How to calculate some link invariants associated to the D_4 planar algebra.

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Abstract

AMS Classification 57M25; 57M27; 57Q45

Keywords Planar Algebras,

Contents

todo list

(1) Write this todo list.
(2) Write an abstract.
(3) Write an introduction.
(4) `top_matter.tex` – contact details, AMS classification
(5) Explain why the principal graph of the D4 subfactor really is D4.

To ask Vaughan about

(1) What adjectives do we need to algebraic bimodules? Finitely generated as 1-sided modules? Should they be positive definite inner product spaces (not necessarily complete)? Or can they just be vector spaces?
(2) What does irreducibility translate to for bimodules? Extremality?
(3) What's the deal with 'bimodules generated by M'? How much do you miss?
(4) What's the reference for the algebraic bimodules generated by M being the same as the L^2 bimodules generated by $L^2(M)$?

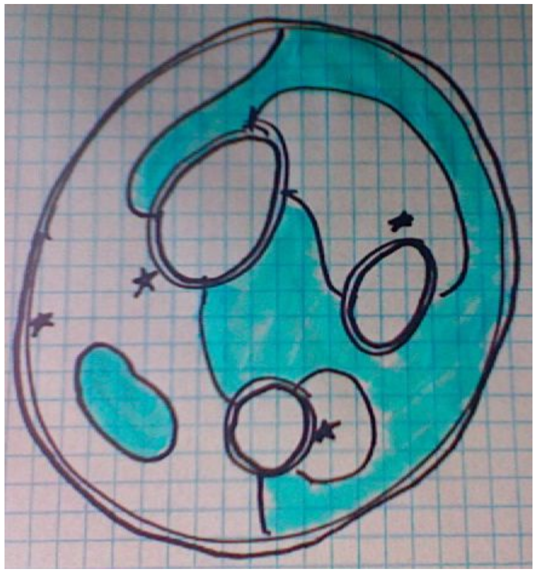
1 Introduction

2 A recipe

2.1 Planar Algebras

A planar algebra is a gadget specifying how to combine objects in planar ways. They were introduced in [**?**] to study subfactors, and have since found more general use.

In the simplest version, a shaded planar algebra $\mathcal{P}$ associates a vector space $\mathcal{P}_k$ to each $k \in +, -, 1, 2, 3, \ldots$ (thought of as a disc in the plane with $2k$ marked points on its boundary; elements of $\mathcal{P}_k$ will be refered to as k-boxes and the space $\mathcal{P}_k$ as the k-disc space) and a linear map $\mathcal{P}(T) : \mathcal{P}_{k_1} \otimes \mathcal{P}_{k_2} \otimes \cdots \otimes \mathcal{P}_{k_r} \rightarrow \mathcal{P}_{k_0}$ to each planar tangle[1] T, for example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with internal discs with $2k_1, 2k_2, \ldots, 2k_r$ marked points, and $2k_0$ marked points on the external disc (evenness of the number of boundary points is required so that the regions can be given a checkerboard shading); this particular tangle corresponds to a map $V_2 \times V_4 \times V_4 \rightarrow V_2$. These maps (the 'planar operations') must satisfy certain properties: "radial" tangles induce identity maps, and composition of the maps $\mathcal{P}(T)$ is compatible with the obvious composition of planar diagrams by gluing one inside the other. In fact, one more piece of data is required here: a small $\star$ for every boundary circle, in one of its adjacent unshaded regions – this tells us, when we place once planar tangle inside another one, how to line up the strings. For the exact details, which are somewhat technical, see [**?**].

A subfactor planar algebra is a shaded planar algebra, with some nice properties that allow us to do linear algebra on it.

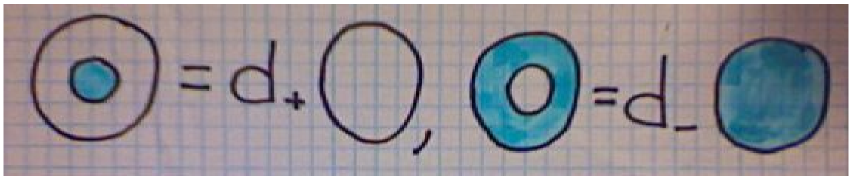
Definition 2.1 *A* subfactor planar algebra *is a shaded planar algebra such that*

- *Each vector space V_i for $i = +, -, 1, 2, \ldots$ has an involution * on it that plays well with tangles*$\star^{(1)}$
- *$dim(V_+) = dim(V_-) = 1$*
- *$d_+ = d_-$: Note that $dim(V_+) = dim(V_-) = 1$ implies that the element of V_+ coming from the tangle with one closed string[2] is a multiple d_+ of the empty*

$\star^{(1)}$ explain this more better –E

[1]Familiarly known as a 'spaghetti and meatballs' diagram.
[2]that is, a spaghetti-O

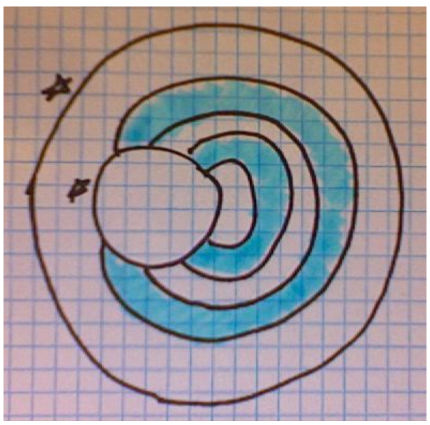
tangle. (There is a corresonding d_- for for V_-.)



