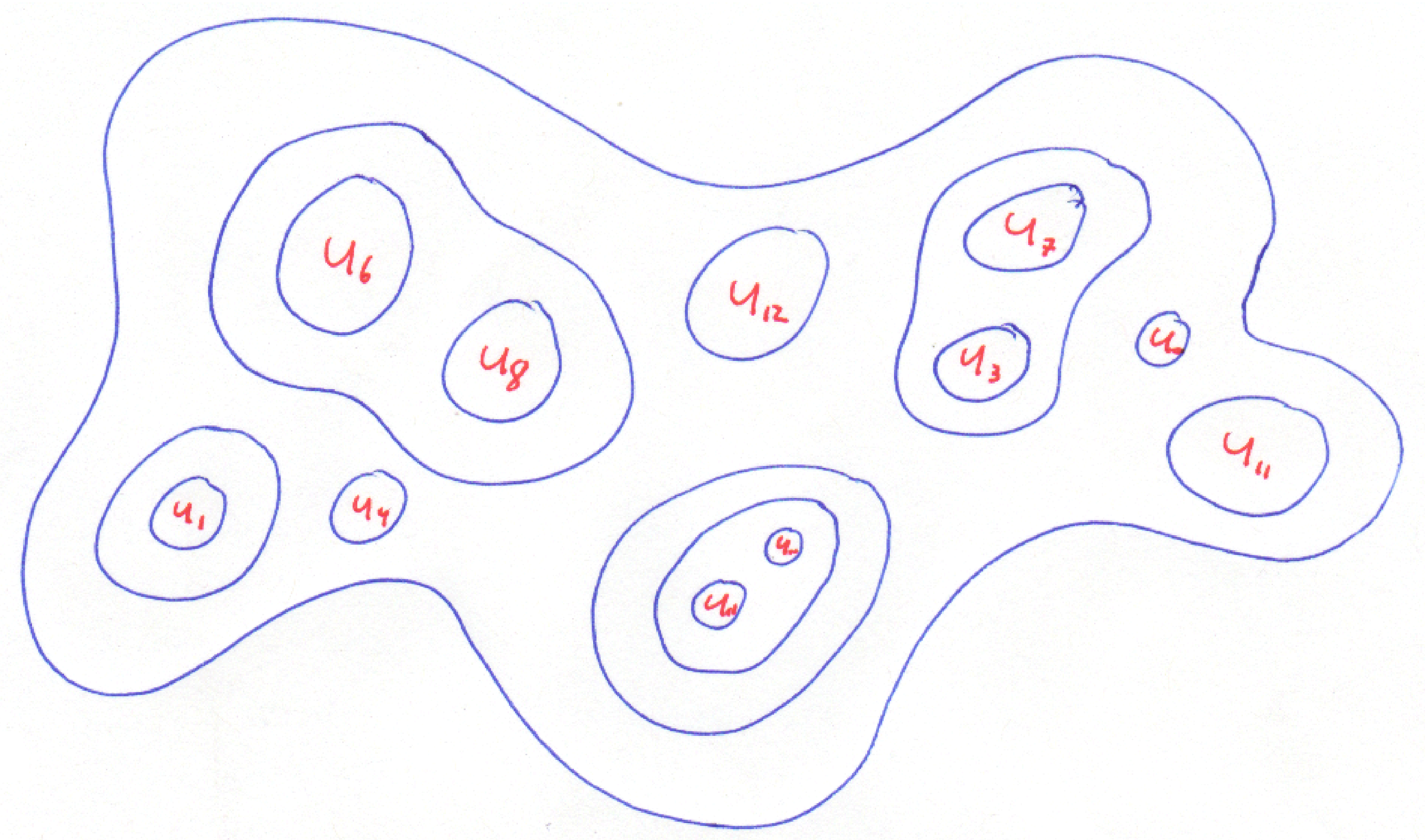
The Blob Complex, part 2

Kevin Walker
(joint work with Scott Morrison)



slides available at canyon23.net/math/

Goals:

- n-category definition optimized for TQFTs
- should be very easy to show that topological examples satisfy the axioms
- as simple as possible (but not simpler)
- both plain and infinity type categories
- also define modules, coends, tensor products, etc.

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- n-category definition optimized for TQFTs
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Main ideas:

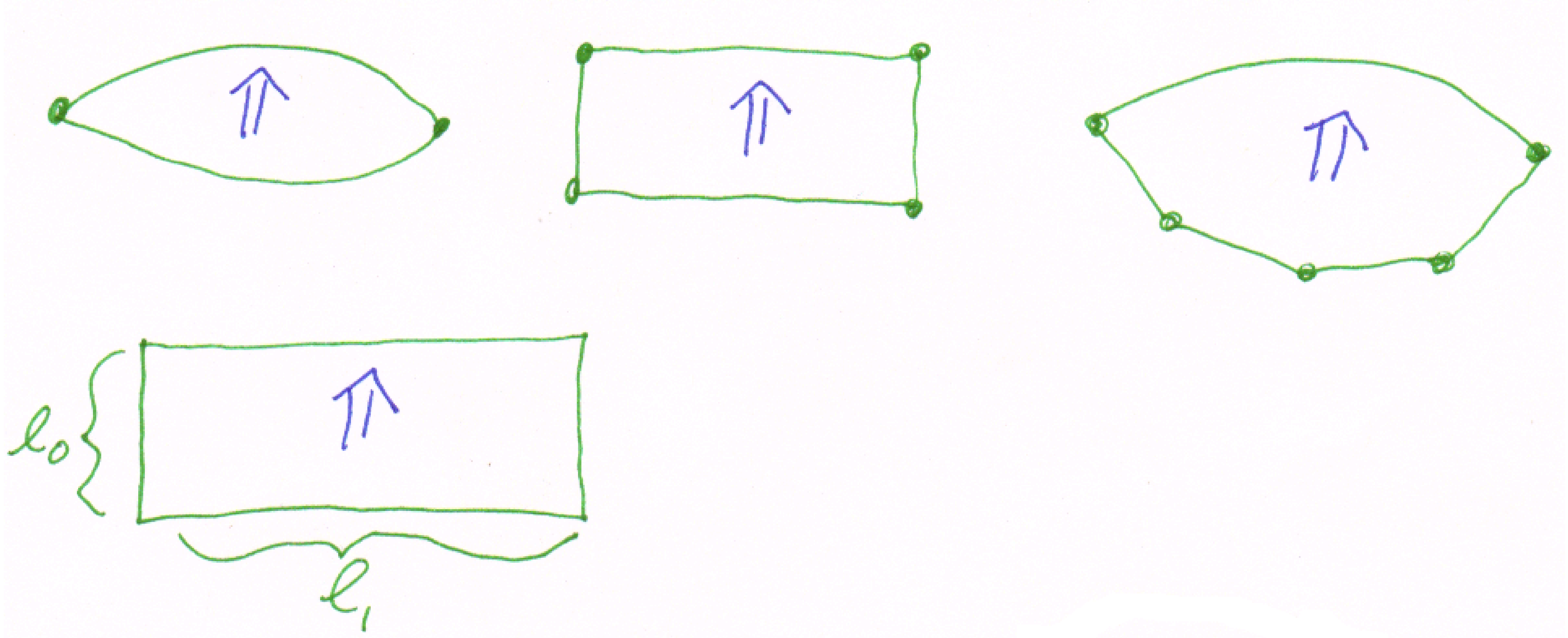
- don’t skeletonize (don’t try to minimize generators, don’t try to minimize relations)
- build in “strong” duality from the start
- non-recursive (don’t need to know what an (n-1)-category is)

Ingredients for an n-category:

1. morphisms in dimensions 0 through n
2. domain/range/boundary
3. composition
4. identity morphisms
5. special behavior in dimension n

Morphisms

- Need to decide on "shape" of morphisms



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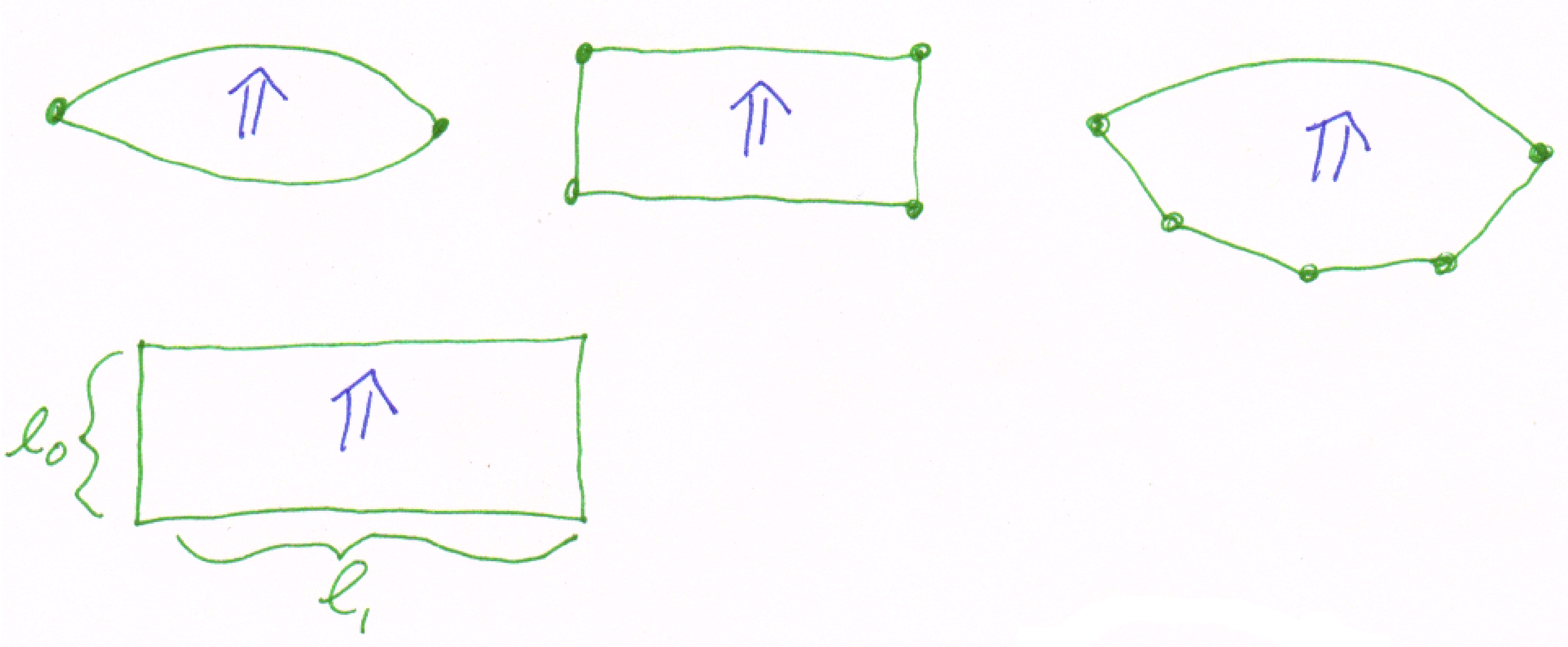

- We will allow morphisms to be of **any** shape, so long as it is homeomorphic to a ball

Morphisms (preliminary version): *For any k-manifold X homeomorphic to the standard k-ball, we have a set of k-morphisms $\mathcal{C}_k(X)$.*

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Balls could be PL, topological, or smooth. Also unoriented, oriented, Spin, $\mathrm{Pin}_\pm$. We will concentrate on the case of PL unoriented balls.

