{aligned}
(\partial h_1 + h_0 \partial)(\{B_1\}, r) &= \partial((\{B_1, B\}, r)) + h_0(r) \\
&= (\{B_1\}, r) - (\{B\}, r) + (\{B\}, r - s(p(r))) \\
&= (\{B_1\}, r) + (\{B\}, -s(p(r))).
\end{aligned}$$

   But note that $p(r) = 0$ by definition of blob homology, so in fact this map is the identity on $\mathcal{B}_1(B;c)$. ☐

   Assuming $(\mathcal{F}, \mathcal{U})$ is enriched over **Vect**, we will always have such a splitting. But note that even when there is no such splitting, we can still let $h_0 = 0$ and get a homotopy equivalence between $\mathcal{B}_*(B;c)$ and $\cdots \to 0 \to \mathcal{U}(B;c) \to \mathcal{F}(B;c)) \to 0 \to \cdots$.

4. *If X is a disjoint union of balls, then $\mathcal{B}_*(X;c)$ is contractible.*

   Proof. This follows directly from Properties 2 and 3. ☐

5. *Suppose $\partial X = Y \cup Y \cup Z$. Let X_{gl} be the result of gluing the two copies of Y together, and write $\partial X_{\mathrm{gl}} = Z_{\mathrm{gl}}$. Suppose $c \in \mathcal{F}(X)$ restricts to the same boundary condition $a \in \mathcal{F}(Y)$ on both copies of Y. Then we can necessarily glue the restriction $b \in \mathcal{F}(Z)$ of c to itself to get $b_{\mathrm{gl}} \in \mathcal{F}(Z_{\mathrm{gl}})$. For any such situation, there is a chain map*

$$\mathrm{gl} : \mathcal{B}_*(X; a, a, b) \to \mathcal{B}_*(X_{\mathrm{gl}}; b_{\mathrm{gl}})$$

   which is natural with respect to the actions of diffeomorphisms and iterated gluings.

4 A combinatorial aside

Blob configurations are rather combinatorial in nature. In this section we describe a functorial, simplicial set-like construction which associates to any blob configuration what we will call a *cone-product polyhedron.* We denote by $\mathcal{P}$ the collection of these, and we denote a typical object by the letter ρ.

Note that this construction is ignorant of whether nested blobs have intersecting boundary.

From a blob configuration b we build a simplicial complex $p(b)$ as follows:

- Let $p(\emptyset) = \mathrm{pt}$, where $\emptyset$ denotes a 0-blob diagram.
- If b and b' are non-overlapping blob diagrams (i.e. the interiors of their blobs are disjoint), let $p(b \sqcup b') = p(b) \times p(b')$ (note that this rule makes the previous rule acceptable).
- If $\bar{b}$ is obtained from b by adding an outer blob which encloses all the others, let $p(\bar{b}) = \mathrm{cone}(p(b))$.

Thus, assuming our diagram has any blobs at all, we start with an edge for each twig blob, take a Cartesian product whenever we need to combine two configurations, and take the cone whenever a new blob encloses the existing configuration. So for example, a diagram of k nested blobs yields a k-simplex, while a diagram of k disjoint blobs yields a k-cube. If two (or more) blobs are equal, this still works if we consider them as being nested in some arbitrary way. However, we might hope to use this to obtain something like a sheaf over the configuration space $\mathrm{Conf}_{\mathcal{B}}(X)$ of blobs on X, in which case the issue might become more serious.

Cone-product polyhedra arising in this way can be given a hint of simplicial flavor as follows. On the vertex of $p(b)$ associated to the blob B_i we can keep track of the local relation $u_i \in \mathcal{U}(B_i)$, and on (whatever arises in $p(b)$ coming from) the interior of a cone associated to adding the blob B_i we can keep track of the field $x_i \in \mathcal{F}(B_i \backslash \{\cdots\})$. On the interior of a product $p(b) \times p(b')$ coming from $b \sqcup b'$ we should probably put something like $r \in \mathcal{F}(X \backslash (\mathrm{supp}(b \sqcup b')))$, but I couldn't prove it. Maybe this is an indication that we shouldn't be taking products, because then we'd end up with disjoint simplicial sets.

Assume in what follows that no two blobs are equal. We can then strengthen the above construction to commute with taking the boundary, but we must make a new definition of "boundary" for our cone-product polyhedra. Let $\Sigma\mathcal{P}$ denote the free abelian group on $\mathcal{P}$. (This should probabl be modified if we decide not to take products after all, as was idly mused in the paragraph above.) Note that throughout the construction of a cone-product polyhedron we can keep track of a distinguished point $*$, namely the point associated to the 0-blob diagram. We define a homomorphism $\delta : \Sigma\mathcal{P} \to \Sigma\mathcal{P}$ by, for any $\rho \in \mathcal{P}$, setting $\delta(\rho)$ to be a signed sum of those faces in $\partial\rho$ which contain $*$, except that we define $\delta(*) = *$. This gives us that $p(\partial b) = \delta p(b)$, since up to signs this is true when we modify b and $p(b)$ using our three operations:

- If $b = \emptyset$ is the 0-blob diagram, then

$$\begin{aligned} p(\partial\emptyset) &= p(\emptyset) = * \\ \delta(p(\emptyset)) &= \delta(*) = *. \end{aligned}$$

- If b and b' are non-overlapping blob diagrams, then

$$\begin{aligned} p(\partial(b \sqcup b')) &= p((\partial b) \sqcup b' + (-1)^{|b|} b \sqcup (\partial b')) = p(\partial b) \times b' + (-1)^{|b|} p(b) \times p(\partial b') \\ \delta p(b \sqcup b') &= \delta(p(b) \times p(b')) = \delta(p(b)) \times p(b') \pm p(b) \times \delta(p(b')) \end{aligned}$$

  so the statement follows by induction on $|b|$ and $|b'|$.

- If $|b| = k$, then

$$\begin{aligned} p(\partial(\bar{b})) &= p\left(\sum_{i=1}^{k} (-1)^{i+1} \overline{b_{\hat{i}}} + (-1)^k b\right) = \sum_{i=1}^{k} (-1)^{i+1} \mathrm{cone}(p(b_{\hat{i}})) + (-1)^k p(b) \\ \delta p(\bar{b}) &= \delta(\mathrm{cone}(p(b))) = \sum_{i=1}^{k} \pm \mathrm{cone}((p(b_{\hat{i}}))) \pm p(b). \end{aligned}$$

Said differently, $p : (\mathcal{B}_*, \partial) \to (\Sigma\mathcal{P}, \delta)$ is a homomorphism of differential groups. Presumably there should be a way to carry over the information on blobs to $\Sigma\mathcal{P}$ in such a way that we can compute $H_*(\mathcal{B}_*(X))$ entirely from its image in (a souped-up version of) $\Sigma\mathcal{P}$.