The blob complex

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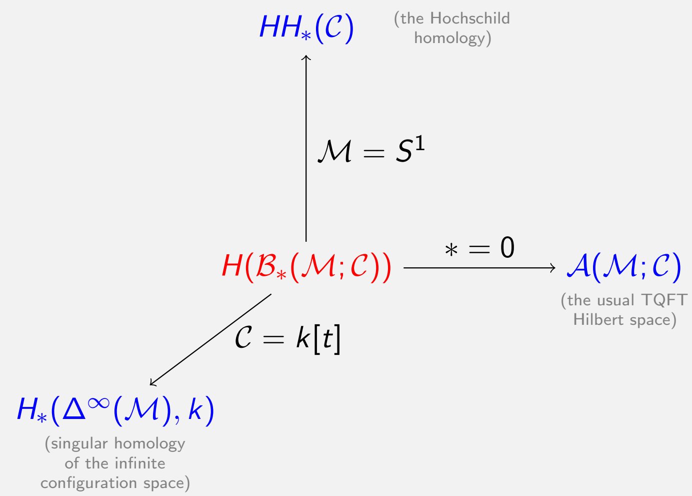
slides: `http://tqft.net/sunysb-blobs`
paper: `http://tqft.net/blobs`

What is *the blob complex*?

The blob complex takes an n-manifold $\mathcal{M}$ and an 'n-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$.



n-categories

There are many definitions of n-categories!

For most of what follows, I'll draw 2-dimensional pictures and rely on your intuition for pivotal 2-categories.

We have another definition!

Many axioms; geometric examples are easy, algebraic ones hard.

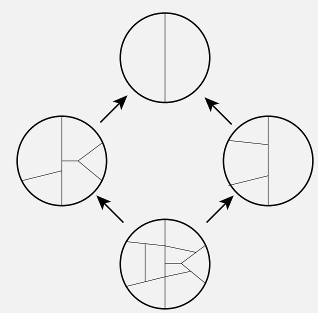
- A vector space $\mathcal{C}(B^n)$ for every n-ball B.
- An associative gluing map: with $B = \bigcup_i B_i$, balls glued together to form a ball,

$$\bigotimes \mathcal{C}(B_i) \to \mathcal{C}(B)$$

  (the $\otimes$ is fibered over 'boundary restriction' maps).
- ...

Cellulations of manifolds

Consider cell(M), the category of cellulations of a manifold M, with morphisms 'antirefinements'.



An n-category $\mathcal{C}$ gives a functor from cell(M) to vector spaces.

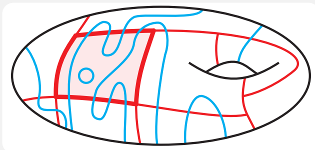
objects send a cellulation to the product of $\mathcal{C}$ on each top-cell, restricting to the subset where boundaries agree

morphisms send an antirefinement to the appropriate gluing map.

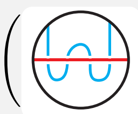
Fields

A field on $\mathcal{M}^n$ is a choice of cellulation and a choice of n-morphism for each top-cell.

Example ($\mathcal{C} = \mathrm{TL}_d$ the Temperley-Lieb category)

 $\in \mathcal{F}(T^2)$

Given a field on a ball, we can evaluate it to a morphism using the gluing map. We call the kernel the *null fields*.

ev( $- \frac{1}{d}$ ) $= 0$

Background: TQFT invariants

Definition

We associate to an n-manifold $\mathcal{M}$ *the skein module*

$$\mathcal{A}(\mathcal{M}) = \mathcal{F}(\mathcal{M}) / \ker ev,$$

fields modulo fields which evaluate to zero inside some ball.

Equivalently, $\mathcal{A}(\mathcal{M})$ is the colimit of $\mathcal{C}$ along $\mathrm{cell}(M)$.