Examples

Let T be a topological space.
$\mathcal{C}_k(X^k) = \text{Maps}(X \to T)$, for $k < n$, X a k-ball.
$\mathcal{C}_n(X^n) = \text{Maps}(X \to T)$ modulo homotopy rel boundary
(fundamental groupoid of T)

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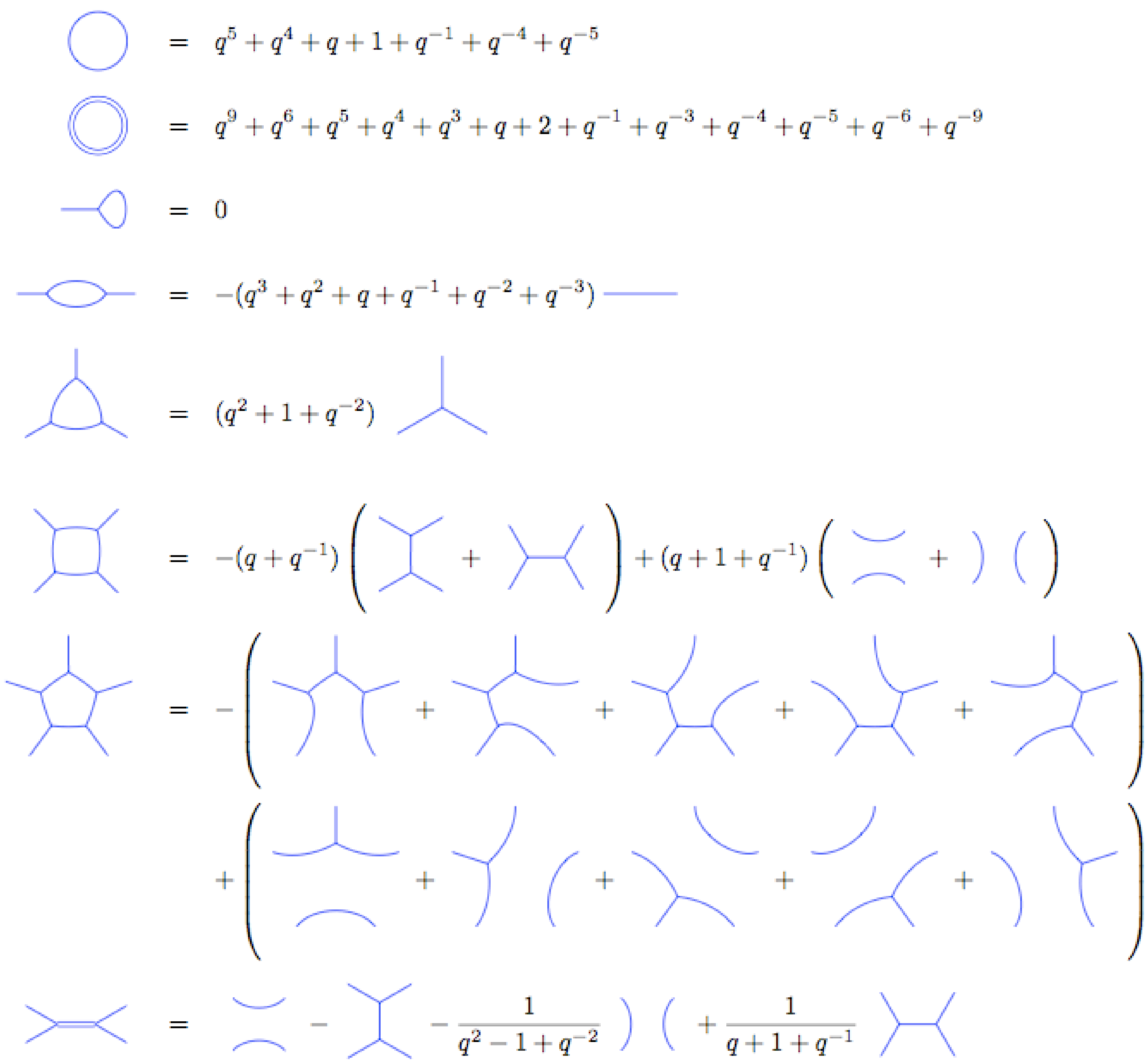
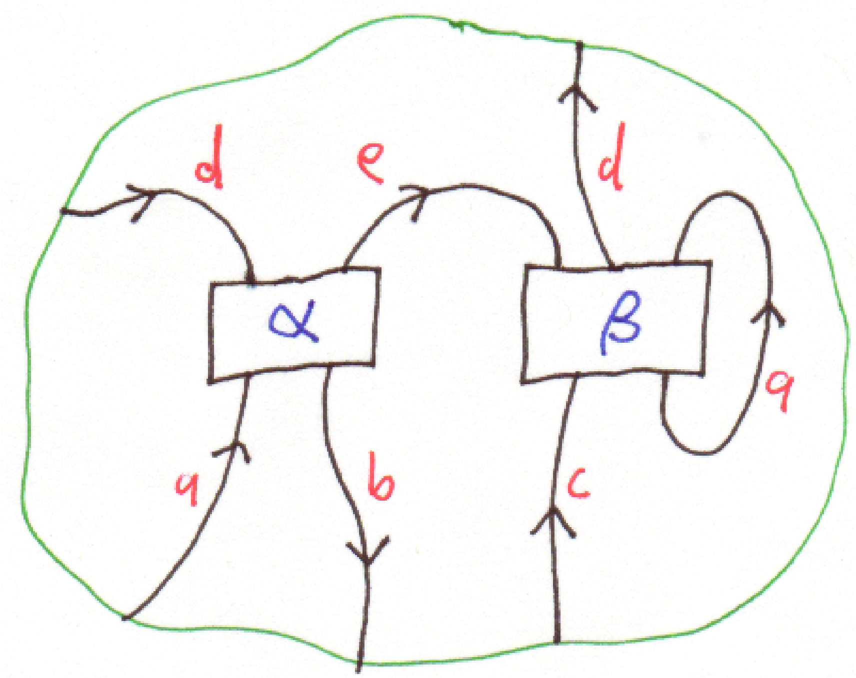
$\mathcal{C}_n(X^n) = C_*(\text{Maps}(X \to T))$ (singular chains)
(∞ version of fundamental groupoid of T)

$\mathcal{C}_k(X^k) = \{$embedded decorated cell complexes in X$\}$, for $k < n$.
$\mathcal{C}_n(X^n) = \{$embedded decorated cell complexes in X$\}$ modulo isotopy and other local relations



Let A be a traditional linear n-category with strong duality (e.g. pivotal 2-category).
$\mathcal{C}_k(X^k) = \{A\text{-string diagrams in } X\}$, for $k < n$.
$\mathcal{C}_n(X^n) = \{\text{finite linear combinations of } A\text{-string diagrams in } X\}$ modulo diagrams which evaluate to zero



$\mathcal{C}_k(X^k) = \{A\text{-string diagrams in } X\}$, for $k < n$.
$\mathcal{C}_n(X^n) =$ blob complex of X based on A-string diagrams

Boundaries (domain and range), part 1: *For each* $0 \leq k \leq n-1$, *we have a functor* $\mathcal{C}_k$ *from the category of* k*-spheres and homeomorphisms to the category of sets and bijections.*

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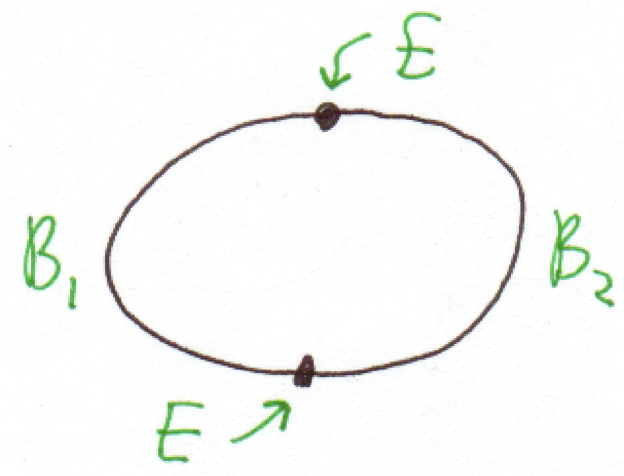
Domain + range → boundary: *Let $S = B_1 \cup_E B_2$, where S is a k-sphere ($0 \le k \le n-1$), B_i is a k-ball, and $E = B_1 \cap B_2$ is a k−1-sphere. Let $\mathcal{C}(B_1) \times_{\mathcal{C}(E)} \mathcal{C}(B_2)$ denote the fibered product of the two maps $\partial : \mathcal{C}(B_i) \to \mathcal{C}(E)$. Then (axiom) we have an injective map*

$$\mathrm{gl}_E : \mathcal{C}(B_1) \times_{\mathcal{C}(E)} \mathcal{C}(B_2) \to \mathcal{C}(S)$$

which is natural with respect to the actions of homeomorphisms.



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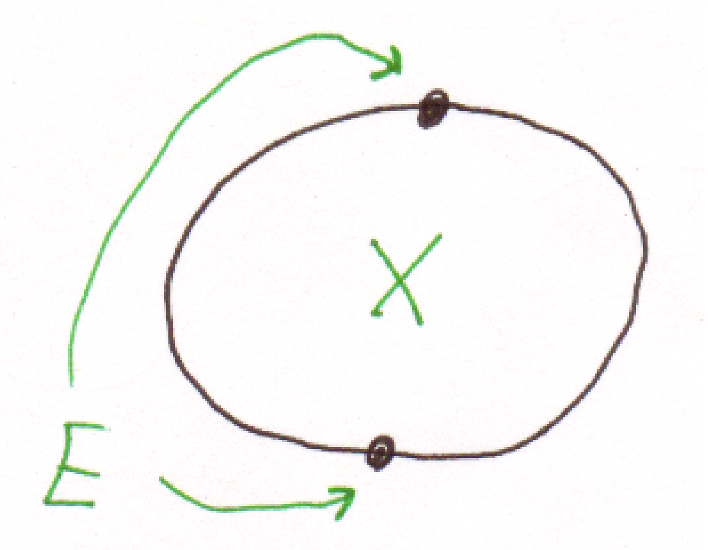

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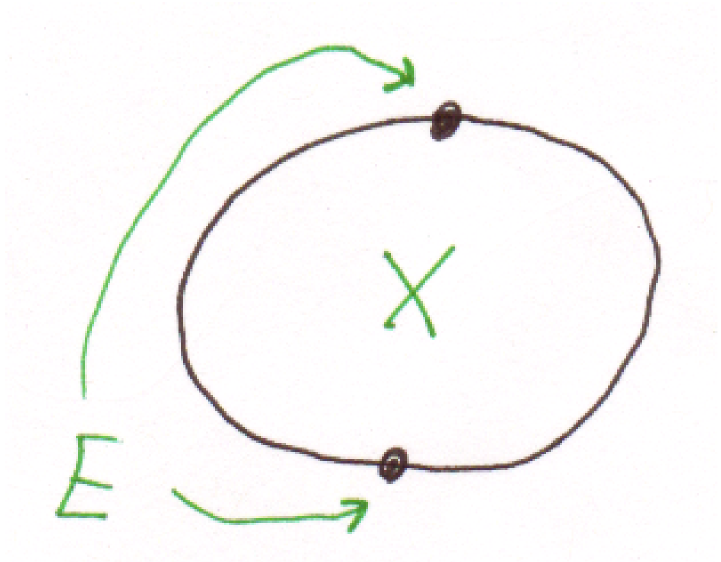