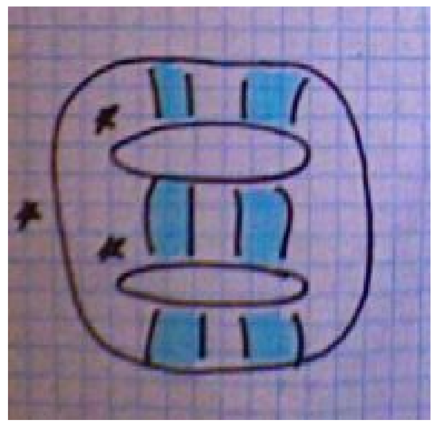
In a subfactor planar algebra we require $d_+ = d_-$, and call this value the modulus d.$\star^{(2)}$

$\star^{(2)}$ can't we call this the dimension? –N

- *Positivity: Another consequence of dim$(V_+) = $ dim$(V_-) = 1$ is that the tangles*



and



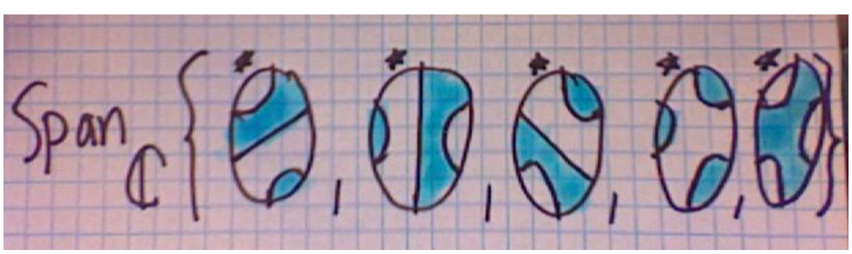
define a trace $V_i \to V_+$ and multiplication $V_i \times V_i \to V_i$; these can be used to define a bilinear form via $\langle X, Y \rangle :=$ tr$(Y^* X)$. This bilinear form must be positive definite in a subfactor planar algebra.

- *Sphericality: Two tangles that are spherically isotopic induce the same maps.*

For more details about subfactor planar algebras, see [**?**, §4].

2.2 Temperley-Lieb and the $\mathcal{D}4$ planar algebra

Everyone's favorite example of a subfactor planar algebra is $\mathcal{T}L(\delta)$, the Temperley-Lieb planar algebra with modulus δ, which is defined for all $\delta \in \{2\cos\frac{\pi}{n} | n \geq 3\} \cup [2, \infty)$. If $\delta > 2$, then $\mathcal{T}L(\delta)_n$ is the vector space with basis consisting of tangles with no inputs, n outputs and no closed strings – for instance, $\mathcal{T}L(\delta)_3$ is

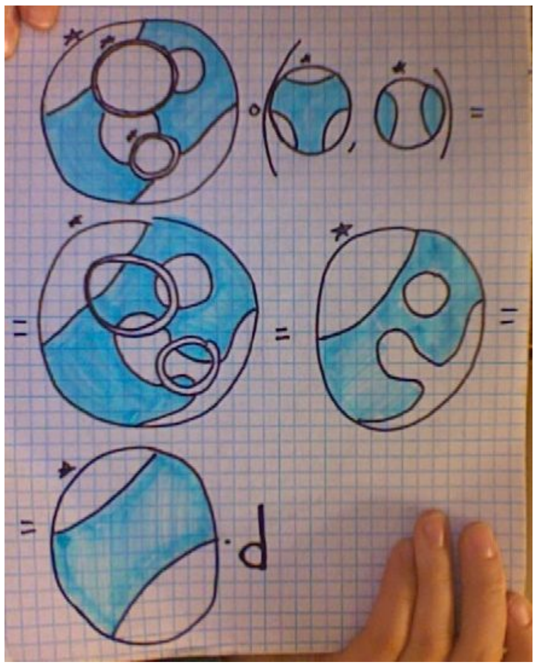


For δ in the range $\{2\cos\frac{\pi}{n} | n \geq 3\}$, we first define a planar algebra TL as above – that is, $TL(\delta)_n$ is the vector space with basis consisting of tangles with no inputs,

n outputs and no closed strings; then we make this a subfactor planar algebra by forcing positive definiteness; that is, $\mathcal{T}L(\delta)$ is

$$\frac{\mathcal{T}L(\delta)}{\langle X \text{ such that for some } Y,\ tr(Y^*X) = 0\rangle}.$$

The action of a general tangle on the pictures generating $\mathcal{T}L(\delta)$ is pretty straightforward – put the $\mathcal{T}L(\delta)$ pictures into the inner disks of the tangle, erase boundaries that are no longer boundaries, and throw out closed loops by multiplying the resulting picture by d. Here's an example:



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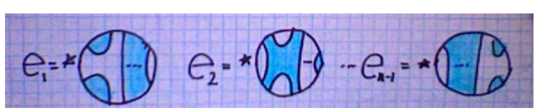
Here are some useful facts about Temperley-Lieb:

- Any subfactor planar algebra with modulus d contains $TL(d)$ as a sub-planar algebra
- $\mathcal{T}L_n$ is multiplicatively generated by elements e_i for $i = 1 \ldots n-1$



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- For each n, $\mathcal{T}L_n$ has one new minimal central idempotent; it is known as the Jones-Wenzl idempotent and is the unique element having $e_i f^{(n)} = f^{(n)} e_i = 0$ for $i = 1 \ldots n-1$ and $f^{(n)} f^{(n)} = f^{(n)}$.
- $\mathrm{tr}(f^{(n)}) = [n]_q$ where $[n]_q := -\frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}$, with q a solution of $-q^2 - q^{-2} = d$.

It's hard to overstate the importance of understanding Temperley-Lieb; most of the structure of a general planar algebra is already visible in $\mathcal{T}L$, and understanding the interaction between the $\mathcal{T}L$ subalgebra and the rest of the algebra can be extremely useful.

