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Khovanov homology of rational tangles

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Khovanov homology for tangles

(following Bar-Natan)

- Bar-Natan defined a planar algebra of categories (aka 'a 'canopolis')

$$\mathcal{B}_k = \left\{ \begin{array}{c} \text{[Diagram 1]} \\ \text{[Diagram 2]} \\ \text{[Diagram 3]} \\ \dots \end{array} \right\}$$

2k boundary points



$$\text{Hom}(D_1, D_2) = \mathbb{Z} \left[\frac{1}{2} \right] \left\{ \begin{array}{c} \text{surfaces in} \\ \text{[Cylinder Diagram]} \\ \text{modulo relations} \end{array} \right\}$$

Surface relations:

$$\text{[Cap Diagram]} = \text{[Circle Diagram]}$$

$$\text{[Cup Diagram]} = 2 \text{[Circle Diagram]}$$

$$\text{[Torus Diagram]} = \frac{1}{2} \text{[Torus Diagram]} + \text{[Circle Diagram]} + \frac{1}{2} \text{[Circle Diagram]} + \text{[Cup Diagram]}$$

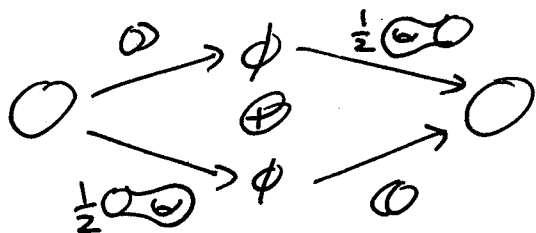
- The Khovanov invariant of a tangle is a complex in the category of matrixes over this category.

$$\text{Kh}(T \in \text{Tangles}_k) \in \text{Kom}(\text{Mat}(\mathcal{B}_k)).$$

Isomorphisms and decategorification

(2)

In \mathcal{B} , there is an isomorphism $0 \cong \phi \oplus \phi$



In fact $K_0(\mathcal{B}_k) \cong TL_k(\delta=2)$.

(And by keeping track of gradings we can get $\delta = q + q^{-1}$.)

The structure of \mathcal{B}_2 and \mathcal{B}_4

Every object in \mathcal{B}_2 is isomorphic to a direct sum of copies of (---) .

$$\text{Hom}(\text{---} \rightarrow \text{---}) = \mathbb{Z}[\frac{1}{2}] \left\{ \square, \square \begin{array}{c} \text{---} \\ \text{---} \end{array}, \square \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}, \dots \right\}$$

$$= \mathbb{Z}[\frac{1}{2}, \text{---}] \left\{ \square, \square \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}$$

(since $\square \begin{array}{c} \text{---} \\ \text{---} \end{array} = \frac{1}{2} \square \text{---} + \frac{1}{2} \square \begin{array}{c} \text{---} \\ \text{---} \end{array}$
and $\text{---} = 0$.)

Complexes in B_2 .

(3)

Over \mathbb{Q} , every complex in B_2 decomposes uniquely into a direct sum of the indecomposable complexes:

$$E =)$$
$$C_{k \geq 1} =) \xrightarrow{\text{Diagram}})$$

The diagram shows a square with a circle inside, and a horizontal arrow pointing to the right from the circle.

You can recover the s -invariant, and both the reduced and unreduced homologies from this decomposition.



There's an implementation (Morrison & Shumakovitch / Bar-Natan / Green)

```
Mathematica
<< KnotTheory`
sInvariant[Knot[8,19]]
6
UniversalKh[Knot[8,19]]
q^6 E + q^12 t^3 C_1 + q^16 t^5 C_2
```