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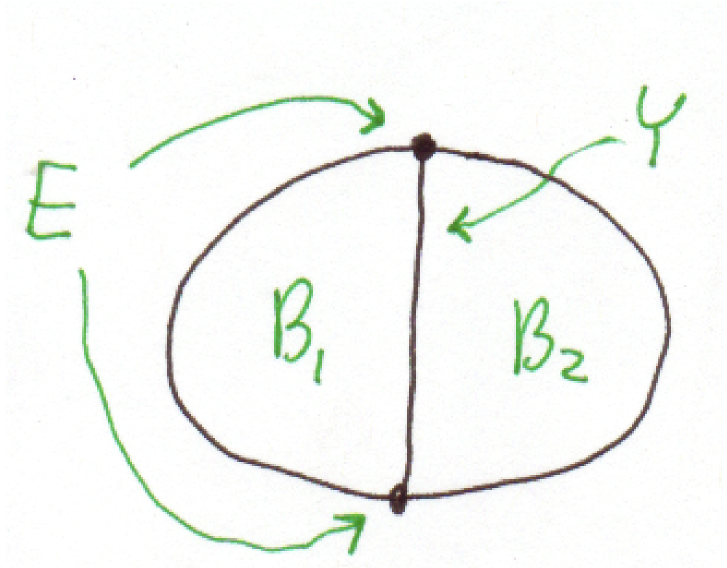
    - Given $E \subset \partial X$, let $\mathcal{C}(X)_E \stackrel{\text{def}}{=} \partial^{-1}(\mathcal{C}(\partial X)_E)$
      - In most examples, we require that the sets $\mathcal{C}(X;c)$ (for all n-balls X and all boundary conditions c) have extra structure, e.g. vector space or chain complex

Composition: *Let $B = B_1 \cup_Y B_2$, where B, B_1 and B_2 are k-balls ($0 \le k \le n$) and $Y = B_1 \cap B_2$ is a $k{-}1$-ball. Let $E = \partial Y$, which is a $k{-}2$-sphere. Note that each of B, B_1 and B_2 has its boundary split into two $k{-}1$-balls by E. We have restriction (domain or range) maps $\mathcal{C}(B_i)_E \to \mathcal{C}(Y)$. Let $\mathcal{C}(B_1)_E \times_{\mathcal{C}(Y)} \mathcal{C}(B_2)_E$ denote the fibered product of these two maps. Then (axiom) we have a map*

$$\mathrm{gl}_Y : \mathcal{C}(B_1)_E \times_{\mathcal{C}(Y)} \mathcal{C}(B_2)_E \to \mathcal{C}(B)_E$$

which is natural with respect to the actions of homeomorphisms, and also compatible with restrictions to the intersection of the boundaries of B and B_i. If $k < n$ we require that gl_Y is injective. (For $k = n$, see below.)



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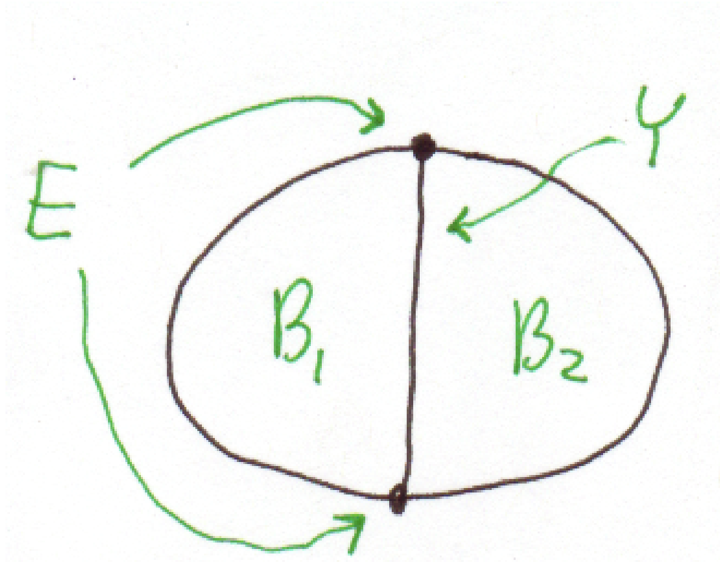
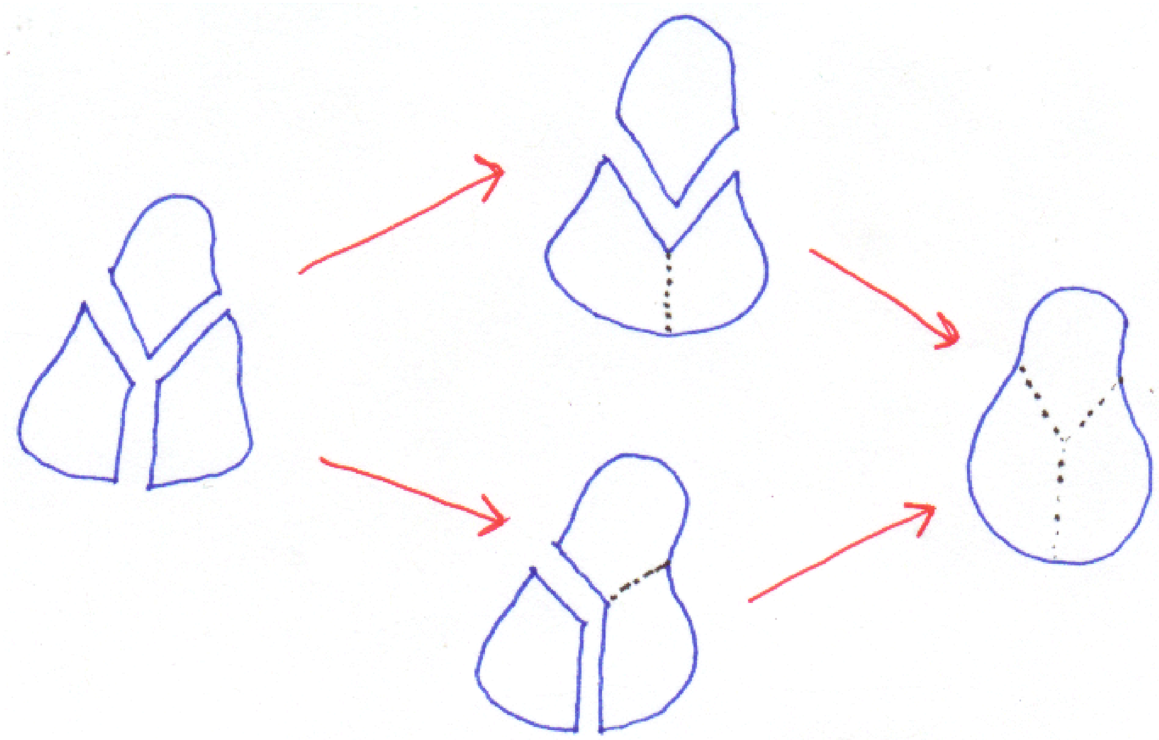
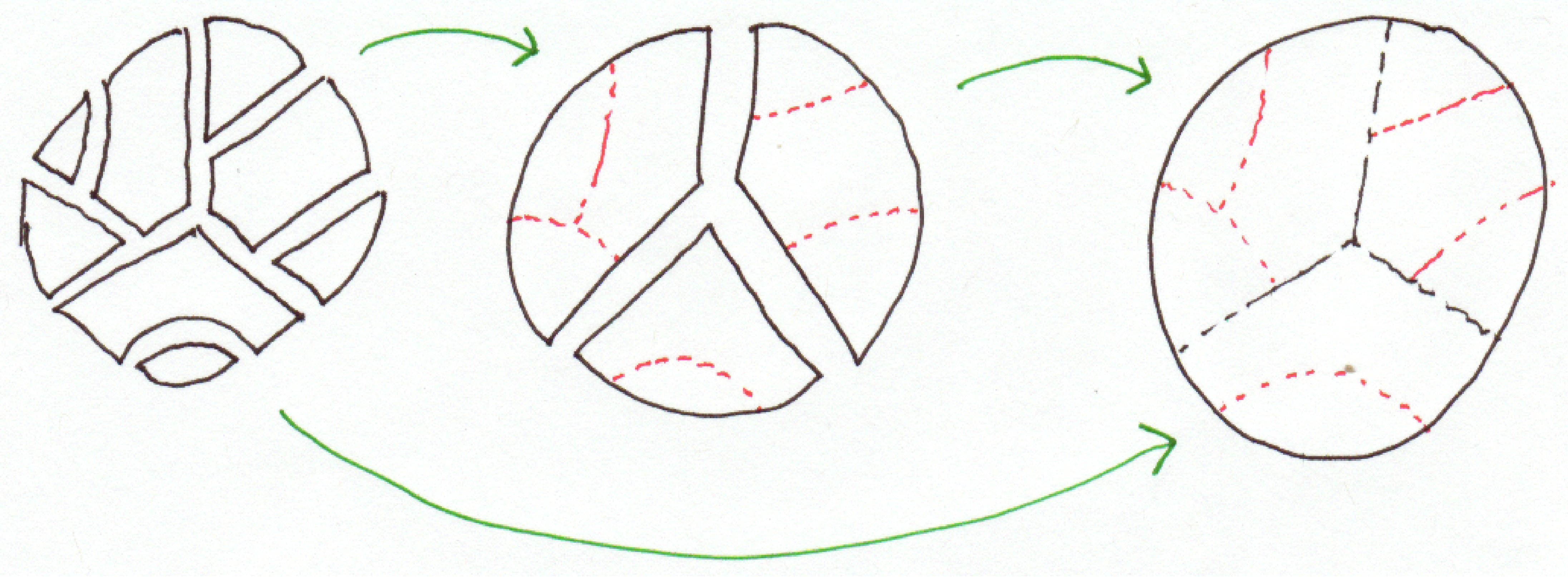




Strict associativity: *The composition (gluing) maps above are strictly associative.*

Multi-composition: *Given any decomposition $B = B_1 \cup \cdots \cup B_m$ of a k-ball into small k-balls, there is a map from an appropriate subset (like a fibered product) of $\mathcal{C}(B_1) \times \cdots \times \mathcal{C}(B_m)$ to $\mathcal{C}(B)$, and these various m-fold composition maps satisfy an operad-type strict associativity condition.*



Product (identity) morphisms: *Let X be a k-ball and D be an m-ball, with $k+m \le n$. Then we have a map $\mathcal{C}(X) \to \mathcal{C}(X \times D)$, usually denoted $a \mapsto a \times D$ for $a \in \mathcal{C}(X)$. If $f : X \to X'$ and $\tilde{f} : X \times D \to X' \times D'$ are maps such that the diagram*

$$\begin{tikzcd} X \times D \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & X' \times D' \arrow[d, "\pi"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

commutes, then we have

$$\tilde{f}(a \times D) = f(a) \times D'.$$

Product morphisms are compatible with gluing (composition) in both factors:

$$(a' \times D) \bullet (a'' \times D) = (a' \bullet a'') \times D$$

and

$$(a \times D') \bullet (a \times D'') = a \times (D' \bullet D'').$$

Product morphisms are associative:

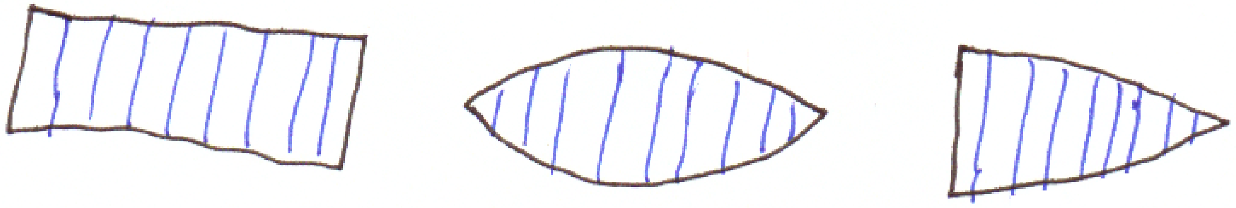
$$(a \times D) \times D' = a \times (D \times D').$$

(Here we are implicitly using functoriality and the obvious homeomorphism $(X \times D) \times D' \to X \times (D \times D')$.) Product morphisms are compatible with restriction:

$$\mathrm{res}_{X \times E}(a \times D) = a \times E$$

for $E \subset \partial D$ and $a \in \mathcal{C}(X)$.