On to our next example: the $\mathcal{D}4$ planar algebra. This is a planar algebra which is planarly generated by $\mathcal{TL}(\sqrt{3})$ and a single 2-box R. By "generated as a planar algebra," we mean that any element of the planar algebra is the image of some tangle applied to some sequence of the generators. The generator R satisfies

- $\rho(R) = -R$ ⋆(3)
- $R^* = R$.
- $\epsilon_1(R) = \epsilon_2(R) = \epsilon_3(R) = \epsilon_4(R) = 0$ ⋆(4)
- and $\mathrm{tr}(R^2) = [3]_q$ ⋆(5)

⋆(3) pic –E

⋆(4) pic –E

⋆(5) pic –E

(Note that the last three of these are mostly harmless, as they simply specify that the generator of the non-$\mathcal{TL}$ part of our planar algebra should be chosen to be self-adjoint, orthogonal to TL and normalized a certain way.)

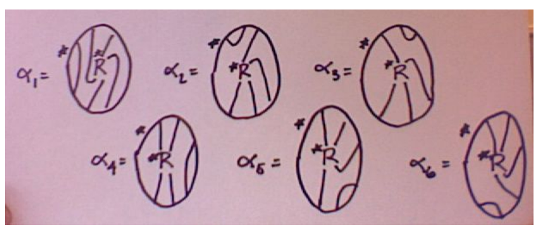
See [Jones, Annular Tangles] for more information about why the $\mathcal{D}4$ planar algebra is generated in this way.

Here are some more relations on $\mathcal{D}4$ which we will need:

Lemma 2.2 (1) $f^{(5)} = 0$

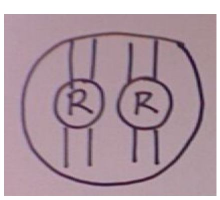
(2) *$R^2 = f^{(2)}$ and Comult(R,R)=$\frac{1}{\sqrt{3}}Id - e_1$.*

(3) *The "annular consequences" of R, that is,*



satisfy the relations $\alpha_i - \sqrt{3}\alpha_{i+1} + 2\alpha_{i+2} - \sqrt{3}\alpha_{i+3} + \alpha_{i+4} = 0$, with the convention that for subscripts outside of $\{1,2,3,4,5,6\}$, $\alpha_{i+6n} := (-1)^n\alpha_i$.

(4) *The box consisting of two R's placed next to each other is in $\mathcal{TL}$; specifically*



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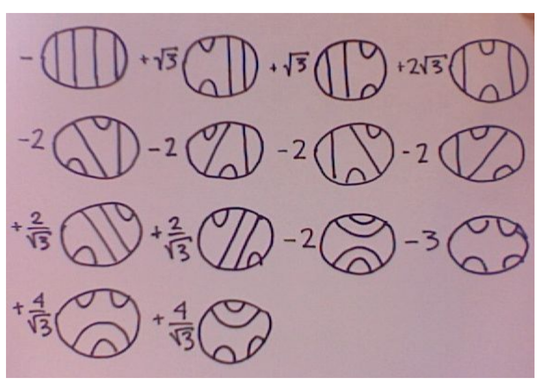
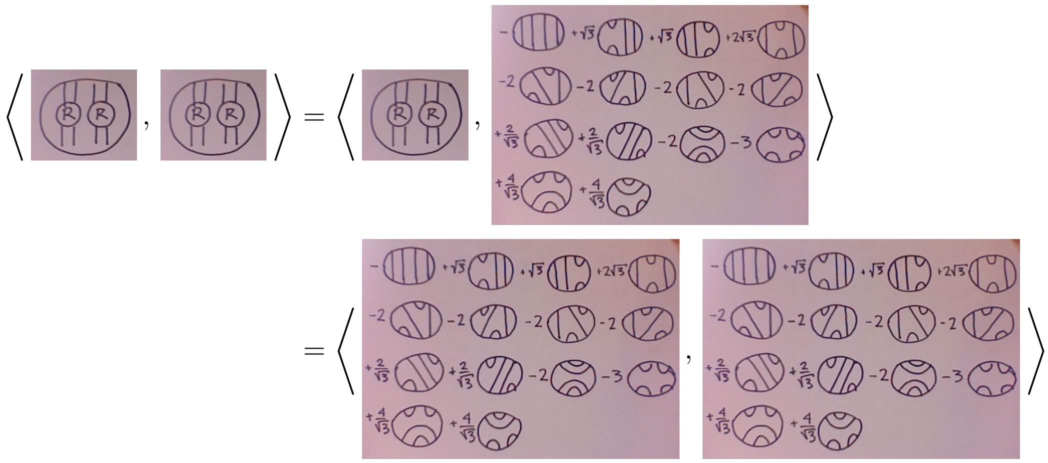


Proof (1) We have $\mathrm{tr}(f^{(5)}) = [6]_q = 0$, hence by positivity $f^{(5)} = 0$.

(2) Rotational eigenvalue considerations tell that this is in $\mathcal{TL}$; then knowing that either top and bottom caps, or both side caps, is zero tells us which $\mathcal{TL}$ element.

(3) Let $\sigma_i = \alpha_i - \sqrt{3}\alpha_{i+1} + 2\alpha_{i+2} - \sqrt{3}\alpha_{i+3} + \alpha_{i+4}$; we can check that $\langle \sigma_i, \sigma_i \rangle = 0$.

(4) We could check that



but this would require us to compute about 200 traces. Instead, we use the fact $f^{(5)} = 0$ to write that the five-strand identity is equal to a weighted sum of the other elements of $\mathcal{TL}_5$ (with coefficents as givenin [Morrison, A formula for the Jones-Wenzl Projections].) This seems not to have made our lives that much simpler; we replace one picture with 41 pictures.

However, if we consider the picture where two Rs are conected by one string, and replace 5 of the remaining strings with minus the rest of the Jones-Wenzl projection, then most of these 41 pictures cap off an R in some way, giving zero; only 9 don't. So we get

$$\Rightarrow$$

and we have a Comult(R,R) in each of these. Using the previous relation we have that this is actually a $\mathcal{TL}$ element:

Now, we do the same thing (replacing 5 strings with 9 pictures involving caps) to two Rs next to each other, and get the desired result.

