4-manifold invariants from Khovanov homology (joint with Chris Douglas & Kevin Walker)

- Today I'll define $\underrightarrow{Kh}(W^4)$, a vector space valued invariant of a smooth 4-manifold.
- There's also a relative version $\underrightarrow{Kh}(W^4, L)$, where L is a link in the 3-dimensional boundary.
- It ~~generalizes~~ is a natural extension of Khovanov homology:
  $\underrightarrow{Kh}(B^4, L) \cong Kh(L)$
- We first build a 4-category from Khovanov homology, then use a standard recipe (topological quantum field theory) to build invariants of manifolds.

Why new 4-manifold invariants?

- The "last man standing" of geometric topology is the
  - Smooth 4-d Poincaré conjecture:
    Every manifold homeomorphic to the 4-sphere is diffeomorphic to the 4-sphere.
- There are many potential counterexamples, although opinions vary on the likelihood of the conjecture.
- Gauge theoretic invariants (SW, Donaldson) "can't detect smooth structure near a point".
- Khovanov homology
  - depends on smooth structure
  - is not obviously weaker than gauge theory
  - may provide ~~the~~ new tools to detect a counterexample.

Previously (with Freedman, Gompf and Walker) we tried to use the "s-invariant" to show the Cappell-Shaneson spheres were exotic

- Each CS-sphere provides a link $L \subset S^3$ which is slice (bounds a disc) in some homotopy 4-ball.
- The s-invariant provides a lower bound on the slice genus in the standard 4-ball
- If $s(L) > 0$, the CS-sphere is exotic.

• We failed, and recently Kronheimer & Mrowska proved the approach is doomed — the s-invariant is "determined by gauge theory".

• But there is much more information in Khovanov homology than just the s-invariant.

- Today, we'll shortcut the general

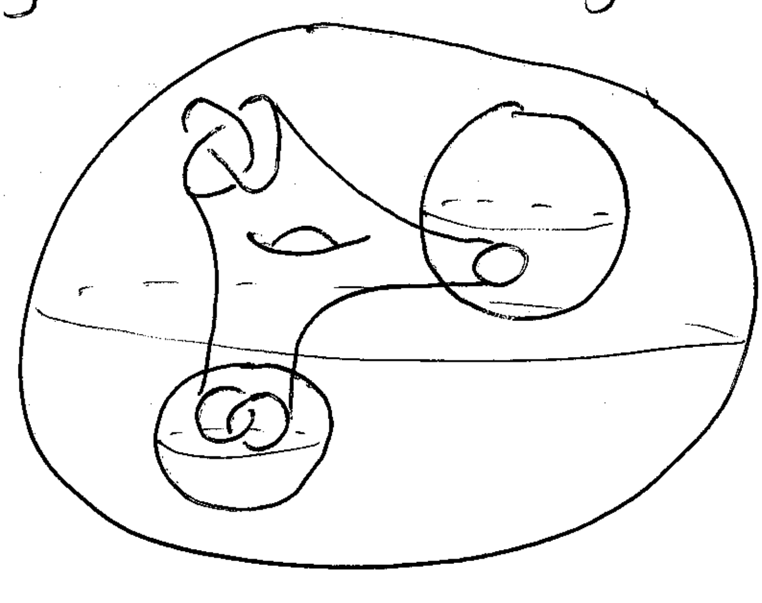
$$\{n\text{-category}\} \rightsquigarrow \{\text{invariants of } n\text{-manifolds}\}$$

formalism.

Instead, I'll show how Khovanov homology provides a lasagna algebra (neatly axiomatizing a particular class of 4-categories) and directly construct ~~the~~ an invariant from a lasagna algebra.

A lasagna diagram $(B_0, \{B_i\}, \Sigma)$ consists of 3

- A 4-ball B_0;
- Disjoint 4-balls $\{B_i\}$ in the interior;
- A surface Σ in the complement $B_0 \setminus \cup B_i$, meeting each B_i transversely in a link L_i.



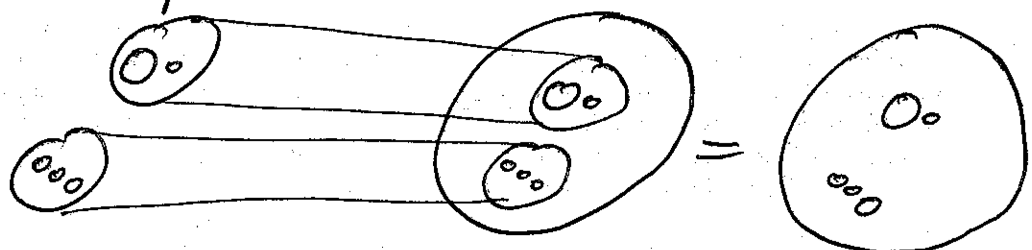
A *lasagna algebra* $\mathcal{A}$ consists of (4)