We need something a little more general than plain products



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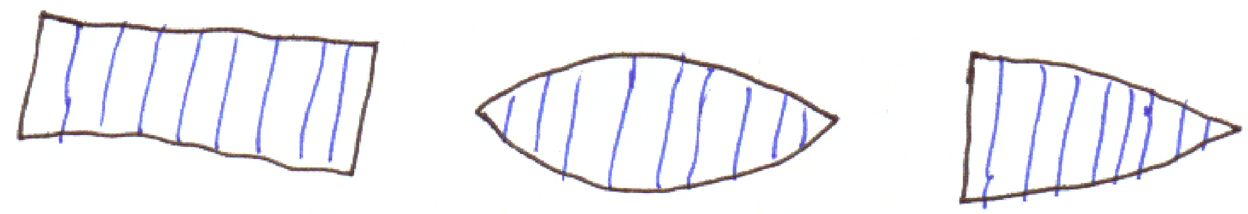

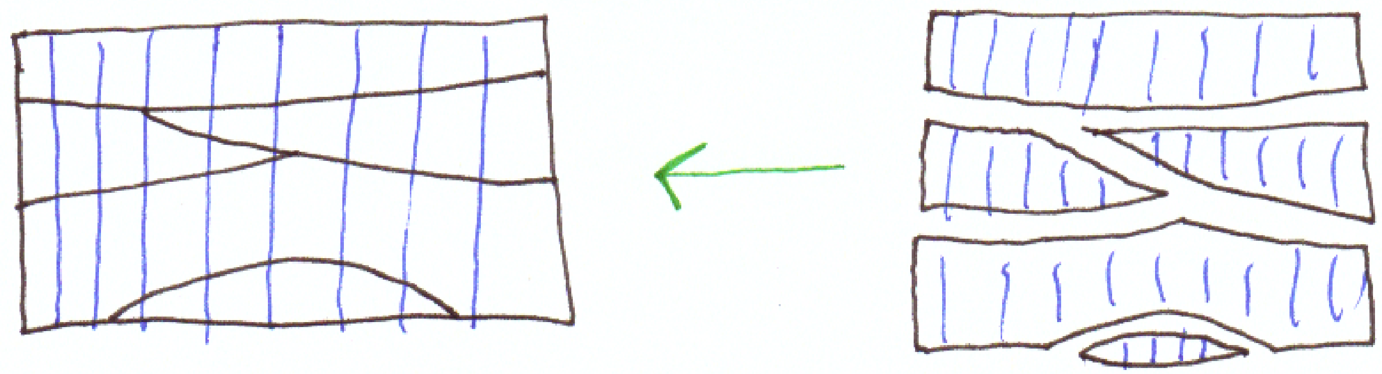

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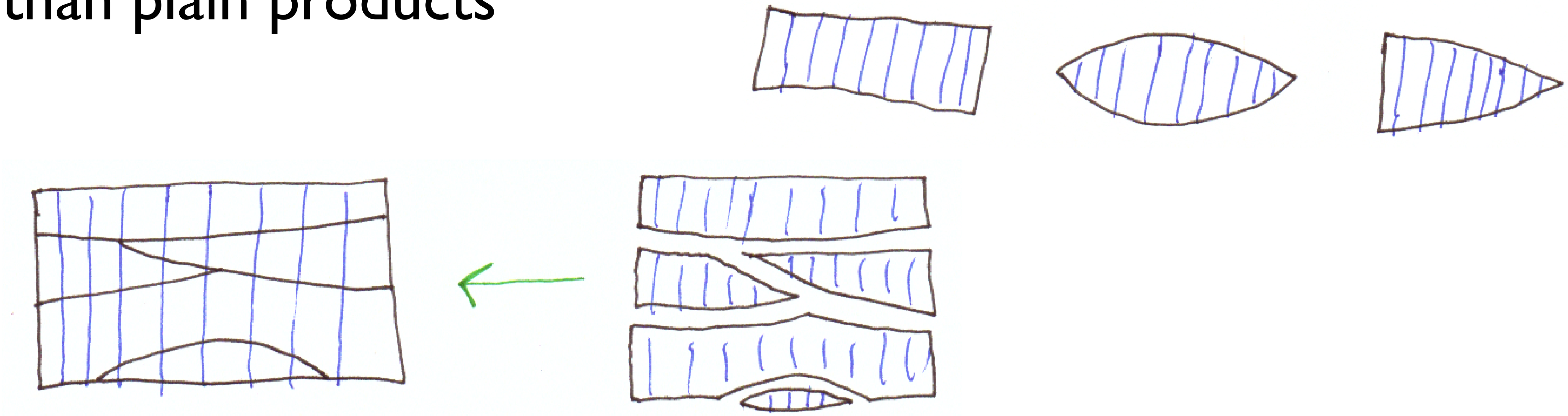
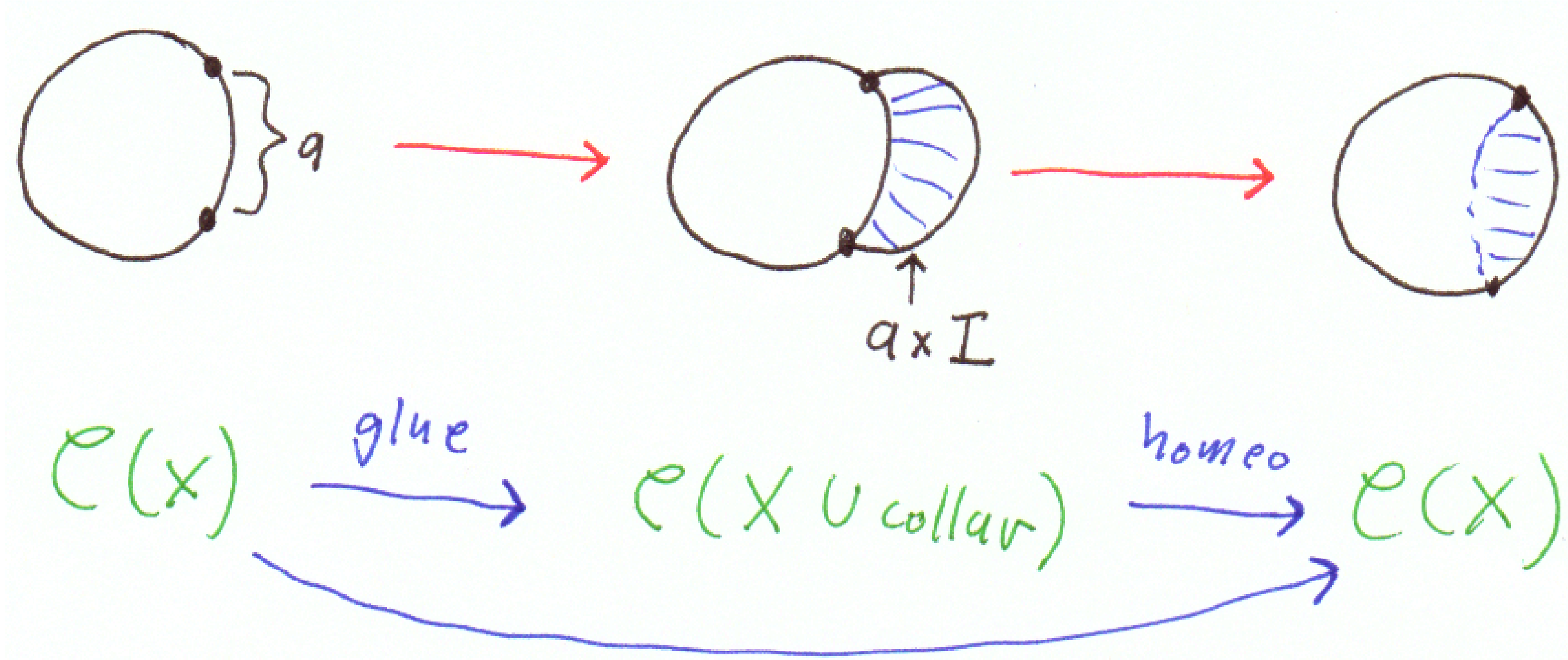


"extended isotopy"



Plain n-cat:

Extended isotopy invariance in dimension n: *Let X be an n-ball and $f : X \to X$ be a homeomorphism which restricts to the identity on ∂X and is extended isotopic (rel boundary) to the identity. Then f acts trivially on $\mathcal{C}(X)$.*

