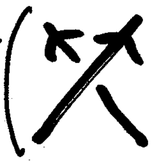
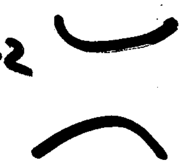
The Jones polynomial

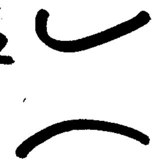
Conventions:

$J : \{\text{oriented tangles}\} \longrightarrow \mathbb{Z}[q,q^{-1}]\left\{\begin{array}{l}\text{unoriented tangles}\\ \text{without crossings}\\ \text{mod } \bigcirc = q+q^{-1}\end{array}\right.$

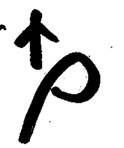
$\cup$

$\{\text{oriented links}\} \longrightarrow \mathbb{Z}[q,q^{-1}]\cdot\phi$

$J\Big($  $\Big) = q^{+1}\,)\,(\;-\; q^{+2}$ 

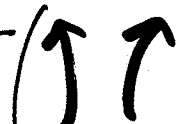
$J\Big($  $\Big) = -q^{-2}$  $+\, q^{-1}\,)\,($

It really is an invariant of tangles:

$J\Big($  $\Big) = q\,)\,\circ \;-\; q^{2}$  $= (q(q+q^{-1}) - q^{2})\,)$

$= \,) = J\Big($  $\Big)$

$J\Big($  $\Big) = \,)\,\Big(-(q+q^{-1})$  $+$ 

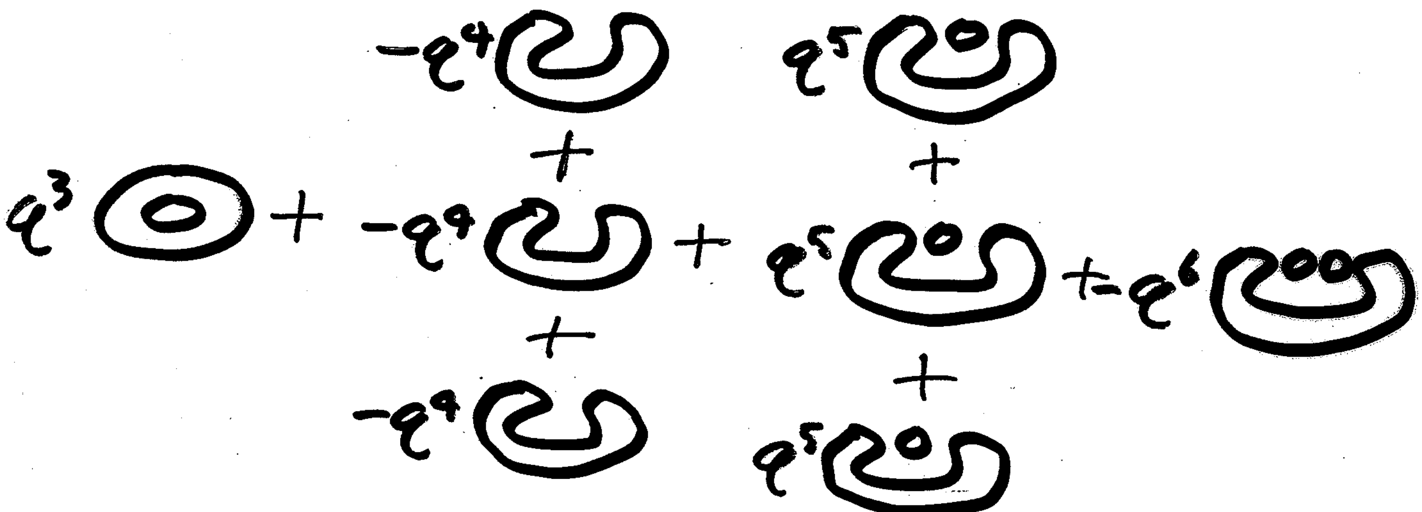
$= \,)\,(= J\Big($  $\Big)$

etc..

You can glue together the invariants of small tangle to find the invariant of a big tangle: "J is a map of planar algebras

The Jones polynomial of the trefoil.

$J\Big($  $\Big) =$



$$\|$$

$$+q^3(q+q^{-1})^2 - 3q^4(q+q^{-1}) + 3q^5(q+q^{-1})^2 - q^6(q+q^{-1})^3$$

Thought of like this, the Jones polynomial is always an <u>alternating sum</u>. Perhaps it's the Euler characteristic of some complex?

- We'd want the columns above to be ~~re~~placed by 'chain groups', whose dimensions (or graded dimensions) give the terms above.
- We'd need differentials, too?

A category for the complex — "Cob(su_2)".

Objects

Let's begin with what we've got:
unoriented tangles without crossings,
along with a 'factor' of q^k

Examples: $)\circ\circ$, $q^2)\circ($, $q^3\emptyset$

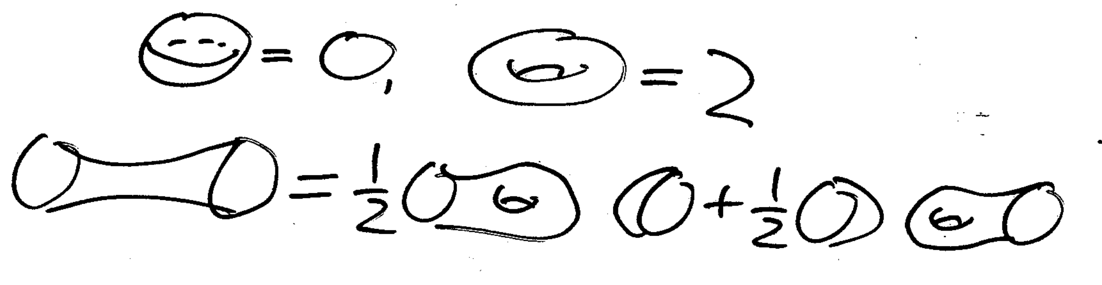
Morphisms

For diagrams D_1 and D_2, let's declare

$$\mathrm{Hom}_{\mathrm{Cob}(su_2)}(D_1, D_2) =$$

R { cobordisms from D_1 to D_2 rel ∂
modulo relations:

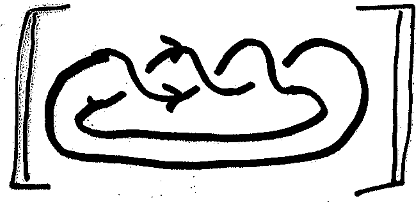
(some ring with $\frac{1}{2}$)

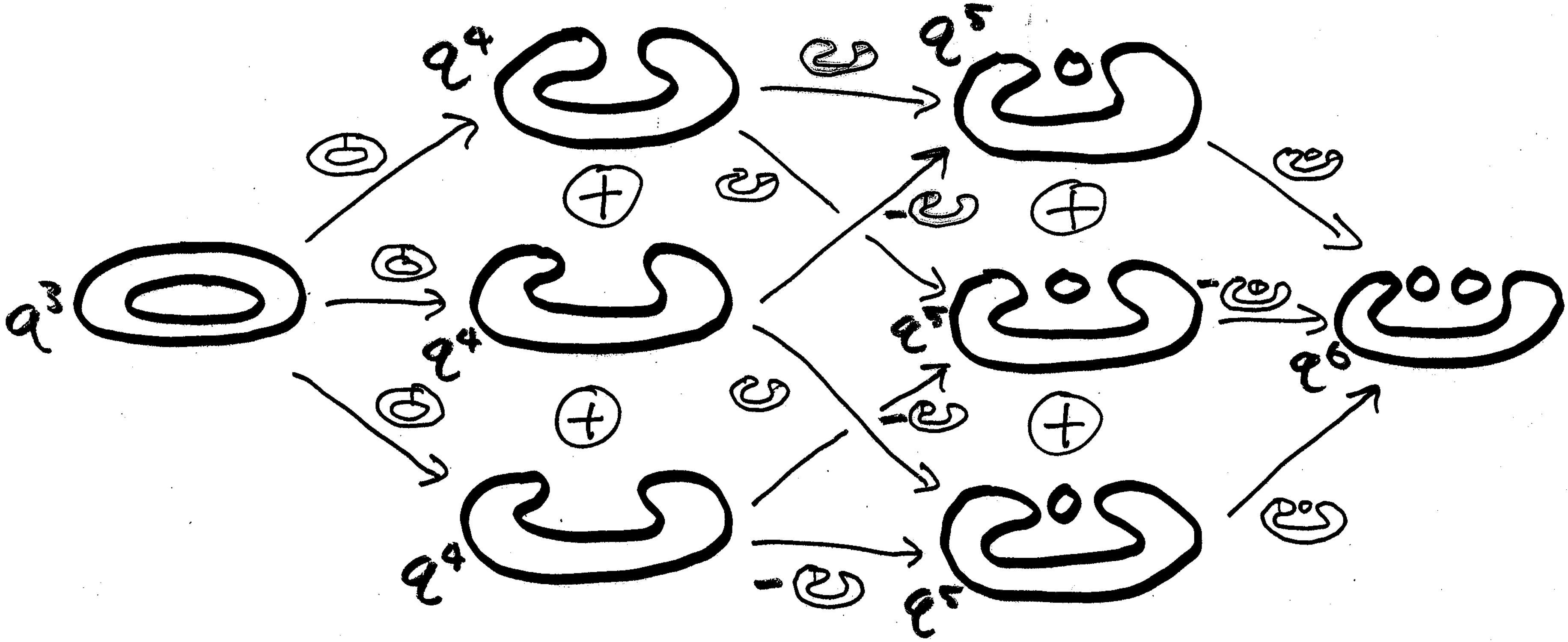


}

We'll come back to these relations in a moment!

The category Cob(su_2) can't be quite right, however: we want to have 'chain groups' which are 'sums' of diagrams

 =



$$[\![\cdot]\!]^0 \xrightarrow{d^0} [\![\cdot]\!]^1 \xrightarrow{d^1} [\![\cdot]\!]^2 \xrightarrow{d^2} [\![\cdot]\!]^3$$

Let's try $Mat(Cob(su_2))$ instead.

- For any category $\mathcal{C}$ whose morphism spaces are additive, we can define $Mat(\mathcal{C})$.

  Objects 'formal ~~direct~~ sums' of objects in $\mathcal{C}$.

  Morphisms matrices of morphisms in $\mathcal{C}$.
  (each matrix entry is a morphism from a direct summand of the source to a direct summand of the target)

In this category, there's a natural complex to write down for a knot!

"Chain groups" as in the calculation of the Jones polynomial.