□

TODO: The D4 planar algebra really has principal graph D4; see later for what this means, and even later for the proof.

2.3 Planar Algebras, knots and links

Take any knot or link, lay it down flat, and replace each crossing with the element of $\mathcal{TL}(\delta)_2$ which is $q \cdot Id + q^{-1} \cdot e_1$ for $q^2 + q^{-2} = -\delta\star^{(6)}$ – what you get is a polynomail in q. This is the Jones polynomial.

$\star^{(6)}$ minus delta right? –E

Now do the same thing, but first cable the knot/link – ie replace each strand by n strands – and on each component of the knot/link (ie, each piece of string between two crossings) place a Jones-Wenzl idempotent $f^{(n)}$. Resolving the crossings gives the colored Jones polynomial.

This process seems generalizable, doesn't it?

3 Why did that work?

If you've spent some time with people who enjoy their category theory, you'll have picked up the idea that "links invariants come from braided tensor categories". While this isn't always the case — there are plenty of link invariants which don't seem to arise in this way — there are many pretty examples. In particular, all of the link invariants defined using quantum groups, via R-matrices, fit into this pattern: the category of representations of a quantum group $U_q(\mathfrak{g})$ is a braided tensor category. Sadly, the invariant we've given a recipe for above doesn't quite fit the mould. On the other hand, happily, it doesn't fall too far outside it. Below, we'll describe quite explicitly something not-too-far from a braided tensor category, which is still sufficient to produce a link invariant, and explain how we can sometime produce examples from a subfactor, and in particular from the D_4 subfactor.

3.1 Subfactors give 'bi-oidal categories'.

If we were going to attempt the impossible, and show you how to produce a braided tensor category from a subfactor, we'd try this in two steps: describe a tensor category first, and then give it a braiding. There are difficulties at both steps. First, subfactors don't exactly give tensor categories, in the usual sense, but something we'll call a 'bi-oidal category', and define in this section. (A common synonym for 'tensor category' is 'monoidal category', and our made up name is intended as an analogue for this.) Second, once we've adapted the notion of 'braided' to bi-oidal categories, we still might not know how to put such a braided structure on the category. However, there's always something we can try, which we describe in §**??**, and explain why it actually works for the D_4 subfactor in §**??**.

3.1.1 What is a subfactor?

You should happily skip ahead to §**??** if any of this sounds familiar!

TODO: Write this!

3.1.2 Bimodules for von Neumann algebras

Understanding bimodules for von Neumann algebras, and their tensor products, can be tricky. Essentially there are two different versions of the theory, depending whether one asks for bimodules to be Hilbert spaces, or just vector spaces. Happily, the Hilbert space version is the difficult one, and for our purposes we can get away with using the simpler 'algebraic' version, where we just ask for vector spaces. The expense, however, is having to refer to some papers with some high-powered analysis for a few results.

Thus, if A and B are von Neumann algebras, the category $A - \mathrm{Mod}' - B$ of 'algebraic' $A - B$ bimodules has as objects all vector spaces X with commuting actions of A and B^{op}, and as morphisms all linear maps commuting with both actions.

(In the Hilbert space version, we'd unsurprisingly ask for for Hilbert spaces and bounded linear maps.) There's an unnecessary looking prime in this notation simply because in the next section we'll pass to a smaller, more useful subcategory which we'll want to be able to refer to without the prime.

Given three von Neumann algebras A, B and C, and two bimodules $X \in A - \mathrm{Mod}' - B$, and $Y \in B - \mathrm{Mod}' - C$, we can form the tensor product $X \otimes_B Y \in A - \mathrm{Mod}' - C$, defined easily[3] as

$$X \otimes_B Y = \frac{X \otimes_{\mathbb{C}} Y}{\{xb \otimes y - x \otimes by \mid x \in X, y \in Y, b \in B\}}.$$

3.1.3 The bi-oidal category of bimodules for a subfactor

When we're looking at a II_1 subfactor $N \subset M$, we might think about the four categories $N - \mathrm{Mod}' - N$, $N - \mathrm{Mod}' - M$, $M - \mathrm{Mod}' - N$ and $M - \mathrm{Mod}' - M$, and take their disjoint union as categories, which we'll call $\mathrm{Mod}'(N \subset M)$ (soon we'll define a smaller subcategory — the prime is there in the name just because we'll end up being more interested in the subcategory, and so reserve the name for it). On $\mathrm{Mod}'(N \subset M)$ we have a 'partially defined' tensor product. Each object, being in one of the four categories, has two 'labels'; the algebra it's a left module over is its 'left label', and the algebra it's a right module over is its 'right label'. We can take the tensor product of X and Y exactly if the right label of X agrees with the left label of Y, in which case the tensor product just means the algebraic tensor product over that algebra. When the labels don't agree, there simply isn't a tensor product.

This structure is the first example of what we're going to call a bi-oidal category, so we'll give the definition now.

Definition 3.1 *A bi-oidal category $\mathcal{C}$ with labels A and B is a category in which*

- *each object $\mathcal{O} \in \mathcal{C}$ has a 'left label' and a 'right label', each either A or B, often indicated by notation like ${}_A\mathcal{O}_B$,*
- *and with ${}_X\mathcal{C}_Y$ denoting the full subcategory of objects with left label X and right label Y, there are a 'tensor product over a label' functors*

$$\otimes_Y : {}_X\mathcal{C}_Y \times {}_Y\mathcal{C}_Z \to {}_X\mathcal{C}_Z$$

  which are associative whenever this makes sense, that is

$$\otimes_X \circ (\mathbf{1}_{\mathcal{C}_X} \times \otimes_Y) = \otimes_Y \circ (\otimes_Y \times \mathbf{1}_{{}_Y\mathcal{C}})$$

  on the sub-category $\mathcal{C}_X \times {}_X\mathcal{C}_Y \times {}_Y\mathcal{C}$ of $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$,
- *and there are objects ${}_AA_A$ and ${}_BB_B$ which are identities for the corresponding tensor products, that is, for $X = A$ or B, $U \in XC$ and $V \in CX$,*

$$X \otimes_X U = U,$$
$$V \otimes_X X = V.$$

TODO: Does that make any sense? Is it missing something?