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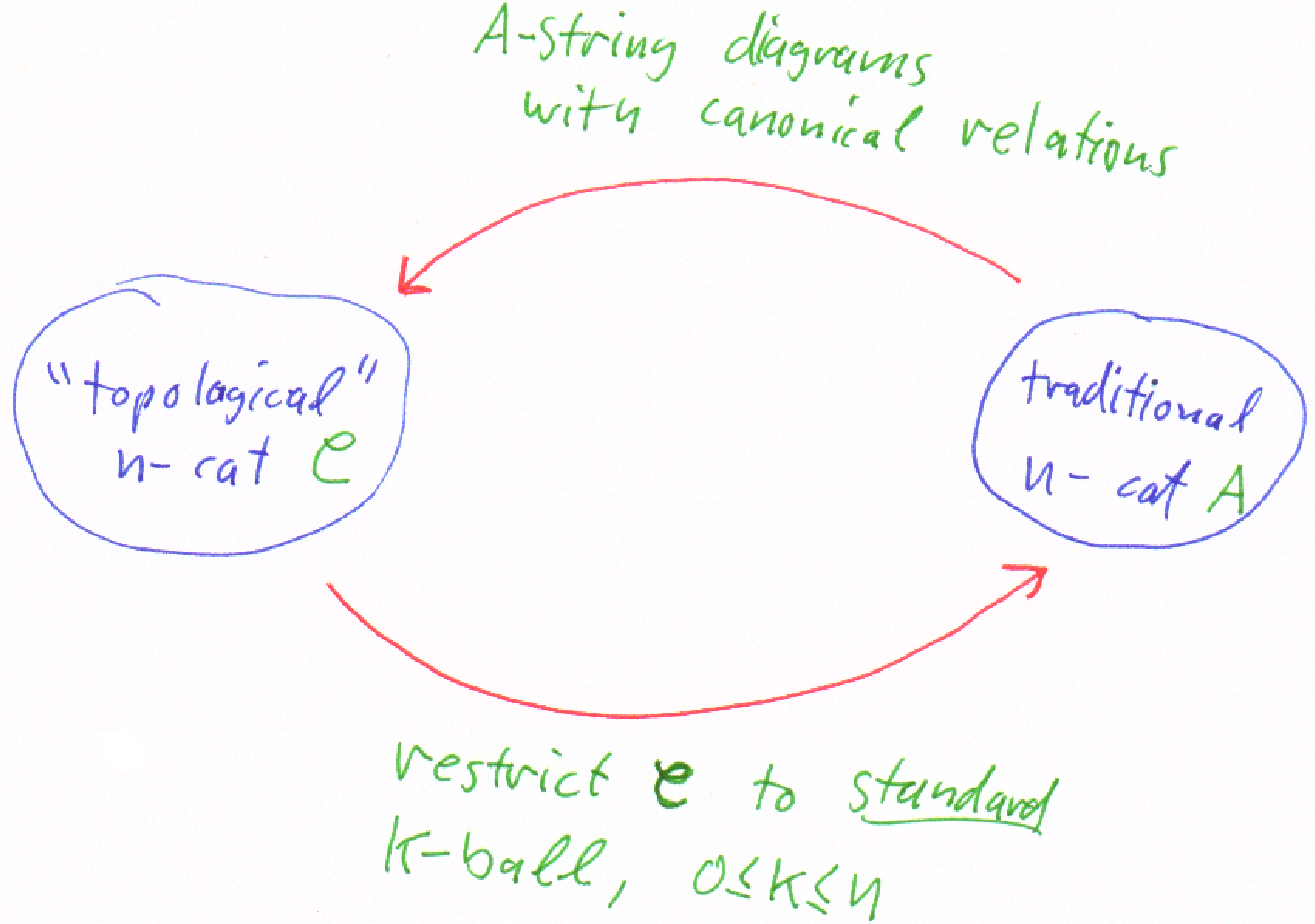
Infinity n-cat:

Families of homeomorphisms act in dimension n. *For each n-ball X and each $c \in \mathcal{C}(\partial X)$ we have a map of chain complexes*

$$C_*(\mathrm{Homeo}_\partial(X)) \otimes \mathcal{C}(X; c) \to \mathcal{C}(X; c).$$

Here C_* means singular chains and $\mathrm{Homeo}_\partial(X)$ is the space of homeomorphisms of X which fix ∂X. These action maps are required to be associative up to homotopy, and also compatible with composition (gluing).

Equivalences between this n-cat definition and more traditional ones (at least for n=1 or 2)



Colimit construction

- Let $\mathcal{C}$ be in n-category.
- We want to extend $\mathcal{C}$ to arbitrary k-manifolds Y, $0 \leq k \leq n$.

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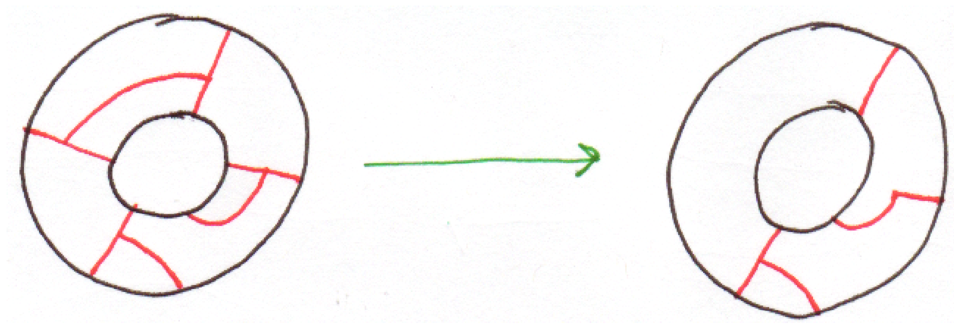

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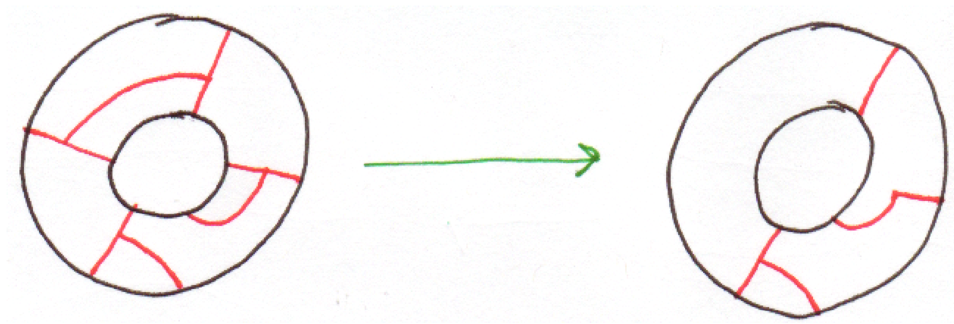

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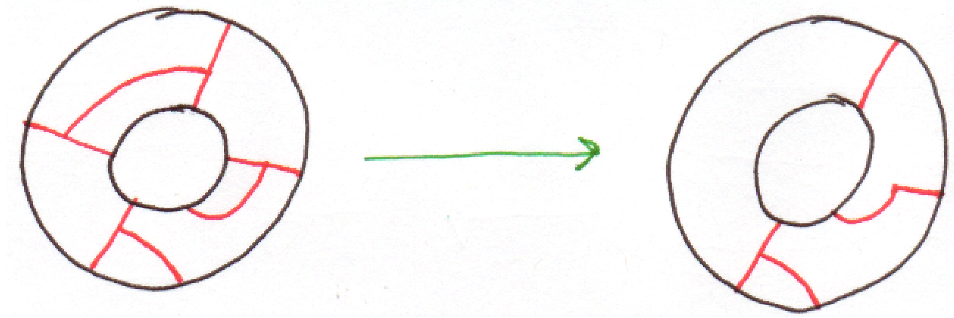

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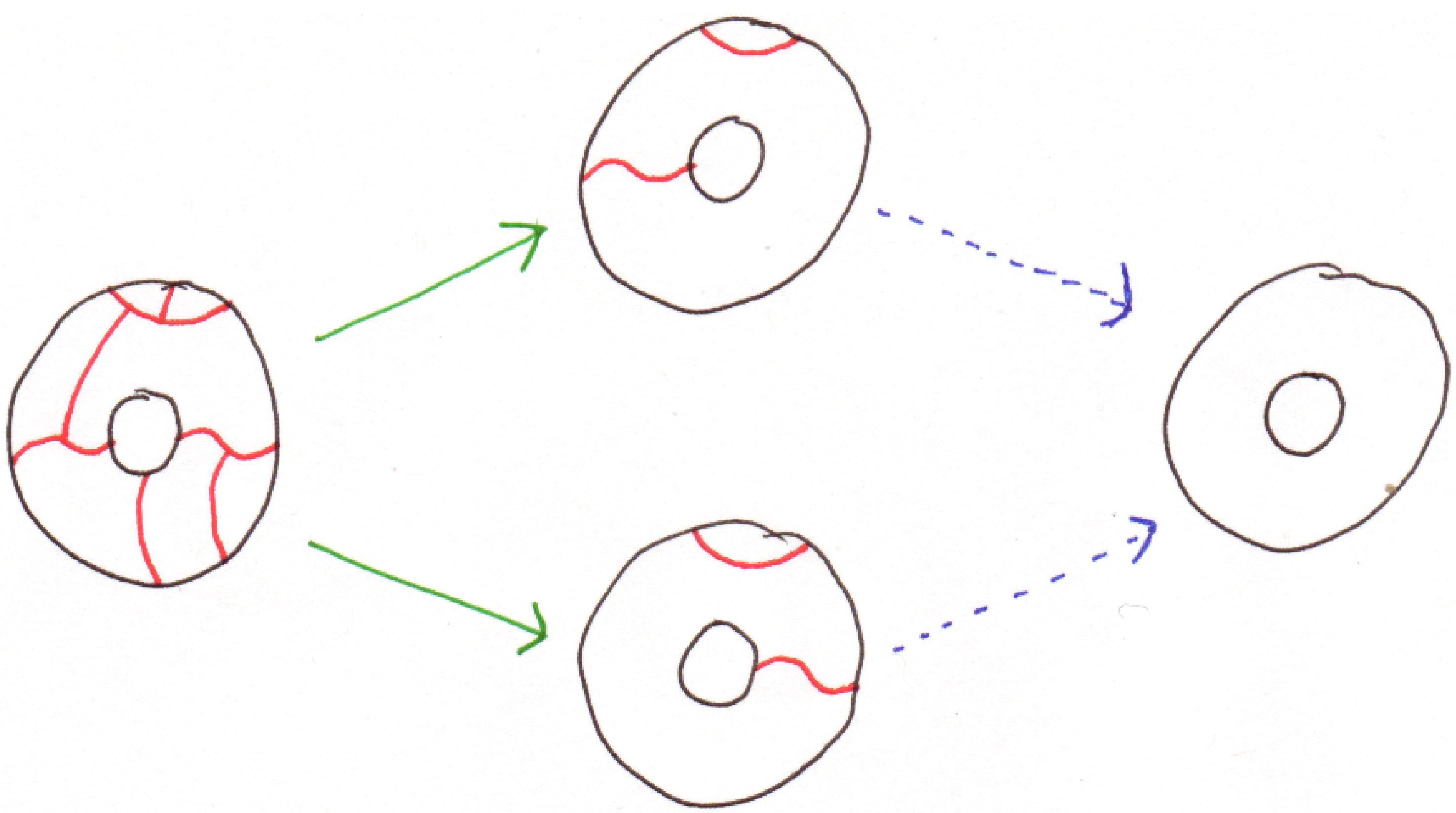
- There is functor which assigns to a decomposition $Y = \bigcup_i X_i$ the set (or vector space or chain complex) $\bigotimes_i \mathcal{C}(X_i)$.
- Define $\mathcal{C}(Y)$ to be the colimit (or homotopy colimit) of this functor.



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    - Let $M^n = F^{n-k} \times Y^k$. Let C be a plain n-category. Let $\mathcal{F}$ be the A_∞ k-category which assigns to a k-ball X the old-fashioned blob complex $\mathcal{B}_*^C(X \times F)$.

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    - Theorem: $\mathcal{F}(Y) \simeq \mathcal{B}_*^C(F \times Y)$.

- Corollary: $\mathcal{D}(M) \simeq \mathcal{B}_*^C(M)$ for any n-manifold M. (Proof: Let F above be a point.) So the old-fashioned and newfangled blob complexes are homotopy equivalent.

