The blob complex

takes an n-category $\mathcal{C}$ and an n-manifold M and produces a chain complex $\mathcal{B}_*(M;\mathcal{C})$.

It is a simultaneous generalization of several interesting constructions.

$$\begin{tikzcd}
 & A(M;\mathcal{C}) \text{ the TFT skein module} & \\
 & \mathcal{B}_*(M;\mathcal{C}) \arrow[u, "H_0"] \arrow[r, "M=S^1,\ n=1"] \arrow[dl, "\mathcal{C}=k[t]"'] & HC_*(\mathcal{C}) \\
 C_*(\Delta^\infty M, k) & &
\end{tikzcd}$$

Hochschild homology

singular chains on configurations

"...homotopical topology and TQFT have grown so close that I have started thinking that they are turning into the language of new foundations"

— Yuri Manin 2008

Outline

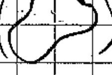
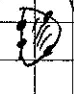
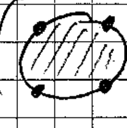
- n-categories
- TFT
- the blob complex
- properties, applications

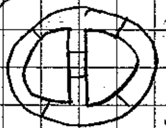
n-categories

There are many definitions. Ours, "disklike n-categories" are adapted to TFT. In particular, strong duality is intrinsic, rather than an optional property.

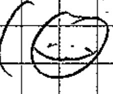
- a space for every n-ball (fibered over 'boundary conditions')
- strictly associative gluing operations

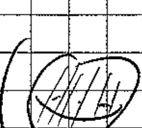
Examples

- Every planar algebra P gives a disklike 2-category Q
  - Boundary conditions: $cQ($  $)$ = finite subsets
  - Morphisms: $Q($  $) = P_n$
  - Gluing: $Q($  $) \otimes Q($  $) \to Q($  $)$

    is $P($  $) : P_4 \otimes P_4 \to P_4$

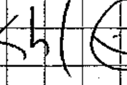
- Contact structures give a disklike 3-category $\mathcal{C}$

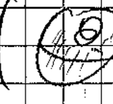
  $\mathcal{C}($  $)$ = { separating curves; circles with an alternating shading }

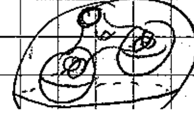
  $\mathcal{C}($  $)$ = { tight contact structures }

  Gluing is gluing.

- Khovanov homology gives a disklike 4-category Kh

  $Kh($  $)$ = { links }

  $Kh($  $) = Kh(L)$ (a vector space)

  Gluing via functoriality:  gives a map from

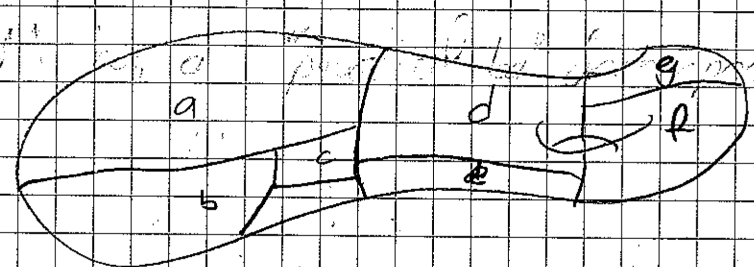
• Maps to a space T gives a disklike n-category $\mathcal{M}$

$$\mathcal{M}(\ )^n = \mathrm{Maps}(\ \to T)/\text{isotopy rel } \partial \qquad (\text{in Set})$$

or $\mathcal{M}(\ )^n = C_* \mathrm{Maps}(\ \to T) \qquad (\text{in Chain})$

or $\mathcal{M}(\ )^n = \mathrm{Maps}(\ \to T) \qquad (\text{in Top})$

What is the TFT skein module?

$A(M, \mathcal{C})^r \ni$  "a field on M"

modulo antirefinement modulo gluing.