[3] It's worth saying 'easily' here, because the corresponding tensor product for the Hilbert space version of bimodules is complicated; it's Connes' fusion. See [**?**] for details.

3.1.4 The bimodules generated by M.

The category $\text{Mod}'(N \subset M)$ has a profusion of objects, and we're about to cut it down. As an example, the algebra M, considered as a vector space with commuting left and right actions of N and M (by multiplication!), is a bimodule. Being a II_1 subfactor, M has a unique normalized trace, and hence an inner product ($\langle x, y\rangle = \text{tr}(y^* x)$), and we can complete M with respect to this inner product to obtain a Hilbert space called $L^2(M)$. This is still a bimodule, and hence is also an object in our category. In some sense, however, it's too closely related to the original bimodule M for our purposes, without actually being isomorphic in the category. Our next definition avoids this sort of problem, amongst several others!

Now pick out two objects in this category, namely the algebra M considered as an $N - M$ bimodule, written ${}_N M_M$, and the algebra M considered as an $M - N$ bimodule, written ${}_M M_N$.

TODO: continue...

3.1.5 Bi-oidal categories with duals

TODO: Mention sphericalness. Define $\widehat{\bigotimes}$.

3.1.6 Temperley-Lieb

The simplest example of a bi-oidal category is the (perhaps familiar!) shaded Temperley-Lieb category.

TODO: Finish this..., having read emily's section on temperley-lieb as a planar algebra above.

3.1.7 The principal graphs of a singly generated bi-oidal category

TODO:

3.2 Translating back and forth: spherical categories and planar algebras.

Having read this far, you'll have noticed the rapidly widening gap between the perspective of §**??**, which described the $\mathcal{D}4$ planar algebra, and §**??**, which explained how to associate a bi-oidal category to a subfactor. This section reunites these perspectives, by explaining the translation back and forth between a singly generated bi-oidal category, with a spherical structure, and a shaded planar algebra.

The fundamental idea is that spherical categories are essentially the same things as a 'general planar algebra' (i.e., not necessarily with the shadings we asked for in the definition of a shaded planar algebra, and certainly not with the dimension and positivity restrictions in the definition of §**??** of a subfactor planar algebra). For each specialization you might make in the definition of a spherical category, there is a corresponding potential variation in the definition of a planar algebra. For the translation in its most general setting, see [**?**, § 5]. For our purposes, however, we're more interested in picking out the class of spherical categories that corresponds exactly to the notion of shaded planar algebra used in the theory of subfactors. As a result we restrict our attention first to bi-oidal categories, to match the shading requirements on the planar algebra side, and secondly, more significantly, to singly generated bi-oidal categories. The description of this more specific translation differs somewhat from that in [**?**], as we'll construct a planar algebra from a category making use of only a certain sub-category, and conversely reconstruct the category in two steps; recovering the sub-category, and then the entire original category by taking its Karoubi envelope, defined below. $\star^{(7)}$

$\star^{(7)}$ I don't much like this paragraph; it sounded better when Noah said it – could you try a partial rewrite? –S

TODO: put in something about this being familiar to analysts; thinking about projections all the time!

3.2.1 From spherical bi-oidal categories to shaded planar algebras

Given a spherical bi-oidal category $\mathcal{C}$, equipped with a generating object ${}_AV_B$ with left label A and right label B, we construct the shaded planar algebra $\mathcal{P}(\mathcal{C})$ as follows:

Definition 3.2 *The shaded planar algebra $\mathcal{P}(\mathcal{C})$ has disc spaces $\mathcal{P}(\mathcal{C})_k = \mathrm{End}\,(C)\widehat{\bigotimes}^k V$. Given a shaded planar tangle T with r internal discs with $2k_1, 2k_2, \ldots, 2k_r$ marked points, and $2k_0$ marked points on the external disc, we construct the linear map*

$$\mathcal{P}(\mathcal{C})(T) : \mathcal{P}(\mathcal{C})_{k_1} \otimes \mathcal{P}(\mathcal{C})_{k_2} \otimes \cdots \otimes \mathcal{P}(\mathcal{C})_{k_r} \to \mathcal{P}(\mathcal{C})_{k_0}$$

*by first choosing an up-to-isotopy representative of T in the manner illustrated in Figure **??**, which we call a 'rectangular planar tangle'. Given elements $f_1, \ldots, f_r$ of the appropriate endomorphism spaces, we form a composition in the category $\mathcal{C}$, turning cups and caps into copairing and pairing maps as usual, translating vertical stacking into composition, and horizontal juxtaposition into tensor product.*

We demonstrate this composition explicitly for the tangle T from Figure **??** in Figure **??**. The resulting composition is then an endomorphism in the appropriate disc space. That the construction is independent of the choice of rectangular planar tangle follows from the axioms for a spherical category. The resulting map is a linear map, simply because composition and tensor product in the category are all appropriately multilinear.

Notice that this construction didn't depend explicitly on the whole category $\mathcal{C}$, but just on a certain subcategory, $\mathcal{C}_{|\widehat{\bigotimes} V}$, the full subcategory in which the only objects are the alternating tensor powers of the generating object V.