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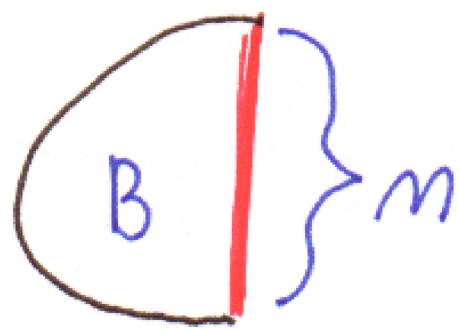

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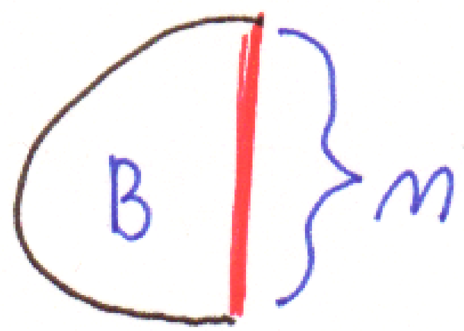

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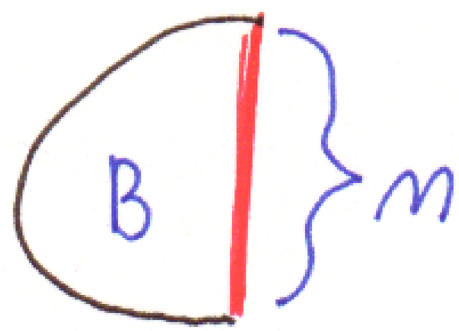

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- In the top dimension n we have the same extra structure as $\mathcal{C}$ (vector space, chain complex, ...).

- Motivating example: Let W be an $m+1$-manifold with non-empty boundary. Let $\mathcal{E}$ be an $m+n$-category.

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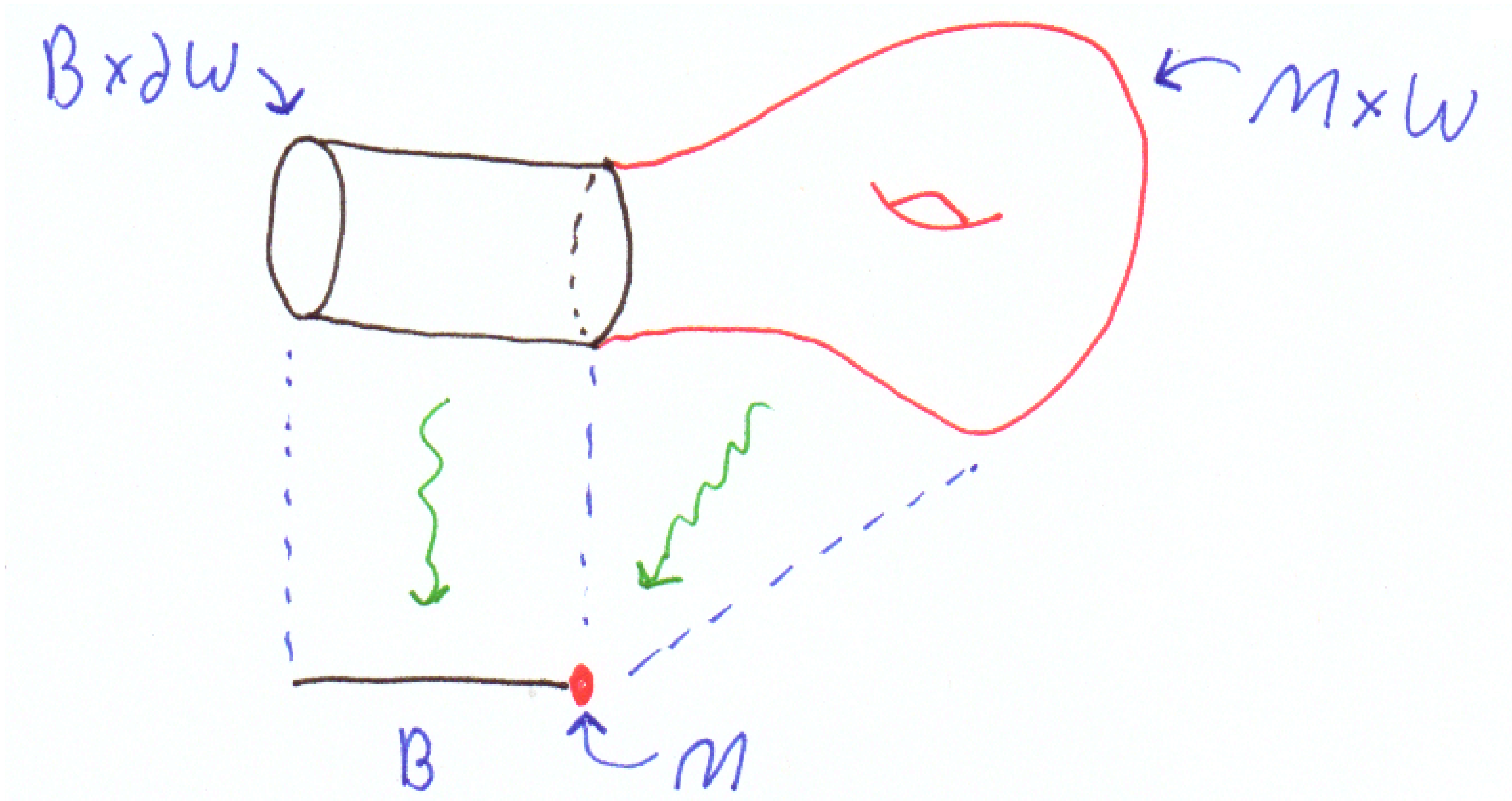
  - Let $\mathcal{C}$ be the n-category with $\mathcal{C}(X) \stackrel{\text{def}}{=} \mathcal{E}(X \times \partial W)$.
    - Define the $\mathcal{C}$-module $\mathcal{M}$ by

$$\mathcal{M}(M, B) \stackrel{\text{def}}{=} \mathcal{E}\left((B \times \partial W) \bigcup_{M \times \partial W} (M \times W) \right).$$



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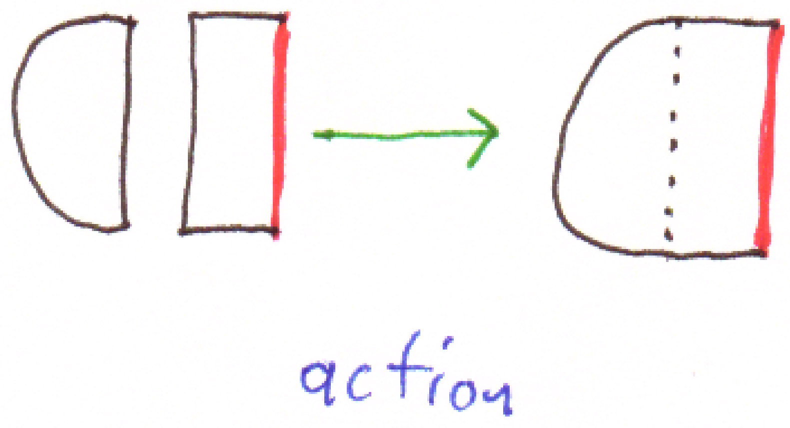