Differentials "pairs of pants" merging or splitting circles
(locally, saddles changing the resolution of a crossing)
(we'll need to sprinkle some minus signs, in order to get $d^2=0$)

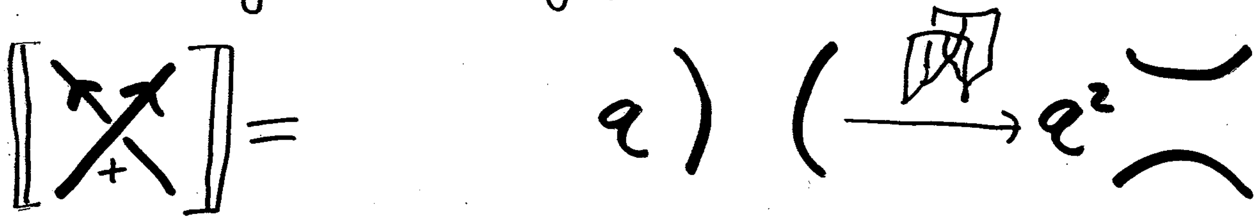
Homology, homotopy, and all that

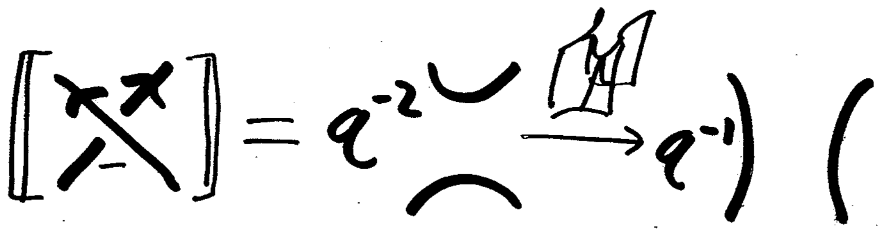
- Obviously, the complex itself can't be a link invariant.
- Complexes are rarely invariants anyway; we expect to have to take homology.
  - That's not an option here. The category isn't abelian (cobordisms don't have kernels or images).
- Instead, we'll settle for the homotopy type of the complex.
  - This isn't as bad as it sounds; it turns out every link has a unique homotopy representative with no differentials ....
- Actually, we can do something simpler — take the simple homotopy type.
  - What is simple homotopy?
    Adding or deleting contractible direct summands:

$$\left(\mathbf{A} \xrightarrow{d \text{ an iso}} \mathbf{B}\right) \underset{\substack{\text{simple}\\ \text{homotopy}}}{\simeq} \bullet$$

The Khovanov/Bar-Natan invariant for tangles.

For single crossings, we make the definitions

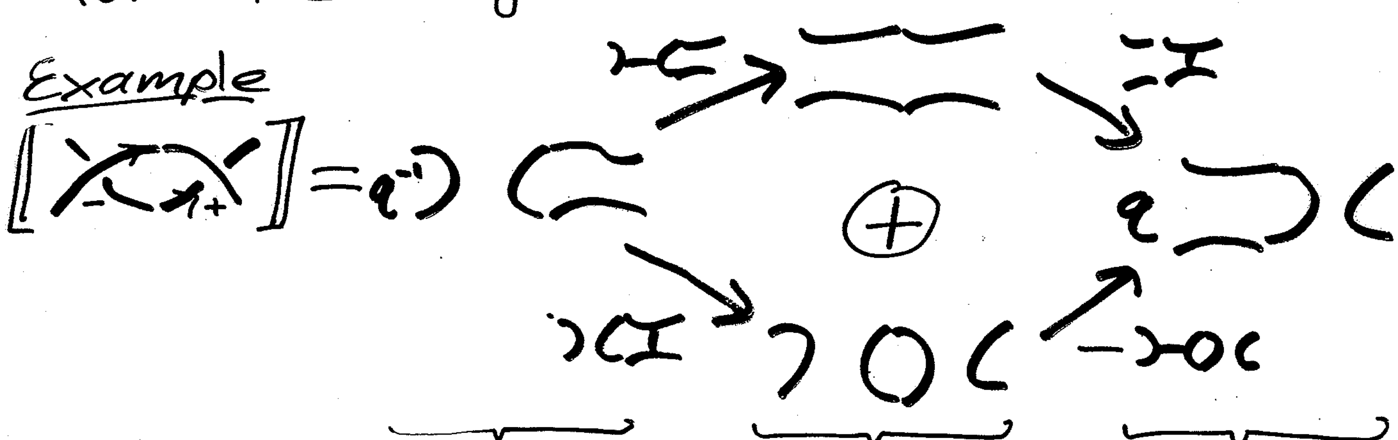




$[\![\cdot]\!]^{-1}$ $\quad$ $[\![\cdot]\!]^{0}$ $\quad$ $[\![\cdot]\!]^{1}$

For a larger tangle (or link), we take the tensor product of the complexes for each crossing, gluing together diagrams and cobordisms in the same way the crossings were glued together to form the tangle.

Example

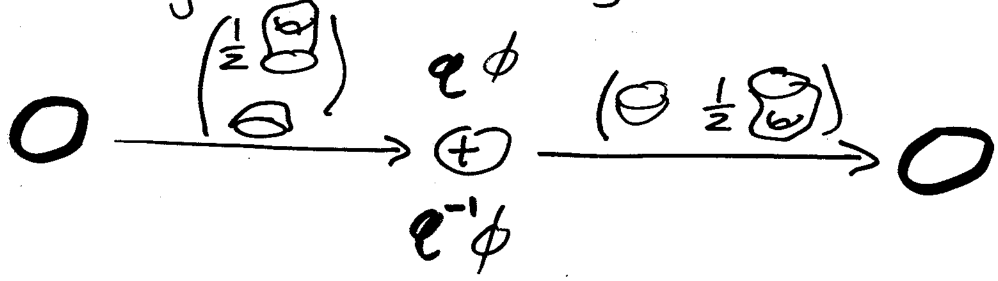


$[\![\cdot]\!]^{-1}$ $\quad$ $[\![\cdot]\!]^{0}$ $\quad$ $[\![\cdot]\!]^{1}$

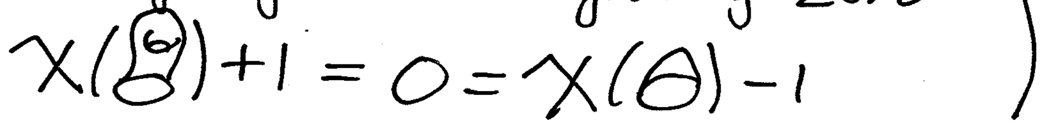
(We say: "$[\![-]\!]$ is a map of planar algebras")