$\mathcal{A}(Y \times [0,1])$ is a 1-category, and when $Y \subset \partial X$, $\mathcal{A}(X)$ is a module over $\mathcal{A}(Y \times [0,1])$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{A}(X \bigcup_Y \circlearrowleft) \cong \mathcal{A}(X) \bigotimes_{\mathcal{A}(Y \times [0,1])} \circlearrowleft.$$

Motivation: Khovanov homology as a 4d TQFT

Theorem

Khovanov homology gives a 4-category:

3-morphisms *tangles, with the usual* 3 *operations,*

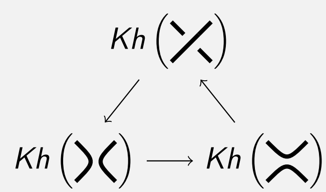
4-morphisms $\mathrm{Hom}_{Kh}(T_1, T_2) = Kh(T_1 \cup \bar{T}_2)$, *composition defined by saddle cobordisms*

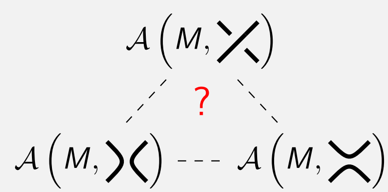
There is a corresponding 4-manifold invariant. Given $L \subset \partial W^4$, it associates a doubly-graded vector space $\mathcal{A}(W, L; Kh)$.

$$\mathcal{A}(B^4, L; Kh) \cong Kh(L)$$

Computations are hard

This invariant is hard to compute, because the TQFT skein module construction breaks the exact triangle for resolving a crossing.





There is a spectral sequence converging to 0 relating the blob homologies for the triangle of resolutions.

Conjecture

It may be possible to compute the skein module by first computing the entire blob homology.

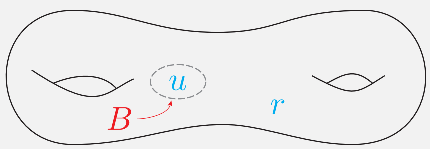
Definition of the blob complex, $k = 0, 1$

Motivation

A *local* construction, such that when $\mathcal{M}$ is a ball, $\mathcal{B}_*(\mathcal{M}; \mathcal{C})$ is a resolution of $\mathcal{A}(\mathcal{M}; \mathcal{C})$.

$$\mathcal{B}_0(\mathcal{M}; \mathcal{C}) = \mathcal{F}(\mathcal{M}), \text{ arbitrary fields on } \mathcal{M}.$$

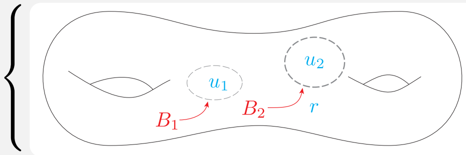
$$\mathcal{B}_1(\mathcal{M}; \mathcal{C}) = \mathbb{C}\left\{ (B, u, r) \;\middle|\; \begin{array}{c} B \text{ an embedded ball} \\ u \in \mathcal{F}(B) \text{ in the kernel} \\ r \in \mathcal{F}(\mathcal{M} \setminus B) \end{array} \right\}.$$



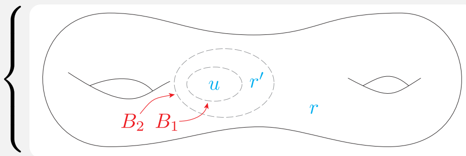
$d_1 : (B, u, r) \mapsto u \circ r$ $\qquad$ $\mathcal{B}_0 / \operatorname{im}(d_1) \cong A(\mathcal{M}; \mathcal{C})$

Definition, $k = 2$

$$\mathcal{B}_2 = \mathcal{B}_2^{\text{disjoint}} \oplus \mathcal{B}_2^{\text{nested}}$$

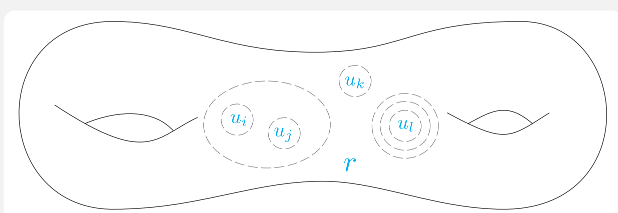
$\mathcal{B}_2^{\text{disjoint}} = \mathbb{C}\Bigg\{$  $\Bigg|\; \mathrm{ev}_{B_i}(u_i) = 0 \Bigg\}$

$$d_2 : (B_1, B_2, u_1, u_2, r) \mapsto (B_2, u_2, r \circ u_1) - (B_1, u_1, r \circ u_2)$$

$\mathcal{B}_2^{\text{nested}} = \mathbb{C}\Bigg\{$  $\Bigg|\; \mathrm{ev}_{B_1}(u) = 0 \Bigg\}$

$$d_2 : (B_1, B_2, u, r', r) \mapsto (B_2, u \circ r', r) - (B_1, u, r \circ r')$$