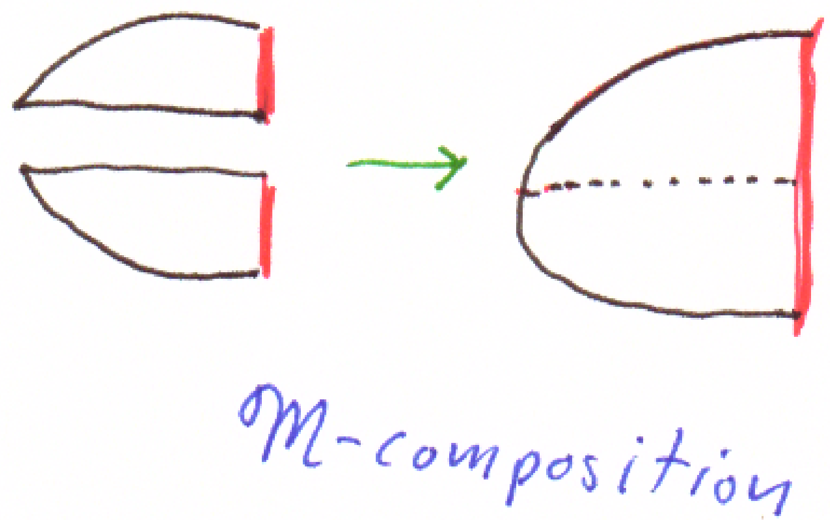

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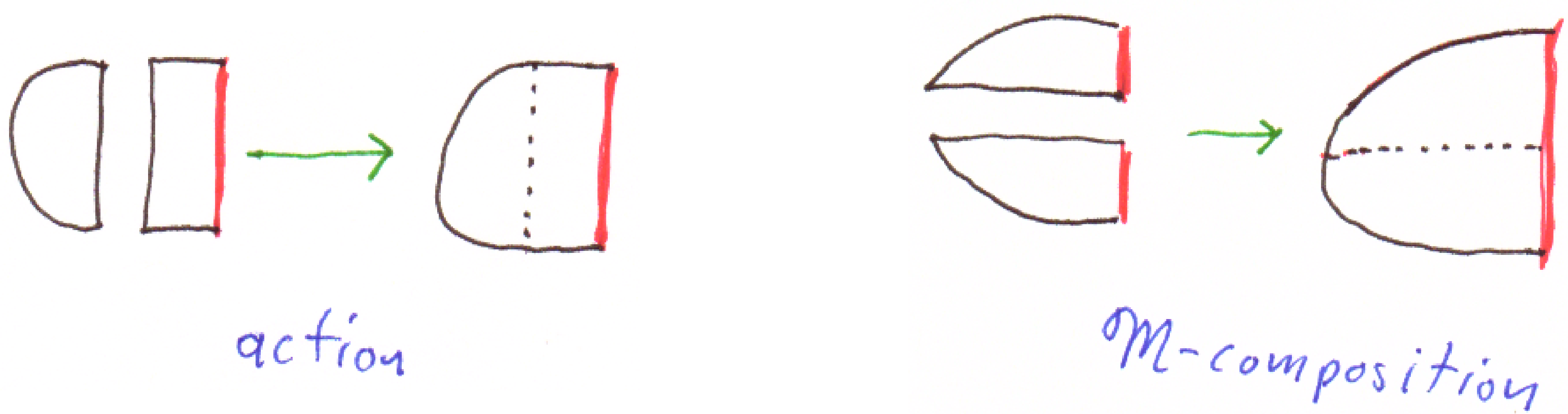
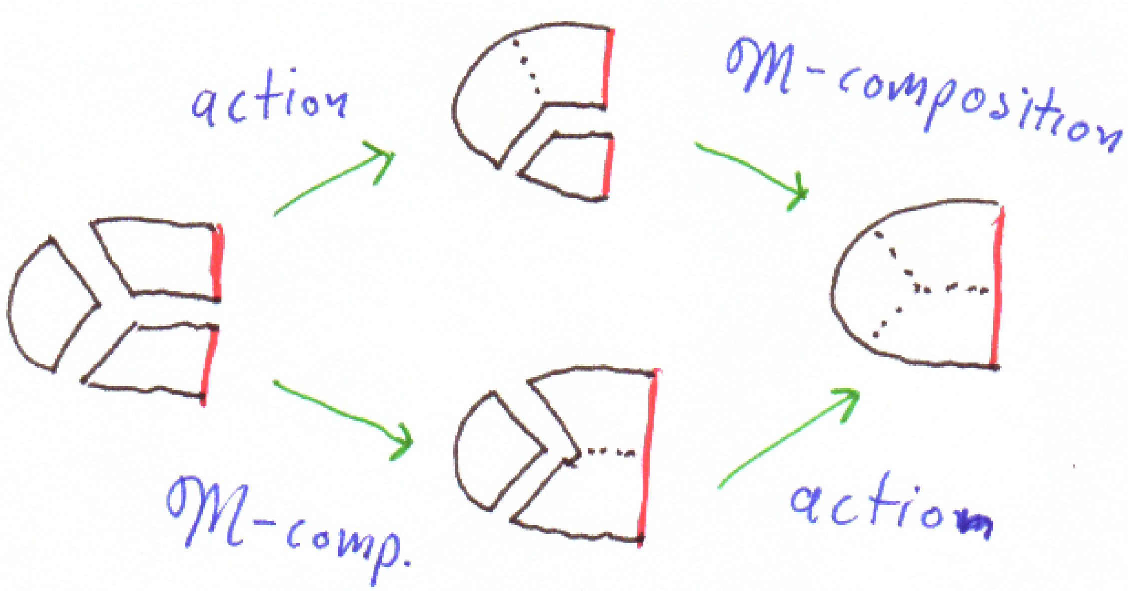
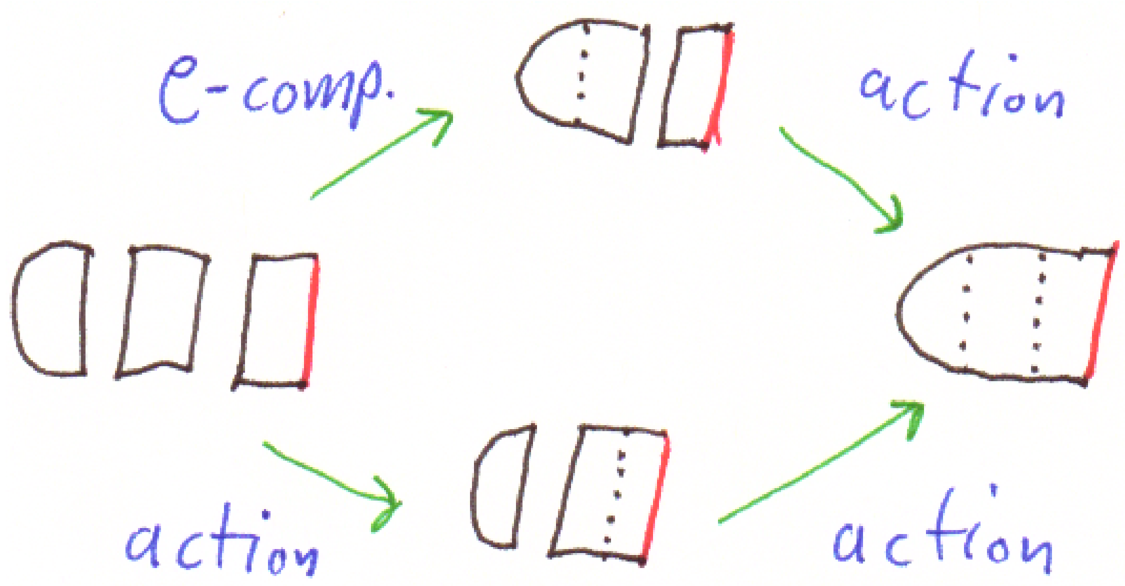


- Various kinds of mixed strict associativity.





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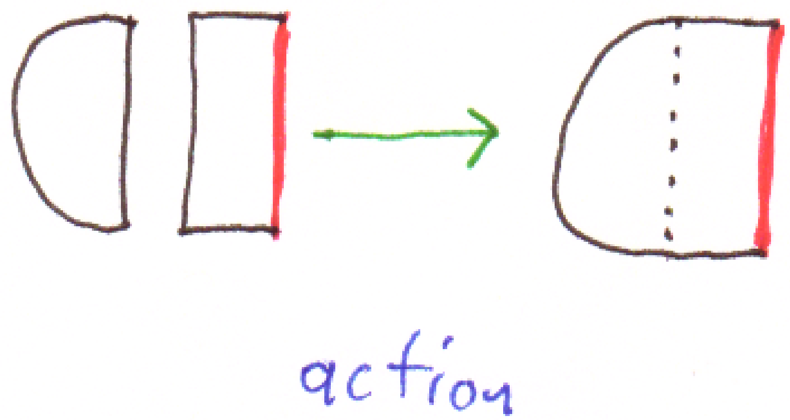

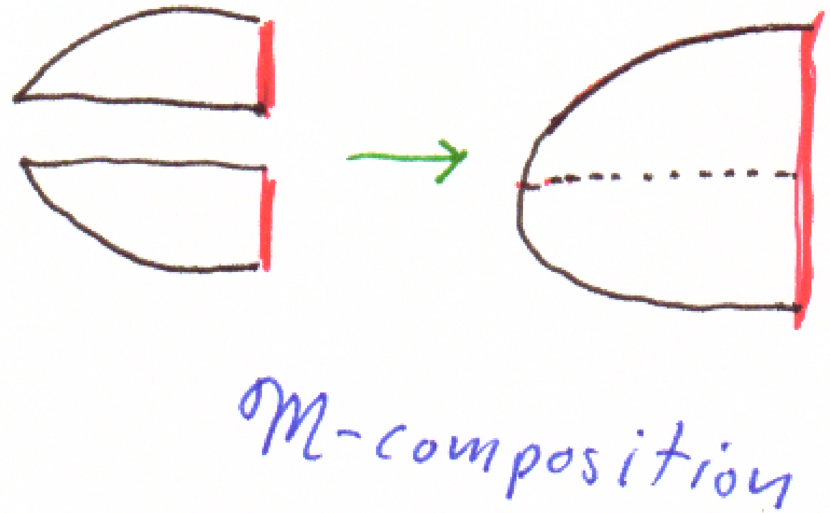

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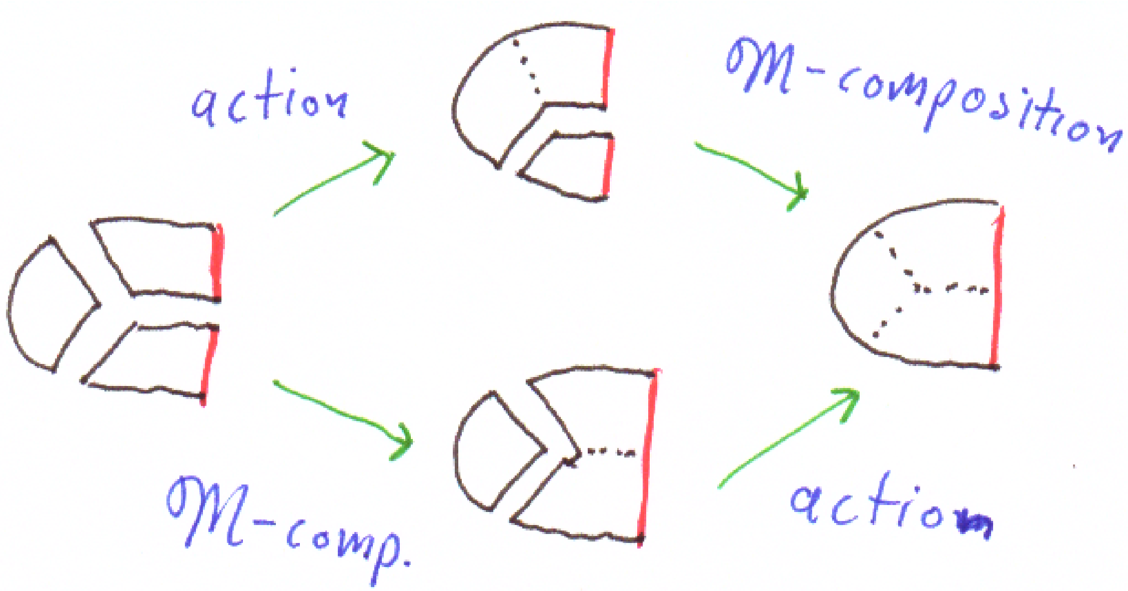

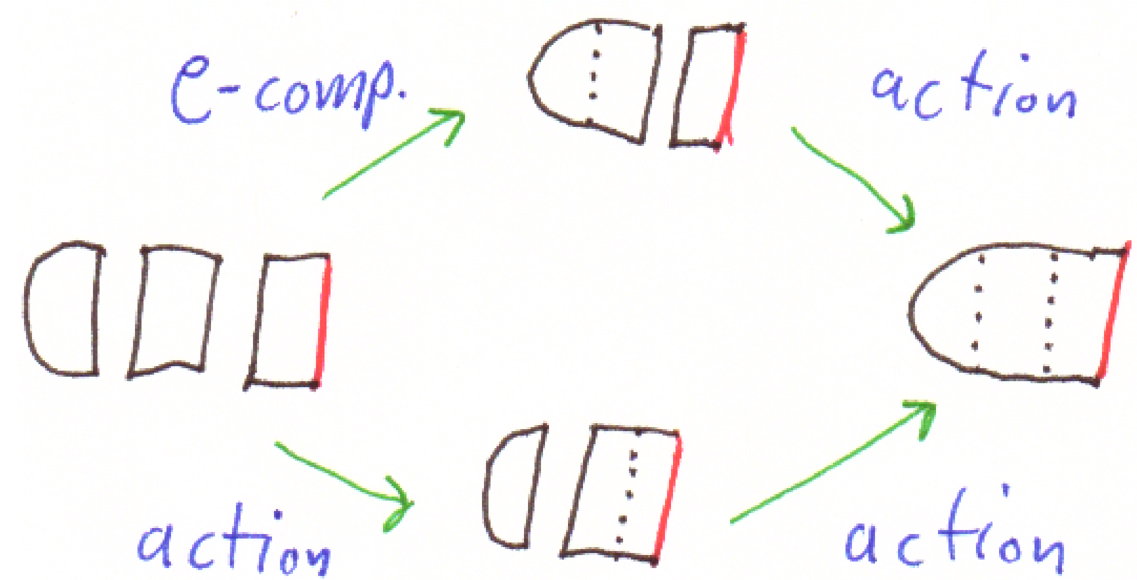

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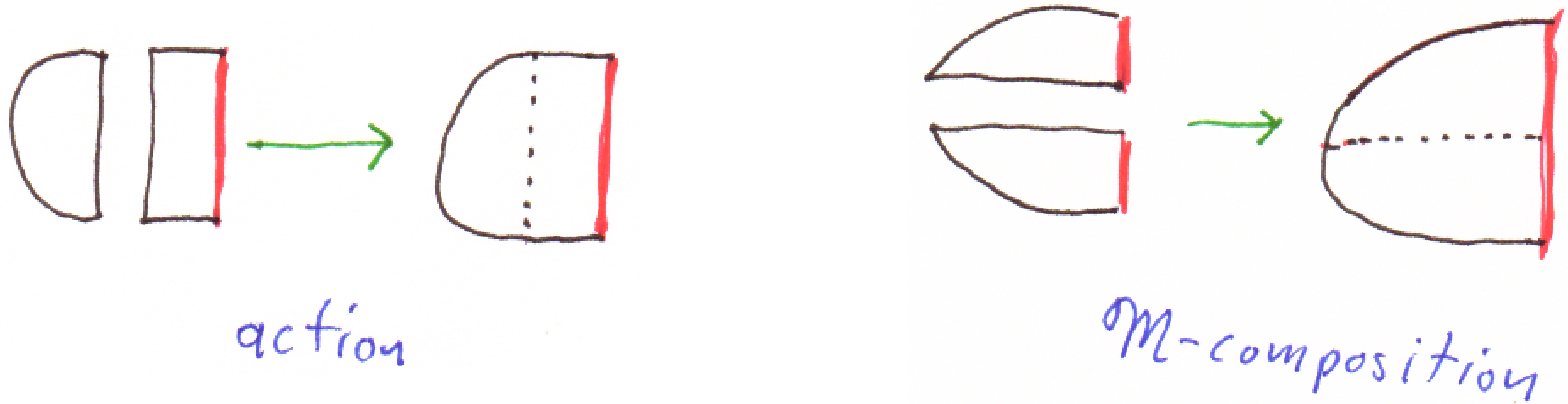