The "delooping" isomorphism

- In Mat(Cob(su_2)) there's an interesting isomorphism between $\bigcirc$ and $q\phi \oplus q^{-1}\phi$ given by the matrices

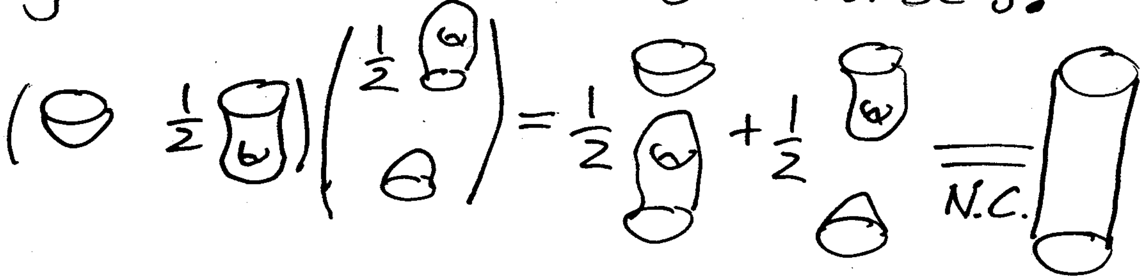


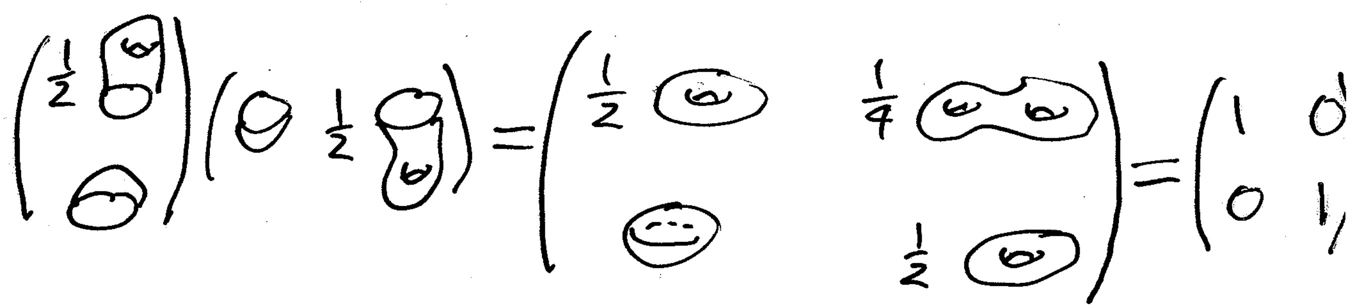
(notice everything here is in grading zero—



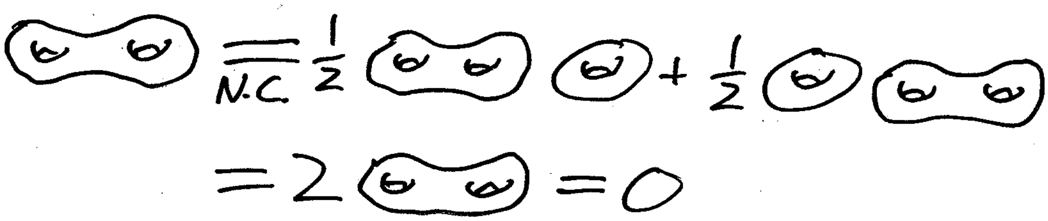
)

- Why are these matrices inverses?

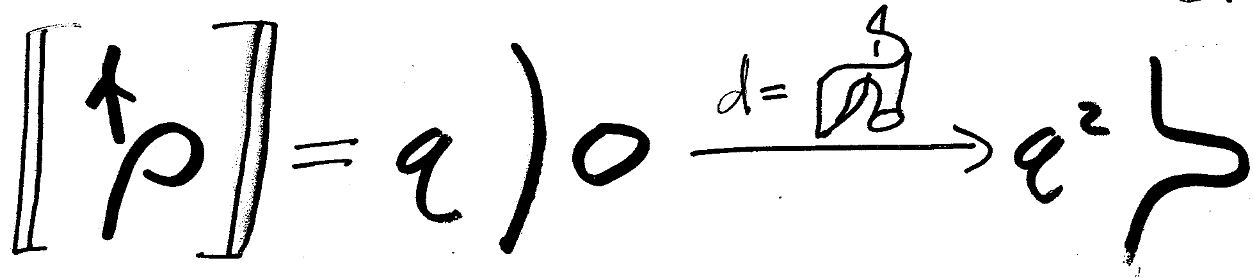




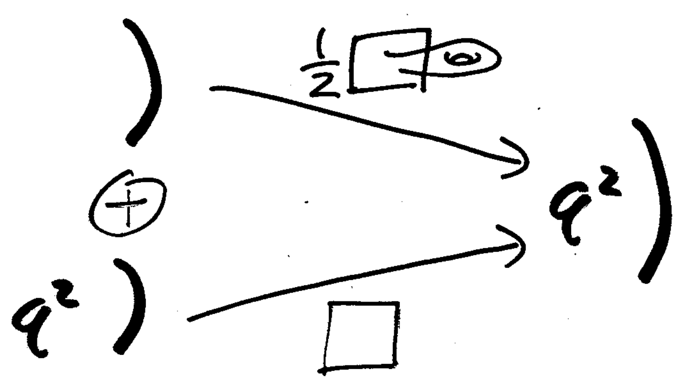
Here we've used  $= 2$,  $= 0$, and derived