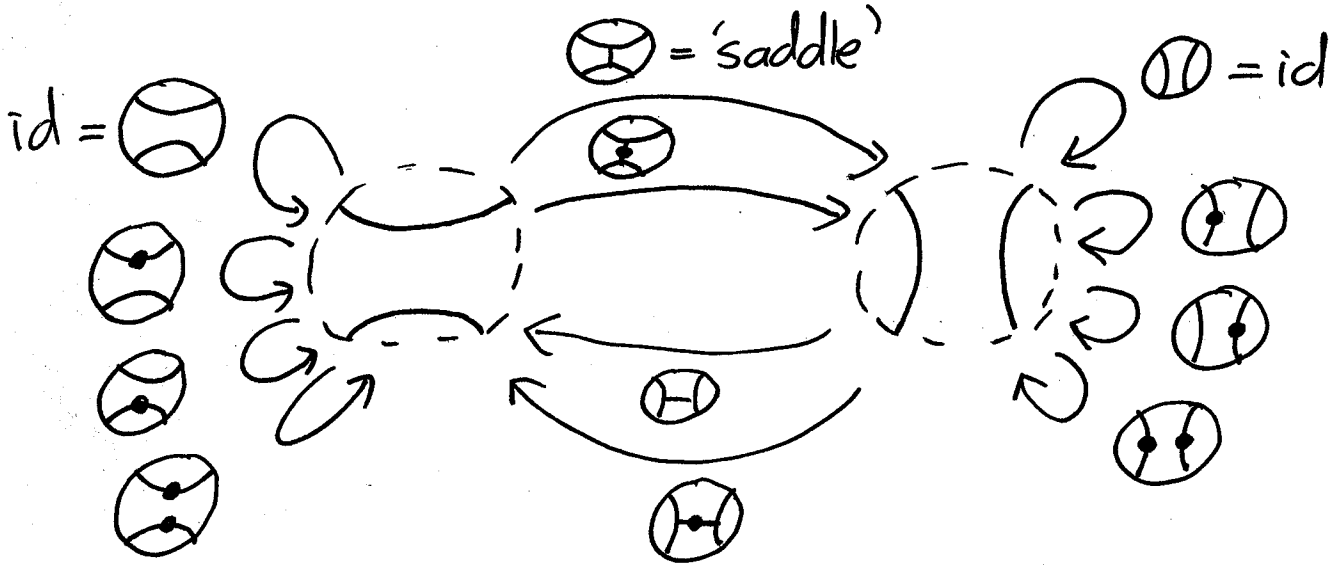
Conjecture

In the complex for a tangle \textcircled{T} , only the summands E , C_1 & C_2 appear.

The structure of B_4

Every object is a direct sum of  & 

As $\mathbb{Z}[\frac{1}{2}, \langle \text{torus} \rangle]$ modules, the morphisms are



Question: Can you describe the indecomposable complexes over B_4 ?

Question: What are the chain maps between indecomposables?
Equivalently, how do tensor products



decompose?

Rational tangles

(5)

$PSL(2, \mathbb{Z})$ acts on $\mathbb{Q} \cup \{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \frac{av+b}{cv+d}$$

and also on tangles via

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \textcircled{T} = \textcircled{T} \cup \text{arc}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \textcircled{T} = \textcircled{T} \cup \text{arc}$$

Rational tangles are just the orbit $PSL(2, \mathbb{Z}) \cdot \textcircled{T}$ and can be identified with $\mathbb{Q} \cup \{\infty\}$ via continued fractions. (Note $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v = v+1$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v = \frac{1}{v}$.)

Theorem (Clark-Morrison-Walker)

The Khovanov complex of a rational tangle has an up-to-homotopy representative which is a "snake" (see below), and we can efficiently describe the action of $PSL(2, \mathbb{Z})$ on snakes.

Examples

$$Kh(\underbrace{\text{[diagram of 4 saddles]}_4}) \approx \text{[diagram of saddle]} \xrightarrow{\text{saddle}} \left(\begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-) \sigma} \end{array} \right) \left(\begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{(+)\sigma} \end{array} \right) \left(\begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-)\sigma} \end{array} \right) \left(\begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-)\sigma} \end{array} \right) \left(\begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-)\sigma} \end{array} \right)$$

(A "straight" complex with only s, a & σ .)