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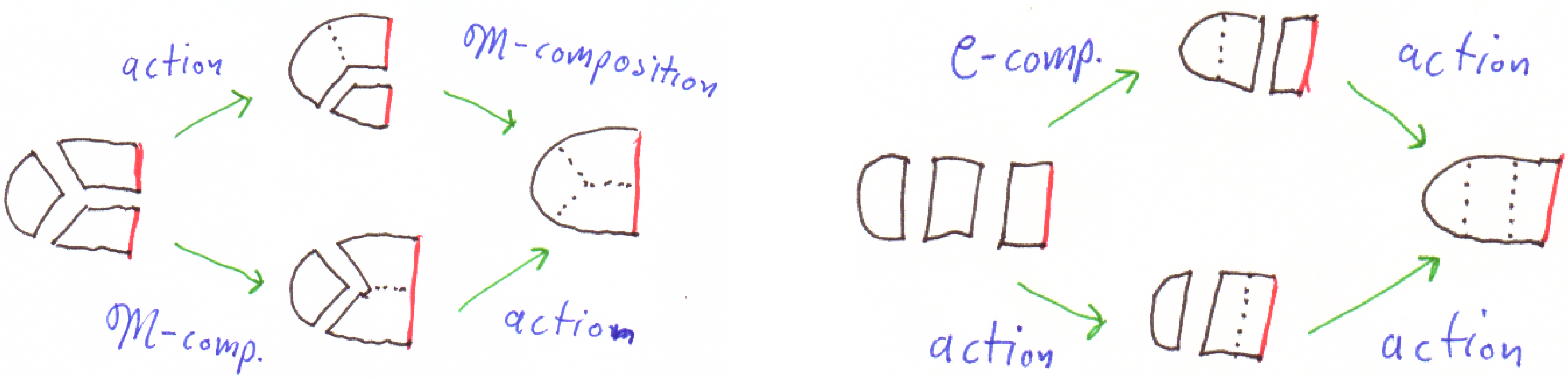

- $\mathcal{M}$ can be thought of as a collection of $n-1$-categories with some extra structure.
- For $n = 1, 2$ this is equivalent to the usual notion of module.

Decorated colimit construction

- Let W be a k-manifold. Let Y_i be a collection of disjoint codimension 0 submanifolds of ∂W.
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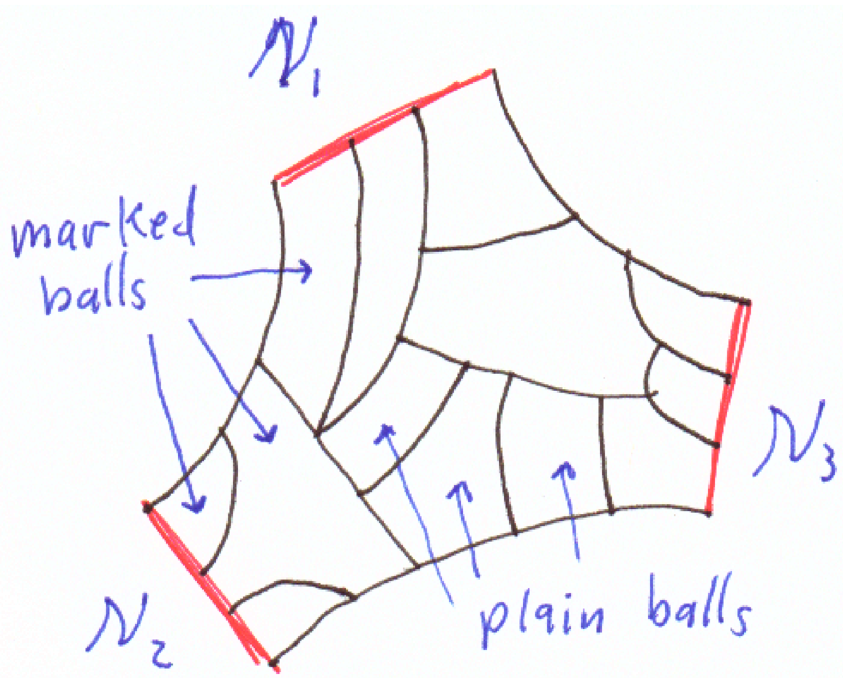

$$\rightsquigarrow \bigotimes_j \mathcal{C}(X_j) \bigotimes_l \mathcal{N}(B_l, M_l)$$

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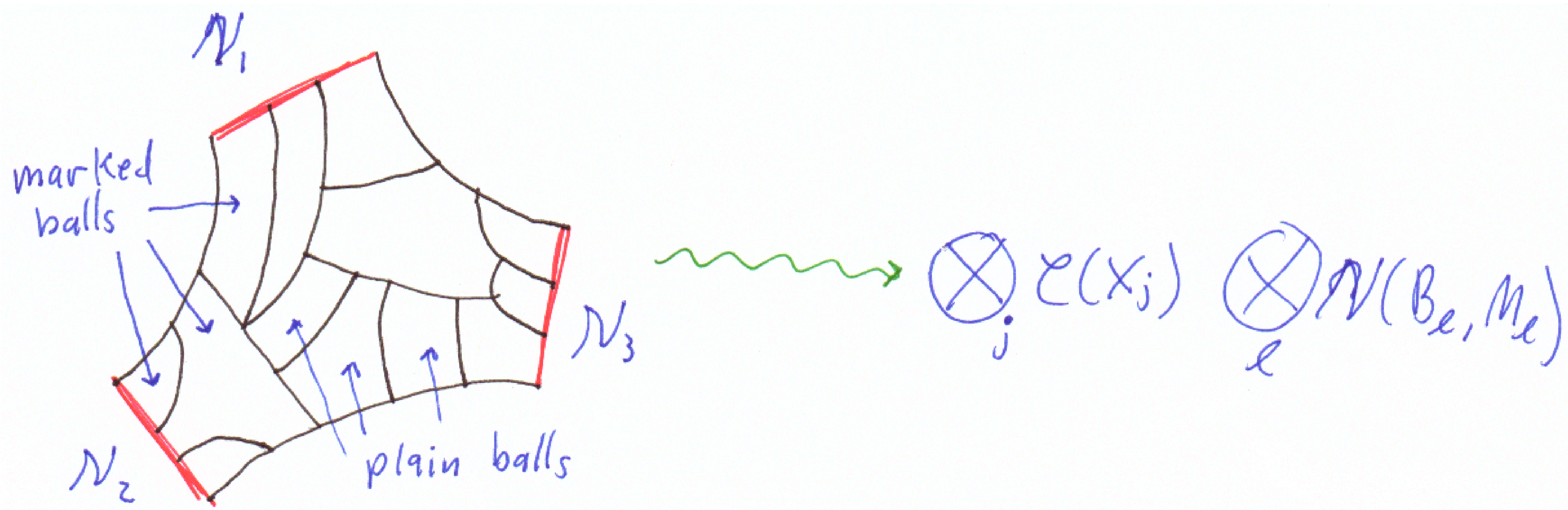
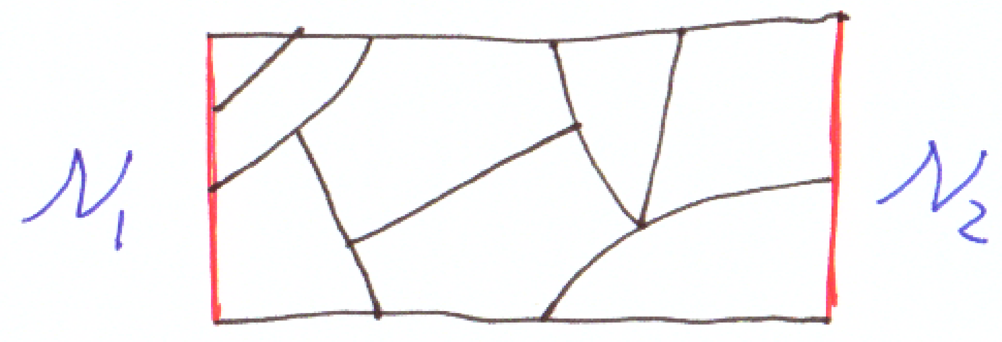


- This defines an $n-k$-category which assigns $\mathcal{C}(D \times W, \mathcal{N})$ to a ball D. (Here $\mathcal{N}_i$ labels $D \times Y_i$.)

Tensor products and gluing

- As a simple special case of this construction, given $\mathcal{C}$-modules $\mathcal{N}_1$ and $\mathcal{N}_2$, define the tensor product $\mathcal{N}_1 \otimes \mathcal{N}_2$ (an $n{-}1$-category) to be the result of taking W to be an interval and letting $\mathcal{N}_1$ and $\mathcal{N}_2$ label the endpoints of the interval.



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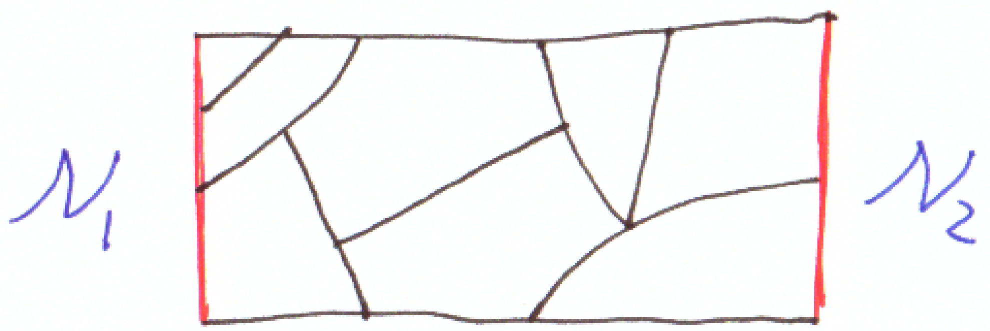

- Gluing theorem: Let $M^{n-k} = M_1 \cup_Y M_2$. Let $\mathcal{C}$ be an n-category. The above constructions give a k-category $\mathcal{C}(M)$, a $k{-}1$-category $\mathcal{C}(Y)$, and two $\mathcal{C}(Y)$-modules $\mathcal{C}(M_i)$. Then

$$\mathcal{C}(M) \simeq \mathcal{C}(M_1) \otimes_{\mathcal{C}(Y)} \mathcal{C}(M_2).$$