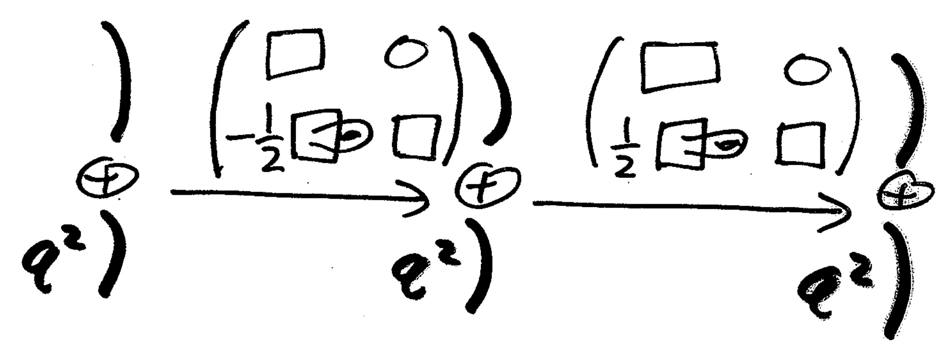
Reidemeister 1 invariance.

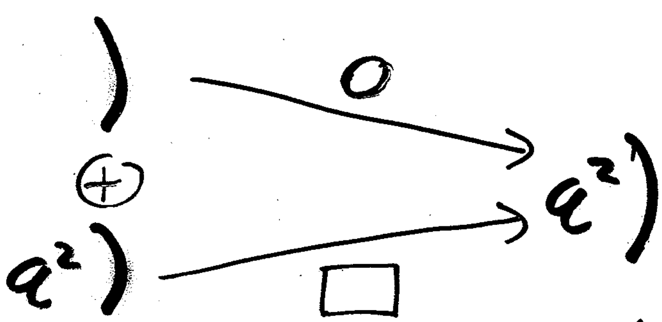


$\cong$ delooping

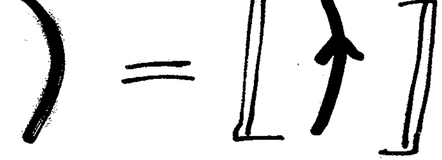


via a "change of basis":





$\cong$ simple homotopy

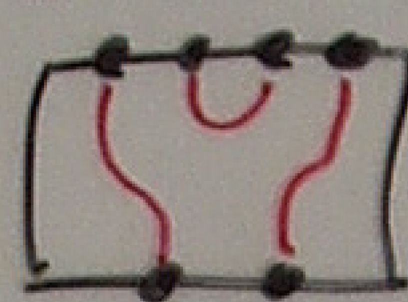


(De)categorification

- Define/recall **Spider(su_2)** is the tensor category

  Objects: $\mathbb{N}$, thought of as points on a line

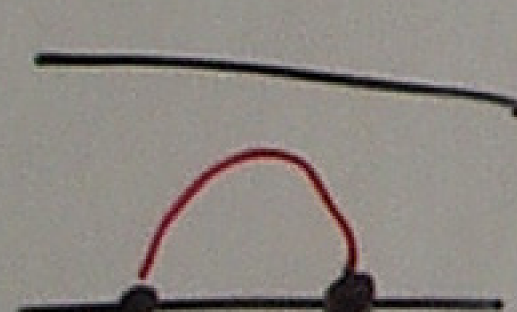
  Morphisms: $\mathbb{Z}[q, q^{-1}]$-linear combinations of diagrams { , , etc.... }

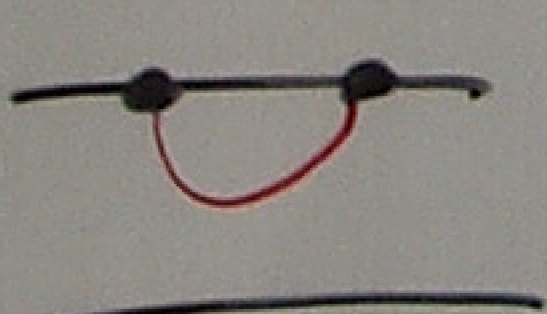
  modulo the relation  $= q + q^{-1}$.

and notice **Spider(su_2)** $\underset{\otimes\text{-cat}}{\cong}$ **FundRep($U_q sl_2$)**

 $\longmapsto$ V the 2-d standard rep.

 $\longmapsto$ id_V

 $\longmapsto$ the pairing map $V \otimes V \longrightarrow \mathbb{C}$.