3.2.2 From shaded planar algebras to spherical bi-oidal categories, step 1.

For the reverse construction, we show how to construct a category $\mathcal{C}'(\mathcal{P})$ from a shaded planar algebra, in which the only objects are alternating tensor powers of

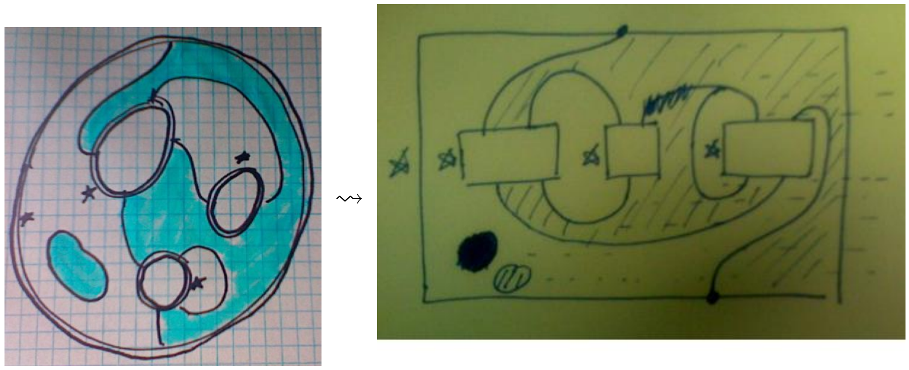


Figure 1: Turning a planar tangle into a 'rectangular planar tangle'. We ask that there is a horizontal strip midway up the rectangle, containing all of the internal boxes, possibly with vertical strands interspersed, and that above and below this strip we see only caps and cups.

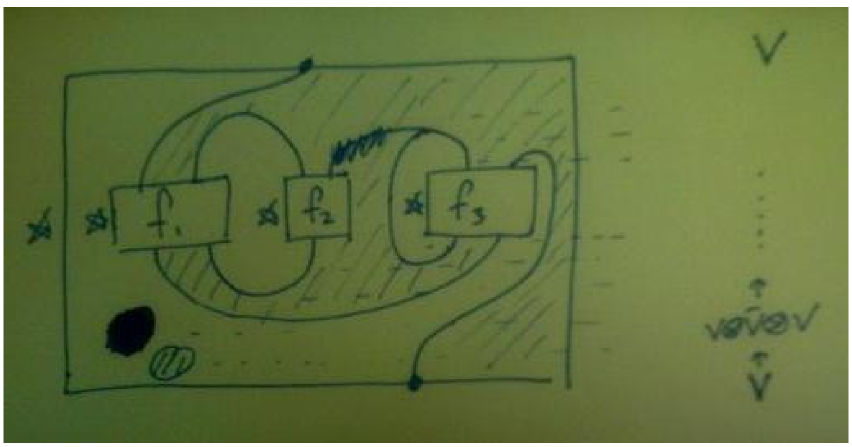


Figure 2: Reading from bottom to top, we compose a copairing or pairing map, for each cup or cap respectively, with the endomorphism

a 'formal' tensor generating object V, and then, after defining a Karoubi envelope in §**??**, explain in §**??** why the category $\mathcal{C}(\mathcal{P}) = \mathbf{Kar}\left(\mathcal{C}'(\mathcal{P})\right)$ is really what we want. That section also proves that the two constructions in either direction really are inverses.

The bi-oidal category $\mathcal{C}'(\mathcal{P})$ has

TODO:

3.2.3 What is the Karoubi envelope?

A *projection* is an endomorphism p satisfying $p^2 = p$. In many contexts in mathematics, if p is a projection then so is $1 - p$, and together these two projections decompose space as a direct sum of the image of p with the image of $1 - p$. Thus a projection often gives rise to a decomposition of space into pieces which are hopefully simpler.

The equation $p^2 = p$ makes sense for an endomorphism in an arbitrary category, so projections make sense in an arbitrary category. And if p is a projection then so is $1-p$ in an arbitrary *additive* category. But in a general (additive or not) category, "the image of p" (or of $1-p$) may or may not make sense.

The *Karoubi envelope*[4] of a category $\mathcal{C}$ is a way of adding objects and morphisms to $\mathcal{C}$ so that every projection has an image and so that if $p : \mathcal{O} \to \mathcal{O}$ is a projection and $\mathcal{C}$ is additive, then (with the proper interpretation) $\mathcal{O} \cong \operatorname{im} p \oplus \operatorname{im}(1-p)$. Thus sometimes complicated objects can be simplified in the Karoubi envelope of $\mathcal{C}$, while in $\mathcal{C}$ they may be indecomposable.

Let us turn to the formal definitions.

Definition 3.3 *Let $\mathcal{C}$ be a category. An endomorphism $p : \mathcal{O} \to \mathcal{O}$ of some object $\mathcal{O}$ in $\mathcal{C}$ is called* a projection *if $p \circ p = p$. The Karoubi envelope $\mathbf{Kar}(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are ordered pairs $(\mathcal{O}, p)$ where $\mathcal{O}$ is an object in $\mathcal{C}$ and $p : \mathcal{O} \to \mathcal{O}$ is a projection. If $(\mathcal{O}_1, p_1)$ and $(\mathcal{O}_2, p_2)$ are two such pairs, the set of morphisms in $\mathbf{Kar}(\mathcal{C})$ from $(\mathcal{O}_1, p_1)$ to $(\mathcal{O}_2, p_2)$ is the collection of all $f : \mathcal{O}_1 \to \mathcal{O}_2$ in $\mathcal{C}$ for which $f = f \circ p_1 = p_2 \circ f$. An object $(\mathcal{O}, p)$ in $\mathbf{Kar}(\mathcal{C})$ may also be denoted by $\operatorname{im} p$.*[5]