$$Kh\left(\underbrace{\text{[diagram of 4 saddles]}_{= \frac{1}{4}}}\right) \approx \left(\begin{array}{c} \xleftarrow{s} \\ \xleftarrow{a} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right) \left(\begin{array}{c} \xleftarrow{\sigma} \\ \xleftarrow{a} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{a} \end{array} \right)$$

$$Kh\left(\underbrace{\text{[diagram of 5 saddles]}_{= \frac{5}{4}}}\right) \approx \left(\begin{array}{c} \xleftarrow{s} \\ \xleftarrow{a} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right) \left(\begin{array}{c} \xleftarrow{\sigma} \\ \xleftarrow{a} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{a} \end{array} \right)$$

$$\downarrow \quad \downarrow s \quad \downarrow s \quad \downarrow s \quad \downarrow s$$

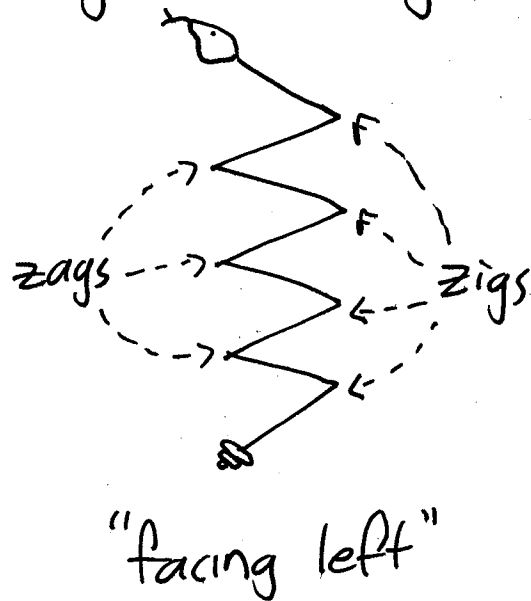
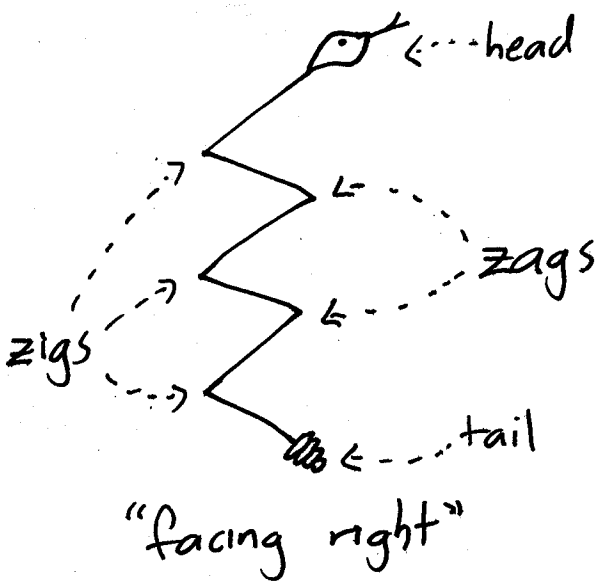
$$\left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{2\sigma} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right)$$

$$\approx \left(\begin{array}{c} \xleftarrow{s} \\ \xleftarrow{a} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{s} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{s} \end{array} \right) \left(\begin{array}{c} \xleftarrow{a} \\ \xleftarrow{s} \end{array} \right)$$

(a "kinky" complex with only s, a & σ .)

Snakes

- All snakes are "kinky" complexes; each summand has at most two differentials starting or ending at it
- All differentials are either s , a or σ .
- There are two types of snakes "facing left" or "facing right". Each consists of a head, a tail, and an alternating sequence of zigs and zags.



zigs $\in \{+, -\}$, + represents $\begin{matrix} \text{---} \xrightarrow{s} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \end{matrix}$ $\begin{matrix} \text{---} \xrightarrow{s} \text{---} \\ \text{---} \xrightarrow{s} \text{---} \end{matrix}$ $\begin{matrix} \text{---} \xrightarrow{s} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \end{matrix}$ $\begin{matrix} \text{---} \xrightarrow{s} \text{---} \\ \text{---} \xrightarrow{s} \text{---} \end{matrix}$

- represents $\begin{matrix} \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{s} \text{---} \end{matrix}$ $\begin{matrix} \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \end{matrix}$ $\begin{matrix} \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{s} \text{---} \end{matrix}$ $\begin{matrix} \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \end{matrix}$

zags $\in \{(2n-1)^{\pm}\}$ eg. 5^+ represents $\begin{matrix} \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{\sigma} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{\sigma} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \end{matrix}$ $\begin{matrix} \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{\sigma} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \\ \text{---} \xrightarrow{\sigma} \text{---} \\ \text{---} \xrightarrow{a} \text{---} \end{matrix}$

(for today, I'll ignore heads & tails)

The action of $PSL(2, \mathbb{Z})$ on snakes.

