Blob Homology

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We define a chain complex $\mathcal{B}_*(M, C)$ (the "blob complex") associated to an n-category C and an n-manifold M. For $n = 1$, $\mathcal{B}_*(S^1, C)$ is quasi-isomorphic to the Hochschild complex of the 1-category C. So in some sense blob homology is a generalization of Hochschild homology to n-categories. The degree zero homology of $\mathcal{B}_*(M, C)$ is isomorphic to the dual of the Hilbert space associated to M by the TQFT corresponding to C. So in another sense the blob complex is the derived category version of a TQFT.

This is work in progress, so various details remain to be filled in.

We hope to apply blob homology to tight contact structures on 3-manfolds ($n = 3$) and extending Khovanov homology to general 4-manifolds ($n = 4$). In both of these examples, exact triangles play an important role, and the derived category aspect of the blob complex allows this exactness to persist to a greater degree than it otherwise would.

$\mathcal{B}_0(M, C)$ is defined to be finite linear combinations of C-pictures on M. (A C-picture on M can be thought of as a pasting diagram for n-morphisms of C in the shape of M together with a choice of homeomorphism from this diagram to M.) There is an evaluation map from $\mathcal{B}_0(B^n, C)$ (C-pictures on the n-ball B^n) to the n-morphisms of C. Let U be the kernel of this map. Elements of U are called null fields. $\mathcal{B}_1(M, C)$ is defined to be finite linear combinations of triples (B, u, r) (called 1-blob diagrams), where $B \subset M$ is an embedded ball (or "blob"), $u \in U$ is a null field on B, and r is a C-picture on $M \setminus B$. Define the boundary map $\partial : \mathcal{B}_1(M, C) \to \mathcal{B}_0(M, C)$ by sending (B, u, r) to $u \bullet r$, the gluing of u and r. $\mathcal{B}_1(M, C)$ can be thought of as the space of relations we would naturally want to impose on $\mathcal{B}_0(M, C)$, and so $H_0(\mathcal{B}_*(M, C))$ is isomorphic to the generalized skein module (dual of TQFT Hilbert space) one would associate to M and C.

$\mathcal{B}_k(M, C)$ is defined to be finite linear combinations k-blob diagrams. A k-blob diagram consists of k blobs (balls) $B_0, \ldots, B_{k-1}$ in M. Each pair B_i

and B_j is required to be either disjoint or nested. Each innermost blob B_i is equipped with a null field $u_i \in U$. There is also a C-picture r on the complement of the innermost blobs. The boundary map $\partial : \mathcal{B}_k(M, C) \to \mathcal{B}_{k-1}(M, C)$ is defined to be the alternating sum of forgetting the i-th blob.

If M has boundary we always impose a boundary condition consisting of an $n-1$-morphism picture on ∂M. In this note we will suppress the boundary condition from the notation.

The blob complex has the following properties:

- **Functoriality.** The blob complex is functorial with respect to diffeomorphisms. That is, fixing C, the association

$$M \mapsto \mathcal{B}_*(M, C)$$

  is a functor from n-manifolds and diffeomorphisms between them to chain complexes and isomorphisms between them.

- **Contractibility for B^n.** The blob complex of the n-ball, $\mathcal{B}_*(B^n, C)$, is quasi-isomorphic to the 1-step complex consisting of n-morphisms of C. (The domain and range of the n-morphisms correspond to the boundary conditions on B^n. Both are suppressed from the notation.) Thus $\mathcal{B}_*(B^n, C)$ can be thought of as a free resolution of C.

- **Disjoint union.** There is a natural isomorphism

$$\mathcal{B}_*(M_1 \sqcup M_2, C) \cong \mathcal{B}_*(M_1, C) \otimes \mathcal{B}_*(M_2, C).$$

- **Gluing.** Let M_1 and M_2 be n-manifolds, with Y a codimension-0 submanifold of ∂M_1 and $-Y$ a codimension-0 submanifold of ∂M_2. Then there is a chain map

$$\mathrm{gl}_Y : \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \to \mathcal{B}_*(M_1 \cup_Y M_2).$$

- **Relation with Hochschild homology.** When C is a 1-category, $\mathcal{B}_*(S^1, C)$ is quasi-isomorphic to the Hochschild complex $\mathrm{Hoch}_*(C)$.

- **Relation with TQFTs and skein modules.** $H_0(\mathcal{B}_*(M, C))$ is isomorphic to $A_C(M)$, the dual Hilbert space of the $n+1$-dimensional TQFT based on C.

- **Evaluation map.** There is an 'evaluation' chain map

$$\mathrm{ev}_M : C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \to \mathcal{B}_*(M).$$

(Here $C_*(\mathrm{Diff}(M))$ is the singular chain complex of the space of diffeomorphisms of M, fixed on ∂M.)

Restricted to $C_0(\mathrm{Diff}(M))$ this is just the action of diffeomorphisms described above. Further, for any codimension-1 submanifold $Y \subset M$ dividing M into $M_1 \cup_Y M_2$, the following diagram (using the gluing maps described above) commutes.

$$\begin{array}{ccc} C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) & \xrightarrow{\mathrm{ev}_M} & \mathcal{B}_*(M) \\ \uparrow {\scriptstyle \mathrm{gl}_Y^{\mathrm{Diff}} \otimes \mathrm{gl}_Y} & & \uparrow {\scriptstyle \mathrm{gl}_Y} \\ C_*(\mathrm{Diff}(M)) \otimes C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) & \xrightarrow{\mathrm{ev}_{M_1} \otimes \mathrm{ev}_{M_2}} & \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \end{array}$$

In fact, up to homotopy the evaluation maps are uniquely characterized by these two properties.

- A_∞ **categories for** $n{-}1$**-manifolds.** For Y an $n{-}1$-manifold, the blob complex $\mathcal{B}_*(Y \times I, C)$ has the structure of an A_∞ category. The multiplication (m_2) is given my stacking copies of the cylinder $Y \times I$ together. The higher m_i's are obtained by applying the evaluation map to $i{-}2$-dimensional families of diffeomorphisms in $\mathrm{Diff}(I) \subset \mathrm{Diff}(Y \times I)$. Furthermore, $\mathcal{B}_*(M, C)$ affords a representation of the A_∞ category $\mathcal{B}_*(\partial M \times I, C)$.

- **Gluing formula.** Let $Y \subset M$ divide M into manifolds M_1 and M_2. Let $A(Y)$ be the A_∞ category $\mathcal{B}_*(Y \times I, C)$. Then $\mathcal{B}_*(M_1, C)$ affords a right representation of $A(Y)$, $\mathcal{B}_*(M_2, C)$ affords a left representation of $A(Y)$, and $\mathcal{B}_*(M, C)$ is homotopy equivalent to $\mathcal{B}_*(M_1, C) \otimes_{A(Y)} \mathcal{B}_*(M_2, C)$.