- a vector space $\mathcal{A}(L \subset S)$ for each link L in a 3-sphere

- a linear map
$$\mathcal{A}(\Sigma) : \bigotimes \mathcal{A}(L_i) \longrightarrow \mathcal{A}(L_0)$$
for each lasagna diagram

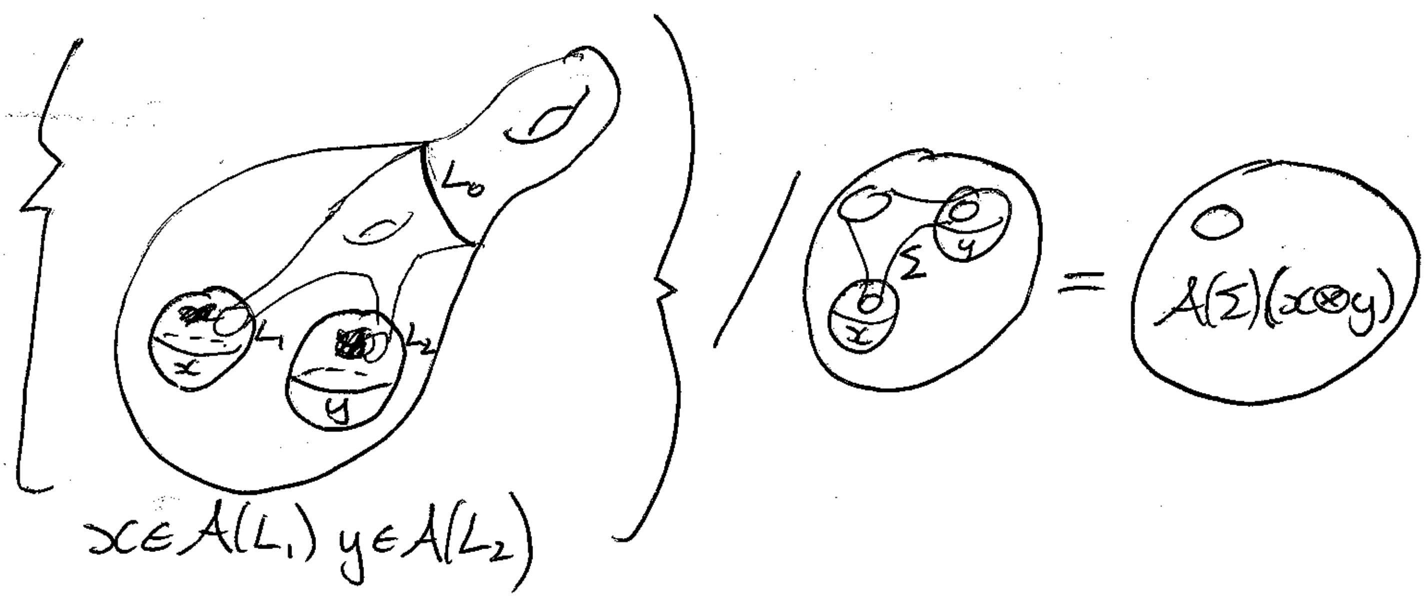
such that:

- isotopic surfaces (rel ∂) give identical maps
- compositions of lasagna diagrams give composite maps

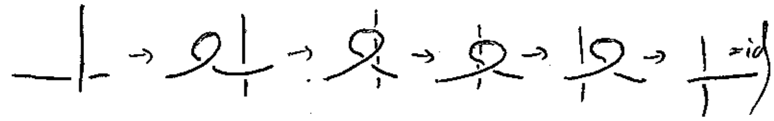


Any lasagna algebra $\mathcal{A}$ gives a 4-manifold invariant (5)

$\underrightarrow{\mathcal{A}}(W^4, L) =$



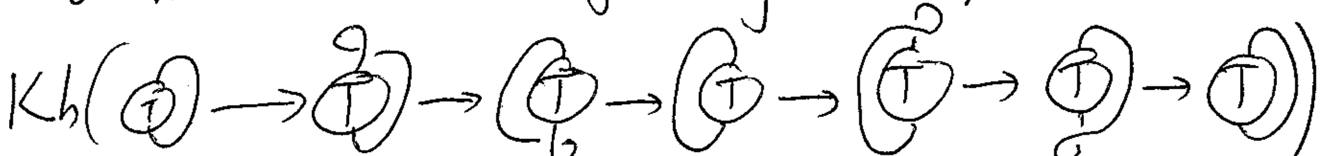
What is Khovanov homology?