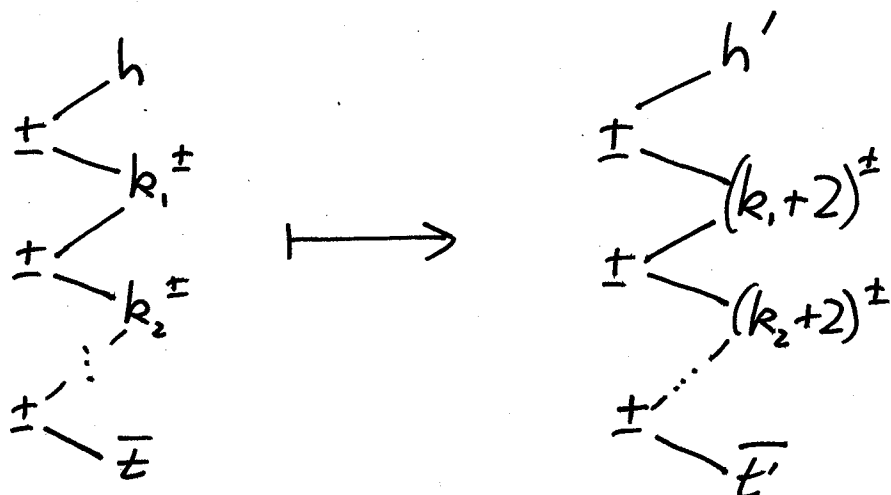
(8)

$$(0 \ 1) : \textcircled{T} \longmapsto \textcircled{\bar{T}}$$

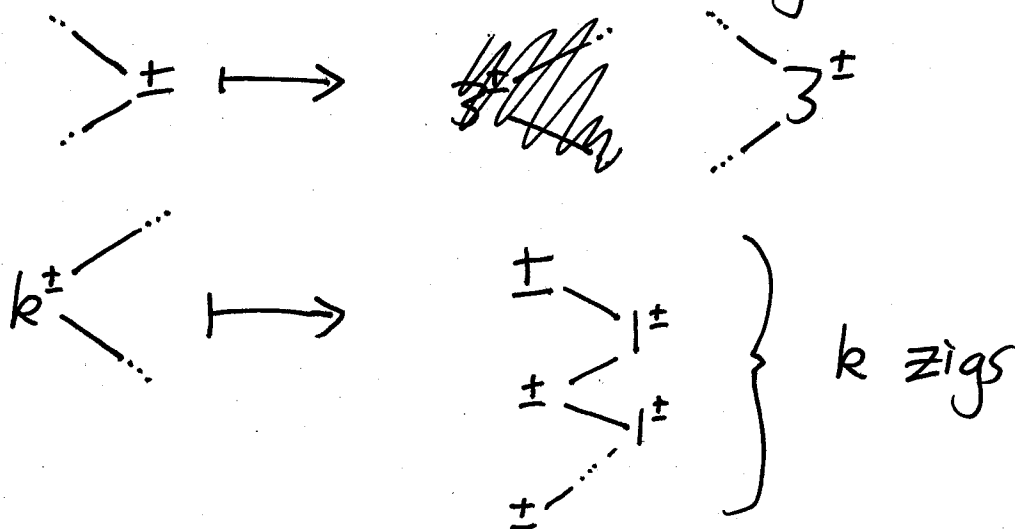
interchanges left-facing and right-facing snakes
(and also $\pm \leftrightarrow \bar{\pm}$).

$$(1 \ 0) : \textcircled{T} \longmapsto \textcircled{T}$$

is easy to describe on right-facing snakes:



and a bit harder on left-facing snakes:



(again, some messy details for heads
and tails omitted.)