The composition of morphisms in $\mathbf{Kar}(\mathcal{C})$ is defined in the obvious way (by composing the corresponding f's). The identity automorphism of an object $(\mathcal{O}, p)$ in $\mathbf{Kar}(\mathcal{C})$ is p itself. It is routine to verify that $\mathbf{Kar}(\mathcal{C})$ is indeed a category. There is an obvious embedding functor $\mathcal{I} : \mathcal{O} \mapsto (\mathcal{O}, I)$ of $\mathcal{C}$ into $\mathbf{Kar}(\mathcal{C})$ and quite clearly, $\operatorname{Hom}_{\mathbf{Kar}(\mathcal{C})}(\mathcal{I}\mathcal{O}_1, \mathcal{I}\mathcal{O}_2) = \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}_1, \mathcal{O}_2)$ for any pair of objects $\mathcal{O}_{1,2}$ in $\mathcal{C}$. Thus we will simply identify objects in $\mathcal{C}$ with their image via $\mathcal{I}$ in $\mathbf{Kar}(\mathcal{C})$.

Below we will assume that $\mathcal{C}$ is an additive category and that direct sums of objects make sense in $\mathcal{C}$. As in [**?**], there is no loss of generality in making these assumptions as formal sums of morphisms and formal direct sums of objects may always be introduced.

Lemma 3.4 *Let $p : \mathcal{O} \to \mathcal{O}$ be an endomorphism in $\mathcal{C}$.*

(1) *If p is a projection then so is $1-p$.*

(2) *In this case, $\mathcal{O} \cong \operatorname{im} p \oplus \operatorname{im}(1-p)$ in $\mathbf{Kar}(\mathcal{C})$.*

Proof (1) $(1-p)^2 = 1 - 2p + p^2 = 1 - 2p + p = 1 - p$ (sorry for the damage to the rainforest).

[4]The Karoubi envelope construction [**?**] was first described in [**?**], a few years before Karoubi. It has previously been used in motivic cohomology [**?**], in diagrammatic representation theory, e.g. [**?**], and by the first author in an application to Khovanov homology in [**?**]. The text of this section is lifted largely verbatim from that paper.

[5]If you're worried about just introducing $\operatorname{im} p$ as notation, when you already know a category-theoretic definition of 'image', don't be; this *is* actually an image.

(2) The isomorphism $\mathcal{O} \to \operatorname{im} p \oplus \operatorname{im}(1-p)$ is given by the 1×2 matrix $\begin{pmatrix} p & 1-p \end{pmatrix}$. Its inverse is the 2×1 matrix $\begin{pmatrix} p \\ 1-p \end{pmatrix}$.

□

Observe that if p is a projection on $\mathcal{O}$ and p' is a projection on $\mathcal{O}'$, then $\operatorname{Hom}((\mathcal{O}, p), (\mathcal{O}', p'))$ may be naturally identified with $p' \operatorname{Hom}(\mathcal{O}, \mathcal{O}') p$. In fact, even before taking the Karoubi envelope, $\operatorname{Hom}(\mathcal{O}, \mathcal{O}')$ can be expressed as a direct sum of 4 'matrix entries', each obtained by pre-composing with p or $1-p$, and post-composing with p' or $1-p'$.

3.2.4 From shaded planar algebras to spherical bi-oidal categories, step 2.

TODO:

TODO: Explain that the constructions are inverses. TODO: Really, we'd explain what morphisms of shaded planar algebras are, and explain that the constructions give an equivalence of categories!

3.3 Braidings

3.3.1 Braidings for monoidal categories

If you already know this stuff, skip it!

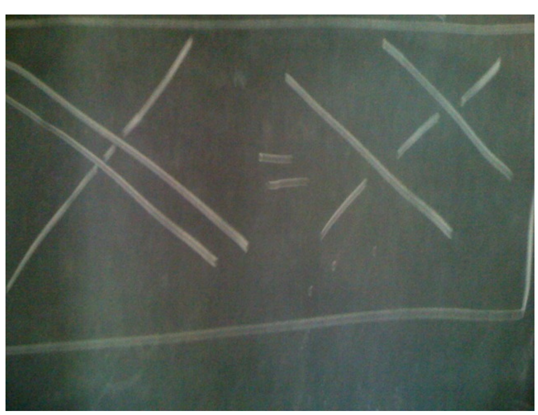
For a monoidal category, a braiding is a family of natural isomorphisms

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying the relations of braids. For example, one requires that

$$c_{X \otimes Y, Z} = (c_{X,Y} \otimes \mathbf{1}) \circ (\mathbf{1} \otimes c_{Y,Z})$$

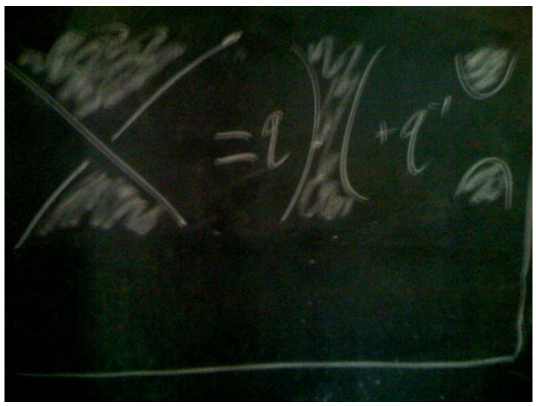
so that the following equation of diagrams holds.



3.3.2 Braidings for bi-oidal categories

What is a braiding for a bi-oidal category? Let's consider a typical two bimodules inside $-\operatorname{Mod}-$, say $X \in N - \operatorname{Mod} - M$ and $Y \in M - \operatorname{Mod} - M$. Notice that $X \otimes_M Y$ makes perfect sense, but there is no way to take the tensor product in the other order! So a braiding can't just be a map $X \otimes Y \to Y \otimes X$. So what are we meant to do?