Definition, general case

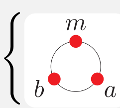
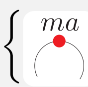
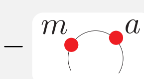
$\mathcal{B}_k = \mathbb{C}\Bigg\{$  $\Bigg\}$

k blobs, properly nested or disjoint, with "innermost" blobs labelled by fields that evaluate to zero.

$$d_k : \mathcal{B}_k \to \mathcal{B}_{k-1} = \textstyle\sum_i (-1)^i (\text{erase blob } i)$$

Hochschild homology

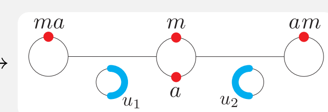
TQFT on S^1 is 'coinvariants'

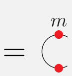
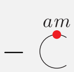
$\mathcal{A}(S^1, A) = \mathbb{C}\Big\{$  $\Big\} \Big/ \Big\{$  $-$  $\Big\} = A/(ab - ba)$

The Hochschild complex is 'coinvariants of the bar resolution'

$$\cdots \to A \otimes A \otimes A \to A \otimes A \xrightarrow{m \otimes a \mapsto ma - am} A$$

Theorem ($\mathrm{Hoch}_*(A) \cong \mathcal{B}_*(S^1; A)$)

$m \otimes a \mapsto$ 

$u_1 =$  $-$  $\qquad u_2 =$  $-$ 

An action of $C_*(\text{Homeo}(\mathcal{M}))$

Theorem

There's a chain map

$$C_*(\text{Homeo}(\mathcal{M})) \otimes \mathcal{B}_*(\mathcal{M}) \to \mathcal{B}_*(\mathcal{M}).$$

which is associative up to homotopy, and compatible with gluing.

Taking H_0, this is the mapping class group acting on a TQFT skein module.

Gluing

$\mathcal{B}_*(Y \times [0,1])$ is naturally an A_∞ category

multiplication (m_2): gluing $[0,1] \simeq [0,1] \cup [0,1]$

associativity up to homotopy (m_k): reparametrising $[0,1]$ using the action of $C_*(\text{Homeo}([0,1]))$.

If $Y \subset \partial X$ then $\mathcal{B}_*(X)$ is an A_∞ module over $\mathcal{B}_*(Y)$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{B}_*(X \bigcup_Y \circlearrowleft) \cong \mathcal{B}_*(X) \bigotimes_{\mathcal{B}_*(Y)}^{A_\infty} \circlearrowleft.$$

In principle, we can compute blob homology from a handle decomposition, by iterated Hochschild homology.

Higher Deligne conjecture

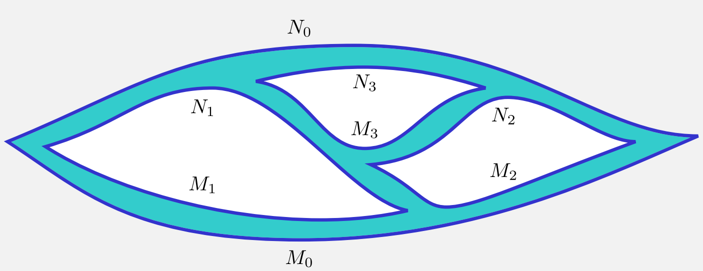
Deligne conjecture

Chains on the little discs operad acts on Hochschild cohomology.

Call $\text{Hom}_{\mathcal{B}_*(\partial M)}(\mathcal{B}_*(\mathcal{M}), \mathcal{B}_*(\mathcal{M}))$ 'blob cochains on $\mathcal{M}$'.

Theorem (Higher Deligne conjecture)

Chains on the n-dimensional fat graph operad acts on blob cochains.



Maps to a space

Fix a target space $\mathcal{T}$. There is an A_∞ n-category $\pi^\infty_{\leq n}(\mathcal{T})$ defined by

$$\pi^\infty_{\leq n}(\mathcal{T})(B) = C_*(\text{Maps}(B \to \mathcal{T})).$$

Theorem

The blob complex recovers mapping spaces:

$$\mathcal{B}_*(\mathcal{M}; \pi^\infty_{\leq n}(\mathcal{T})) \cong C_*(\text{Maps}(\mathcal{M} \to \mathcal{T}))$$

This generalizes a result of Lurie: if $\mathcal{T}$ is $n-1$ connected, $\pi^\infty_{\leq n}(\mathcal{T})$ is an E_n-algebra and the blob complex is the same as his topological chiral homology.