- It is at first defined combinatorially, in terms of ~~link diagrams~~
    - link diagrams
    - Reidemeister (and Morse) moves
    - movie moves (eg.  = id)
- We can think of it as providing
    - a vector space for each $L \subset B^3$
    - a map for each cobordism $\Sigma \subset B^3 \times I$ (invariant under isotopy of Σ)
- But we need to work in S^3, not B^3. How can we lift?

Define $Kh(L \subset S)$ as the flat sections of

$$\{Kh(L \subset S \setminus \{x\})\} \quad \text{with parallel transport along } \gamma \subset S \setminus L$$
$$\downarrow \qquad \text{given by the cobordism}$$
$$S \setminus L \qquad L \times I \subset S \times I \setminus \mathrm{graph}(\gamma)$$

Does this bundle have monodromy? Parallel transport around a meridian of L is given by the map

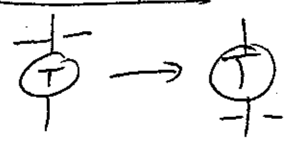


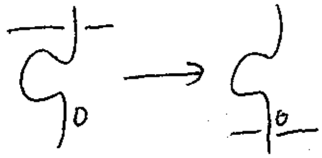
Theorem (Douglas–Morrison–Walker)

This map is ~~(homotopic to)~~ the identity

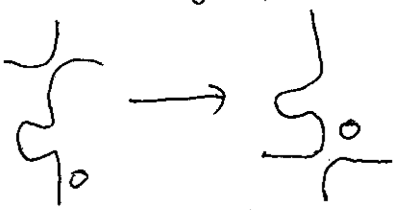
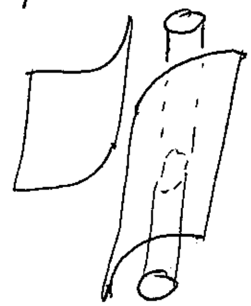
~~Theorem~~ • Evaluating a flat section at a point is an isomorphism.

Sketch proof

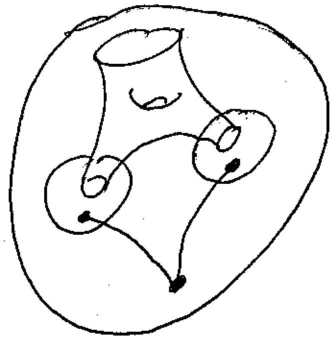
 is 'diagonal' w.r.t. resolutions of T,

with  via Riedemaster 2 moves,

and in fact 'diagonal' w.r.t complete resolutions,

with  via .

We define the lasagna action by picking arcs γ avoiding Σ



$\rightsquigarrow$ a cobordism $\sqcup L_i \to L_0$

in $B_0 \setminus \cup B_i \setminus \gamma \cong B^4$

$$\rightsquigarrow$$

$$\bigotimes Kh(L_i) \longrightarrow Kh(L_0)$$

This is independent of γ by the theorem above.

Thus Khovanov homology provides a lasagna algebra, and hence a 4-manifold invariant.

Computations

- The TQFT framework lets us also define $\underrightarrow{Kh}(M)$, a category to each 3-manifold M.
- If $M \subset \partial W^4$, $\underrightarrow{Kh}(W)$ becomes a module over $\underrightarrow{Kh}(M)$.
- If $M \sqcup M^{op} \subset \partial W^4$, $\underrightarrow{Kh}(W)$ becomes a bimodule over $\underrightarrow{Kh}(M)$, and

__Theorem__ $\underrightarrow{Kh}(W \underset{M}{\cup} \circlearrowleft) \cong \underrightarrow{Kh}(W) \underset{\underrightarrow{Kh}(M)}{\otimes} \circlearrowleft$

- Using this, we have tentative answers for $\underrightarrow{Kh}(B^3 \times S^1, \circledcirc_T)$ for $T = \times, \asymp$ and $)\,($.

(Infinite rank, but finite in each (q,t) grading).

These cannot fit into an exact triangle, as we would expect in B^4.

We also have a "derived" version, via our "blob complex" formalism
(replacing the quotient in the definition of $\underrightarrow{Kh}$ by $Kh_0 \leftarrow Kh_1 \leftarrow \cdots$)

There there is a spectral sequence abutting zero

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & Kh_{1,s}()\,() \arrow[r] \arrow[d] & Kh_{1,s}(\times) \arrow[r] \arrow[d] & K_{1,s+1}(\asymp) \arrow[r] \arrow[d] & Kh_{1,s+1}()\,() \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Kh_{0,s}()\,() \arrow[r] & Kh_{0,s}(\times) \arrow[r] & Kh_{0,s+1}(\asymp) \arrow[r] & Kh_{0,s+1}()\,() \arrow[r] & \cdots
\end{tikzcd}
$$

(horizontal homology zero, vertical homology has $\underrightarrow{Kh}$ in the bottom row).

We're working on using this for computations.