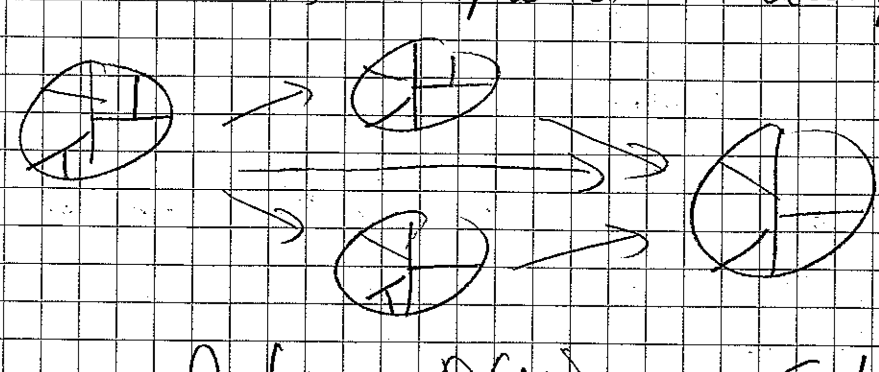
Claim 1 $A(\Sigma^2, Q)$ is the Turaev-Viro vector space for Σ from the planar algebra P

Claim 2 $A(M^3, \mathcal{C}_{\text{contact}})$ = tight contact structures on M

Claim 3 $A(W, \mathcal{M}) = [W \to T]$.

Reformulated:

Associated to M is the "poset of ball decompositions" $\mathcal{D}(M)$



$\mathcal{C}$ gives a functor $\mathcal{D}(M) \longrightarrow \mathrm{Set}$.

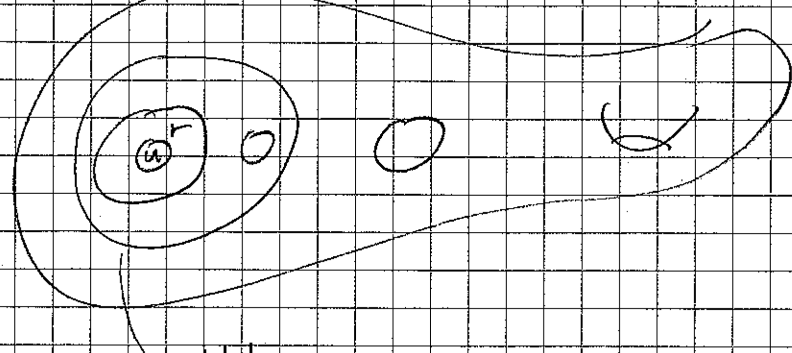
$A(M, \mathcal{C})$ is the colimit of this functor.

Now the blob complex is easy to define:

$$\mathcal{B}_*(M;\mathcal{C}) = \operatorname*{hocolim}_{\mathcal{O}(M)} \mathcal{C}.$$

Let's pull that apart:

$\mathcal{B}_k(M;\mathcal{C}) =$



k blobs

Each region has a field.

The differential is a signed sum of
- forgetting a blob

or
- gluing up an innermost blob.

Example

$$\mathcal{B}_1 \xrightarrow{\partial} \mathcal{B}_0$$

Observe $\mathcal{B}_0 / \operatorname{Im}\partial = A(M;\mathcal{C})$.

Properties

an action: $C_* \mathrm{Homeo}\, M \otimes \mathcal{B}_*(M;\mathcal{C}) \to \mathcal{B}_*(M;\mathcal{C})$

gluing: $\mathcal{B}_*(M_1 \cup_\Sigma M_2;\mathcal{C}) \simeq \mathcal{B}_*(M_1;\mathcal{C}) \hat{\otimes}_{\mathcal{B}_*(\Sigma;\mathcal{C})} \mathcal{B}_*(M_2;\mathcal{C})$

mapping spaces: $\mathcal{B}_*(M, C_* \mathrm{Maps}(- \to T)) \simeq C_* \mathrm{Maps}(M \to T)$

exactness w.r.t ∂: $Kh(X)$ $A(W;X;Kh)$ $\mathcal{B}_*(W,X;Kh)$