The blob complex

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Low-Dimensional Topology and Categorification,
Stony Brook University, June 21-25 2010

slides: `http://tqft.net/talks`

paper: `http://tqft.net/blobs`

The blob complex

> *... homotopical topology and TQFT have grown so close that I have started thinking that they are turning into the language of new foundations.*
>
> — Yuri Manin, September 2008

1. Overview
2. TQFTs
3. Definition
4. Properties

What is *the blob complex*?

The blob complex takes an n-manifold $\mathcal{M}$ and an 'n-category with strong duality' $\mathcal{C}$ and produces a chain complex, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$.

$$\begin{tikzcd}
 & HH_*(\mathcal{C}) & \\
 & H(\mathcal{B}_*(\mathcal{M};\mathcal{C})) \arrow[u, "\mathcal{M} = S^1"'] \arrow[r, "* = 0"] \arrow[dl, "\mathcal{C} = k[t]"] & \mathcal{A}(\mathcal{M};\mathcal{C}) \\
H_*(\Delta^\infty(\mathcal{M}), k) & &
\end{tikzcd}$$

(the Hochschild homology)

(the usual TQFT Hilbert space)

(singular homology of the infinite configuration space)

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n-categories

There are many definitions of *n*-categories!

For most of what follows, I'll draw 2-dimensional pictures and rely on your intuition for pivotal categories.

We have yet another definition: *topological n-categories*

- A vector space $\mathcal{C}(B^n)$ for every *n*-ball B.
- An associative gluing map: with $B = \bigcup_i B_i$, balls glued together to form a ball,

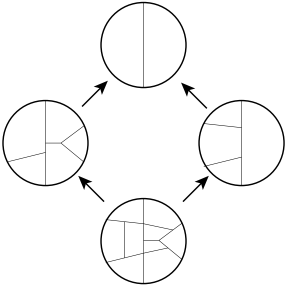
  $$\bigotimes \mathcal{C}(B_i) \to \mathcal{C}(B)$$

  (the $\otimes$ is fibered over 'boundary restriction' maps).
- ...

These are easy to check for geometric examples, hard to check for algebraic examples.

Cellulations of manifolds

Consider cell(M), the category of cellulations of a manifold M, with morphisms 'antirefinements'.



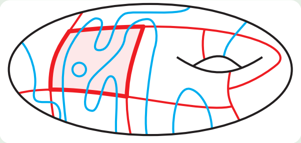
An n-category $\mathcal{C}$ gives a functor from cell(M) to vector spaces.

objects	send a cellulation to the product of $\mathcal{C}$ on each top-cell, restricting to the subset where boundaries agree
morphisms	send an antirefinement to the appropriate gluing map.

Fields

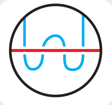
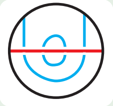
A field on $\mathcal{M}^n$ is a choice of cellulation and a choice of n-morphism for each top-cell (with matching boundaries).

Example ($\mathcal{C} = \mathrm{TL}_d$ the Temperley-Lieb category)



$\in \mathcal{F}\left(T^2\right)$

Given a field on a ball, we can evaluate it to a morphism using the gluing map. We call the kernel the *null fields*.

$\mathrm{ev}\Bigg($  $- \frac{1}{d}$  $\Bigg) = 0$

Background: TQFT invariants

Definition

We associate to an n-manifold $\mathcal{M}$ the skein module

$$\mathcal{A}(\mathcal{M}) = \mathcal{F}(\mathcal{M}) / \ker ev,$$

fields modulo fields which evaluate to zero inside some ball.

Equivalently, $\mathcal{A}(\mathcal{M})$ is the colimit of $\mathcal{C}$ along $\mathrm{cell}(M)$.

$\mathcal{A}(Y^{n-1} \times [0,1])$ is a 1-category, and when $Y \subset \partial X$, $\mathcal{A}(X)$ is a module over $\mathcal{A}(Y \times [0,1])$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{A}(X \bigcup_{Y} \circlearrowleft) \cong \mathcal{A}(X) \bigotimes_{\mathcal{A}(Y \times [0,1])} \circlearrowleft.$$

Motivation: Khovanov homology as a 4d TQFT

Theorem

Khovanov homology gives a 4-category:

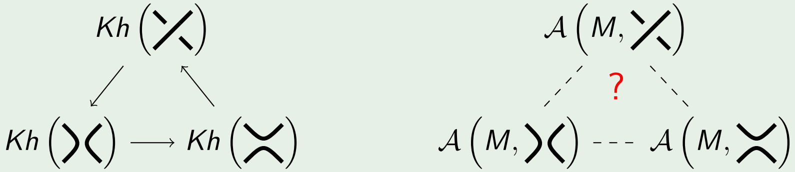
3-morphisms	*tangles, with the usual* 3 *operations,*
4-morphisms	$\mathrm{Hom}_{Kh}(T_1, T_2) = Kh(T_1 \cup \bar{T}_2)$, *composition defined by saddle cobordisms*

There is a corresponding 4-manifold invariant. Given $L \subset \partial W^4$, it associates a doubly-graded vector space $\mathcal{A}(W, L; Kh)$.

$$\mathcal{A}(B^4, L; Kh) \cong Kh(L)$$

Computations are hard

This invariant is hard to compute, because the TQFT skein module construction breaks the exact triangle for resolving a crossing.



There is a spectral sequence converging to 0 relating the blob homologies for the triangle of resolutions.

Conjecture

It may be possible to compute the skein module by first computing the entire blob homology.

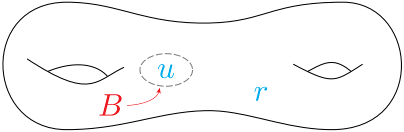
Definition of the blob complex, $k = 0, 1$

Motivation

A *local* construction, such that when $\mathcal{M}$ is a ball, $\mathcal{B}_*(\mathcal{M};\mathcal{C})$ is a resolution of $\mathcal{A}(\mathcal{M};\mathcal{C})$.

$\mathcal{B}_0(\mathcal{M};\mathcal{C}) = \mathcal{F}(\mathcal{M})$, arbitrary fields on $\mathcal{M}$.

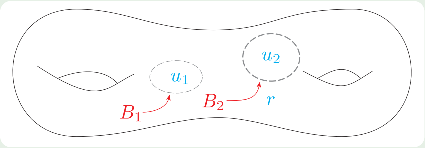
$$\mathcal{B}_1(\mathcal{M};\mathcal{C}) = \mathbb{C}\left\{ (B, u, r) \;\middle|\; \begin{array}{c} B \text{ an embedded ball} \\ u \in \mathcal{F}(B) \text{ in the kernel} \\ r \in \mathcal{F}(\mathcal{M} \setminus B) \end{array} \right\}.$$



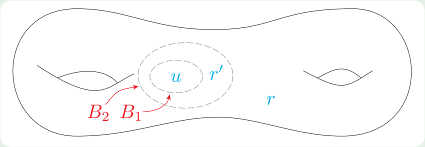
$d_1 : (B, u, r) \mapsto u \circ r$ $\qquad$ $\mathcal{B}_0 / \operatorname{im}(d_1) \cong A(\mathcal{M};\mathcal{C})$

Definition, $k = 2$

$$\mathcal{B}_2 = \mathcal{B}_2^{\text{disjoint}} \oplus \mathcal{B}_2^{\text{nested}}$$

$\mathcal{B}_2^{\text{disjoint}} = \mathbb{C}\Bigg\{$  $\Bigg| \; \mathrm{ev}_{B_i}(u_i) = 0 \Bigg\}$

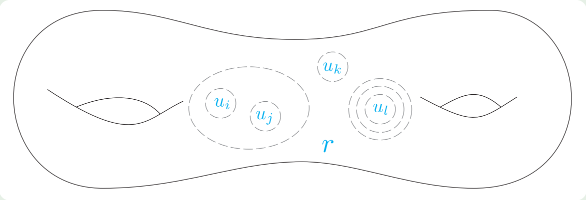
$$d_2 : (B_1, B_2, u_1, u_2, r) \mapsto (B_2, u_2, r \circ u_1) - (B_1, u_1, r \circ u_2)$$

$\mathcal{B}_2^{\text{nested}} = \mathbb{C}\Bigg\{$  $\Bigg| \; \mathrm{ev}_{B_1}(u) = 0 \Bigg\}$

$$d_2 : (B_1, B_2, u, r', r) \mapsto (B_2, u \circ r', r) - (B_1, u, r \circ r')$$

Definition, general case

$\mathcal{B}_k = \mathbb{C}\Bigg\{$



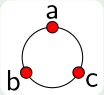
$\Bigg\}$

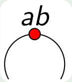
k blobs, properly nested or disjoint, with “innermost” blobs labelled by fields that evaluate to zero.