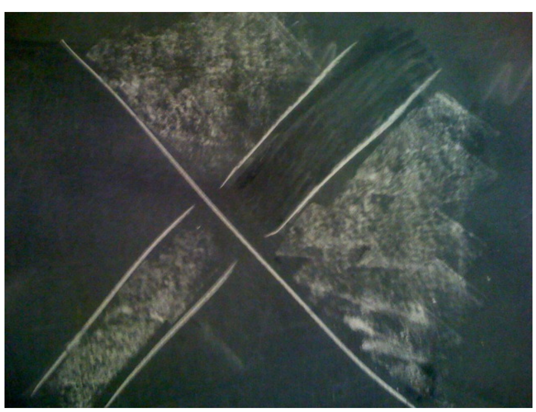
Rather than guessing a new definition, we will return to the motivating example: the Jones polynomial. There is a braiding on the bi-oidal category $\mathcal{TL}_d$ (see **??**) given by the usual Kauffman bracket formula.



This braiding is a map

$$c_{M,\bar{M}} : {}_N M_M \otimes_M {}_M \overline{M}_N \to {}_N M_M \otimes_M {}_M \overline{M}_N.$$

What about using the Kauffman bracket to compute the braiding of more complicated bimodules? For example where does $c_{M \otimes \bar{M}, M}$ live? To decide this question we draw the diagram below and read off what it means.



We read off that the braiding is a map

$$c_{M \otimes \bar{M}, M} : \left({}_N M_M \otimes_M {}_M \bar{M}_N \right) \otimes_N {}_N M_M \to {}_N M_M \otimes_M \left({}_M \bar{M}_N \otimes_n {}_N \bar{M}_M \right).$$

In general, we see that if we braid some object across an odd object, then we need to reverse all the shadings. If we braid some object across an even object, then we leave the shadings the same. Since the braiding is natural, we will also need a way to reverse the shadings on morphisms. For Temperley-Lieb, it is clear what shading reversing does. But for a more general planar algebra a shading-reversing is an additional piece of data.

Let's think about how shading-reversing might work for $\mathcal{D}4$. We need to know how to shading-reverse the 2-box R, the obvious picture doesn't make any sense.

PICTURE

Since we know how things in Temperley-Lieb shading-reverse, we know that the shading-reverse of R must satisfy the defining relations of R. Hence the shading-reverse of R must be one of the following two options.

PICTURE

Since either possibility seems reasonable, we will allow ourselves the flexibility to use either shading change. In the end this will be necessary in order to make the following to pictures work.

PICTURE

Definition 3.5 *A shading reversing for a bi-oidal category is two involutions* * *and* $_\star$ *(read "over" and "under") which reverse all the shadings. We also require that* $\mathbf{1}^\star = \mathbf{1} = \mathbf{1}_\star$*, that* $(\otimes)_\star = {}_\star \otimes {}_\star$*, and that* $(\otimes)^\star = {}^\star \otimes {}^\star$*.*

When something passes under something odd you apply the "under" involution to it, when something passes over something odd you apply the "over" involution to it. We use notation like $X^{\star \deg Y}$ to mean "apply the over involution to X if the degree of Y is odd, but leave X alone if the degree of Y is even."

Definition 3.6 *A braiding for a bi-oidal category is a family of isomorphisms* $c_{X,Y} : X \otimes Y \to Y_{\star \deg X} \otimes X^{\star \deg Y}$ *such that*

(1) c *is natural:*

- $c_{X,\mathbf{1}} = \mathbf{1}_X = c_{\mathbf{1},X}$.
- *For any* $f \in \operatorname{Hom}(X, Z)$*, we have that* $c_{Z,Y} \circ (f \otimes \mathbf{1}) = (\mathbf{1} \otimes f^{\star \deg Y}) \circ c_{X,Y}$.
- *For any* $f \in \operatorname{Hom}(Y, Z)$*, we have that* $c_{X,Z} \circ (\mathbf{1} \otimes f) = (f_{\star \deg X} \otimes \mathbf{1}) \circ c_{X,Y}$.

(2) c *is natural with respect to color changing:*

- $c^\star_{X,Y} = c_{X^\star, Y^\star}$
- $(c_{X,Y})_\star = c_{X_\star, Y_\star}$.

(3) c *is quasitriangular:*

- $c_{X \otimes Y, Z} = (c_{X, Z_{\star \deg Y}} \otimes \mathbf{1}) \circ (\mathbf{1} \otimes c_{Y,Z})$
- $c_{X, Y \otimes Z} = (\mathbf{1} \otimes c_{X^{\star \deg Y}, Z}) \circ (c_{X,Y} \otimes \mathbf{1})$.

INSERT PROOFS OF THE REIDEMEISTER RELATIONS

3.3.3 Where might you look for a braiding?

One nice thing about a braiding (as opposed to more general commutors) is that the quasi-triangularity condition combined with naturality, tells you that the entire braiding is determined by the braiding for $V \otimes V$ where V is a tensor generator. To see this notice that by naturality $c_{X,Y} = c_{V^{\otimes m}, V^{\otimes n}|X \otimes Y}$, and that by quasitriangularity $c_{V^{\otimes m}, V^{\otimes n}}$ can be written in terms of only $c_{V,V}$. In pictures this looks like:

PICTURE

Suppose we're looking at a subfactor planar algebra, all we need to know to determine a braiding is how to braid M with $\bar{M}$. But the Kauffman bracket formula already gives us a braiding for $M \otimes \bar{M}$. In fact, if the principal graph starts out like A_3, you can show that this is the only possible braiding (for example, for D_{2n} for $n > 2$). So rather than working hard to find a braiding, let's just look at the one we already have and see if it works.

What do we need to check to see that this braiding works?

PICTURE

3.4 Back to D_4.

3.4.1 The principal graphs

This paper is available online at arXiv:`arXiv:?????`, and at `http://tqft.net/d4`.