 $\longmapsto$ the copairing map

- Delooping secretly says:

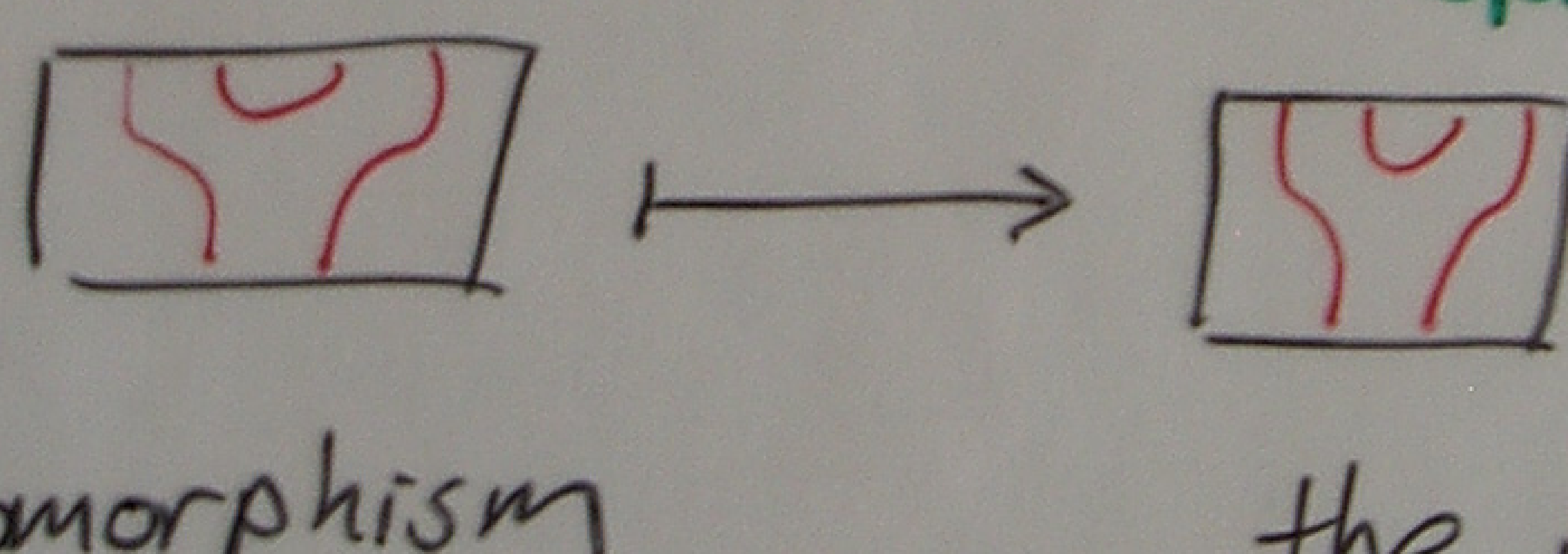
  "**Cob(su_2)** is a categorification of **Spider(su_2)** as a tensor category."

- $\mathrm{Cob}(su_2)$ is a 2-category which replaces the $\mathbb{Z}[q,q^{-1}]$ module $\mathrm{Hom}_{\mathrm{Spider}}(n,m)$ with the graded category of (n,m)-diagrams and cobordisms between them.
- The Grothendieck group of $\mathrm{Cob}(su_2)$ is just $\mathrm{Spider}(su_2)$.

For any additive category $\mathcal{C}$, the "split" Grothendieck group is

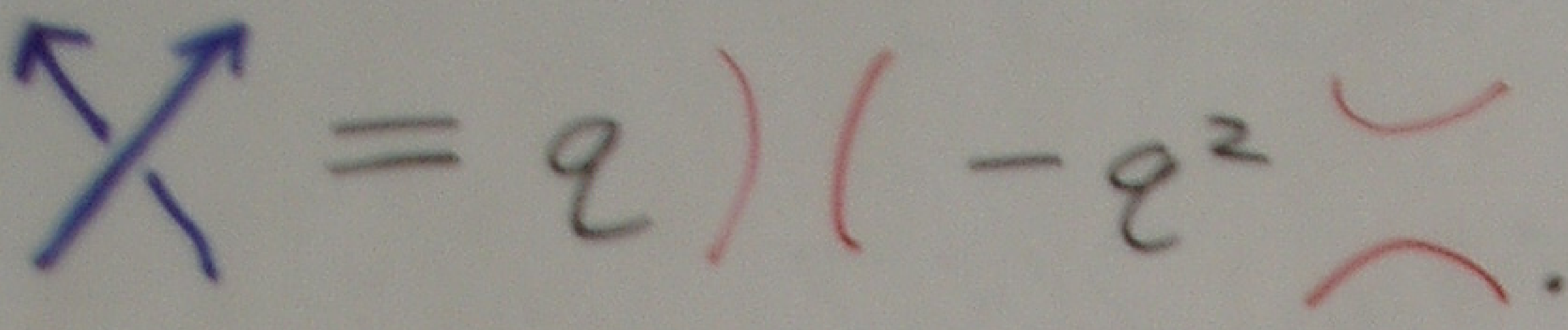
$$K^{split}(\mathcal{C}) = \left\langle \mathrm{Obj}(\mathcal{C}) \;\middle|\; \begin{matrix}[A]=[B]+[C]\\ \text{whenever}\\ A\cong B\oplus C\end{matrix}\right\rangle$$

So $K^{split}(\mathrm{Cob}(su_2)(n\to m)) \cong \mathrm{Hom}_{\mathrm{Spider}}(n,m)$



the isomorphism $\bigcirc \cong q^{-1}\phi \oplus q\phi$ $\rightsquigarrow$ the relation $\bigcirc = q+q^{-1}$

- However $\mathrm{Spider}(su_2)$ is more than a tensor category; it's a <u>braided tensor category</u>:

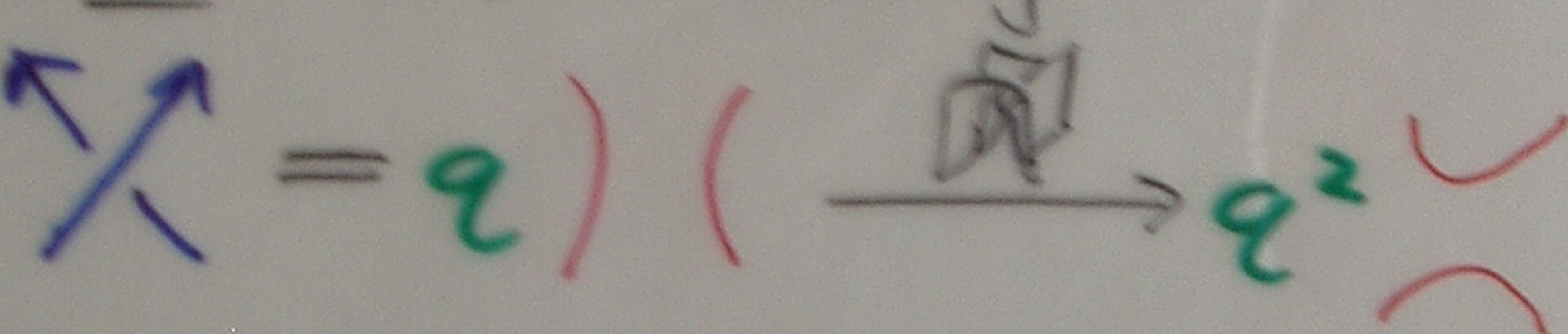


Sadly, $\mathrm{Cob}(su_2)$ is not braided.

- The essence of today's talk has been that

$$\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}(su_2)))$$

<u>does</u> have a braiding:



- Now $K^{\text{triangulated}}(\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}(su_2))))$

$$\cong K^{\text{split}}(\mathrm{Cob}(su_2)) \cong \mathrm{Spider}(su_2).$$

so the category of complexes is <u>also</u> a categorification, but actually a better one, because it's a categorification of $\mathrm{Spider}(su_2)$ as a braided tensor category.

Gaussian elimination for complexes.

- Anytime you see an isomorphism as a matrix entry in a differential, there's a contractible direct summand to strip off!

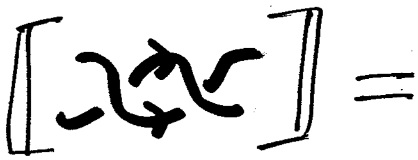
$$
\begin{tikzcd}
 & B \arrow[r, "\varphi\text{, an iso}"] \arrow[dr, "\lambda"' near start] & D \arrow[dr, "\bullet"] & \\
A \arrow[ur, "\bullet"] \arrow[dr, "\alpha"'] & \oplus & \oplus & F \\
 & C \arrow[ur, "\mu" near end] \arrow[r, "\nu"'] & E \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
$$

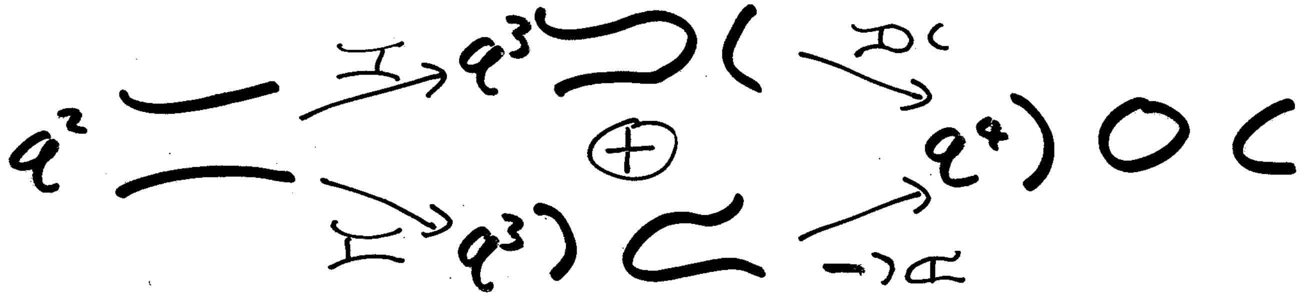
$\cong$ via a change of basis

$$
\begin{tikzcd}
 & B \arrow[r, "\varphi"] \arrow[dr, dashed, "0"' near start] & D \arrow[dr, dashed, "0"] & \\
A \arrow[ur, dashed, "0"] \arrow[dr, "\alpha"'] & \oplus & \oplus & F \\
 & C \arrow[ur, dashed, "0" near end] \arrow[r, "\nu-\lambda\varphi^{-1}\mu"'] & E \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
$$