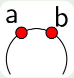
$$d_k : \mathcal{B}_k \to \mathcal{B}_{k-1} = \sum_i (-1)^i (\text{erase blob } i)$$

Hochschild homology

TQFT on S^1 is 'coinvariants'

$\mathcal{A}(S^1, A) = \mathbb{C}\left\{\right.$ 

$\left.\right\} \Big/ \left\{\right.$ 

$-$ 

$\left.\right\} = A/(ab - ba)$

Blob homology on S^1 is Hochschild homology

The Hochschild complex is 'coinvariants of the bar resolution'

$$\cdots \to A \otimes A \otimes A \to A \otimes A \xrightarrow{m \otimes a \mapsto ma - am} A$$

We check universal properties, as it's hard to directly construct an isomorphism.

An action of $C_*(\mathrm{Homeo}(\mathcal{M}))$

Theorem

There's a chain map

$$C_*(\mathrm{Homeo}(\mathcal{M})) \otimes \mathcal{B}_*(\mathcal{M}) \to \mathcal{B}_*(\mathcal{M}).$$

which is associative up to homotopy, and compatible with gluing.

Taking H_0, this is the mapping class group acting on a TQFT skein module.

$$H_0(\mathrm{Homeo}(\mathcal{M})) \otimes \mathcal{A}(\mathcal{M}) \to \mathcal{A}(\mathcal{M}).$$

An action of $C_*(\mathrm{Homeo}(\mathcal{M}))$

Proof.

Uniqueness:

Step 1 If $\mathcal{M} = B^n$ or a union of balls, there's a unique (up to homotopy) chain map, since $\mathcal{B}_*(B^n; \mathcal{C}) \simeq \mathcal{C}$ is concentrated in homological degree 0.

Step 2 Fix an open cover $\mathcal{U}$ of balls.
A family of homeomorphisms $P^k \to \mathrm{Homeo}(\mathcal{M})$ can be broken up in into pieces, each of which is supported in at most k open sets from $\mathcal{U}$.

Existence:

Step 3 Show that all of the choices available above can be made consistently, using the method of acyclic models. ☐

Gluing

$\mathcal{B}_*(Y \times [0,1])$ is naturally an A_∞ category

multiplication (m_2): gluing $[0,1] \simeq [0,1] \cup [0,1]$

associativity up to homotopy (m_k): reparametrising $[0,1]$ using the action of $C_*(\mathrm{Homeo}([0,1]))$.

If $Y \subset \partial X$ then $\mathcal{B}_*(X)$ is an A_∞ module over $\mathcal{B}_*(Y)$.

Theorem (Gluing formula)

When $Y \sqcup Y^{op} \subset \partial X$,

$$\mathcal{B}_*(X \bigcup_{Y} \circlearrowleft) \cong \mathcal{B}_*(X) \overset{A_\infty}{\underset{\mathcal{B}_*(Y)}{\bigotimes}} \circlearrowleft.$$

In principle, we can compute blob homology from a handle decomposition, by iterated Hochschild homology.

Higher Deligne conjecture

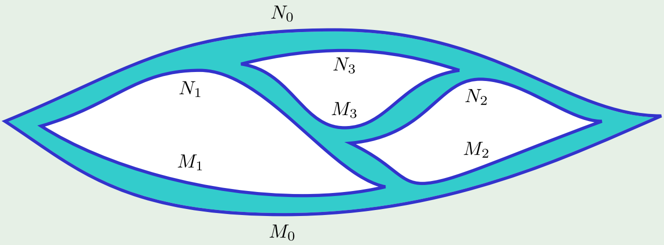
Deligne conjecture

Chains on the little discs operad acts on Hochschild cohomology.

Call $\mathrm{Hom}_{\mathcal{B}_*(\partial M)}(\mathcal{B}_*(\mathcal{M}), \mathcal{B}_*(\mathcal{M}))$ 'blob cochains on $\mathcal{M}$'.

Theorem (Higher Deligne conjecture)

Chains on the *n*-dimensional fat graph operad acts on blob cochains.



Maps to a space

Fix a target space $\mathcal{T}$. There is an A_∞ n-category $\pi^\infty_{\leq n}(\mathcal{T})$ defined by

$$\pi^\infty_{\leq n}(\mathcal{T})(B) = C_*(\mathrm{Maps}(B \to \mathcal{T})).$$

(Here B is an n-ball.)

Theorem

The blob complex recovers mapping spaces:

$$\mathcal{B}_*(\mathcal{M}; \pi^\infty_{\leq n}(\mathcal{T})) \cong C_*(\mathrm{Maps}(\mathcal{M} \to \mathcal{T}))$$

This generalizes a result of Lurie: if $\mathcal{T}$ is $n-1$ connected, $\pi^\infty_{\leq n}(\mathcal{T})$ is an E_n-algebra and in this special case the blob complex is presumably the same as his topological chiral homology.