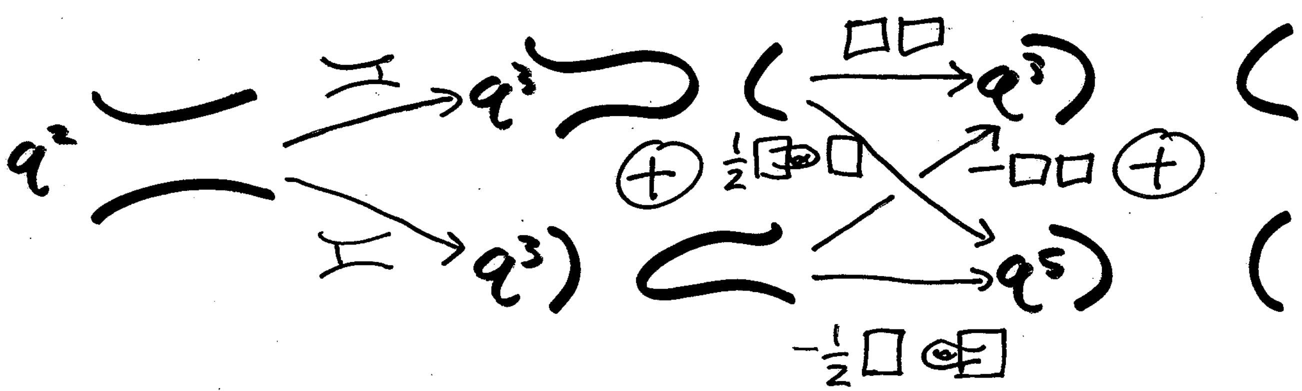
$\simeq$ simple homotopy

$$
A \xrightarrow{\alpha} C \xrightarrow{\nu-\lambda\varphi^{-1}\mu} E \xrightarrow{\varepsilon} F
$$

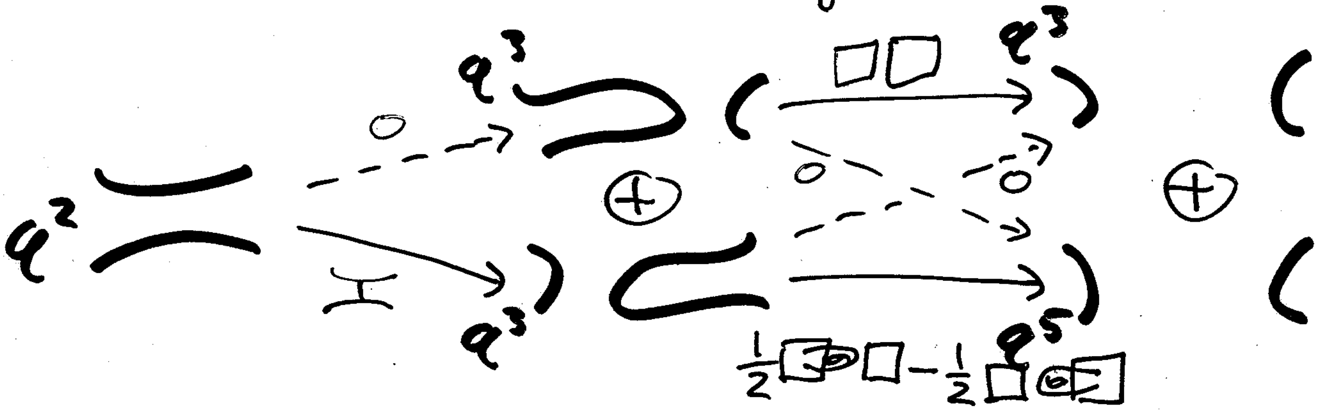




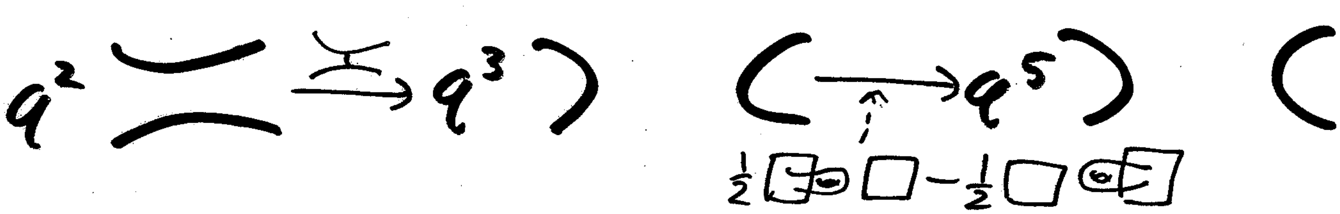
$\cong$ (delooping)



$\cong$ (change of basis)



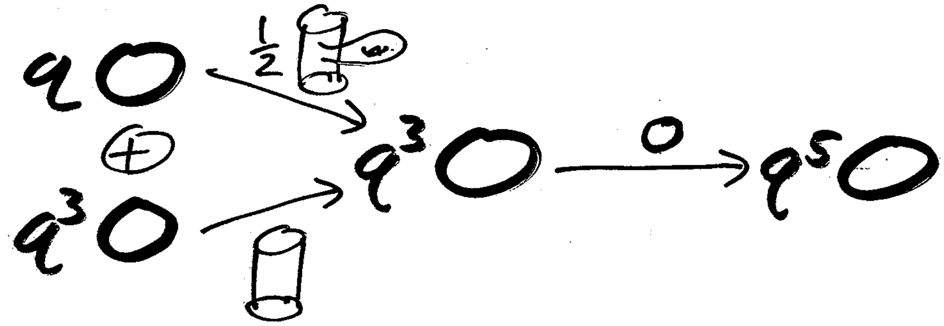
$\cong$ (simple homotopy)



The Hopf link:

 $=$

$\cong$ delooping



$\cong$ simple homotopy

$$q\bigcirc \xrightarrow{0} \cdot \xrightarrow{0} q^5\bigcirc$$

$\cong$ delooping again

$$\begin{matrix}1\\ \oplus\\ q^2\end{matrix} \xrightarrow{0} \cdot \xrightarrow{0} \begin{matrix}q^4\\ \oplus\\ q^6\end{matrix}$$

Kh() $= t^0(1+q^2)+t^2(q^4+q^6)$

J() $= 1+q^2+q^4+q^6$ (set $t=-1$)