Coincidences of tensor categories

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Outline

1. Motivation and overview
   - Goal and motivation
   - The quantum subgroup D_{2n} and its structure
   - Coincidences involving D_{2n}
2. Recognizing braided categories via $X \otimes X$
   - $P \otimes P$
   - Recognizing categories by $X \otimes X$
   - More on G_2
3. Sources of coincidences
   - Level-Rank duality
   - Kirby-Melvin symmetry
   - Coincidences of Dynkin diagrams
4. Examples and applications
   - G_2 and unitarity
   - Knot polynomial identities

> *"One goal [of this conference] is to establish the study of modular category theory as a subject in parallel to that of group theory."*

Along those lines someday we'd like to have a classification of modular categories. What would such a classification look like?

- Families of examples (for example, coming from quantum groups).
- Sporadic examples (for examples, see Scott's talk).
- A description of when the same category appears twice on the above lists (for example, see this talk!).

Examples of coincidences in finite group theory:

- $\mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$
- S_6 has an outer automorphism

Goal of this talk

Find coincidences of modular categories that involve the even part of the subfactor planar algebra D_{2n}. (This is a modular category related to $\mathfrak{so}_3$.)

Methods

- Hands-on technique for recognizing some important families of tensor categories by looking at $X \otimes X$.
- Theoretical explanation of coincidences using Level-Rank duality, Kirby-Melvin symmetry, and coincidences of Dynkin diagrams.

Why should you care?

Classifying modular categories is a long way off, so why should you care about coincidences? Because they have applications!

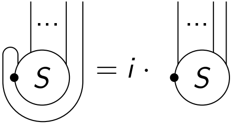
- These coincidences give strange identities between knot polynomials.
- One coincidence is $\frac{1}{2}D_{14} \cong (G_2)_{\frac{1}{3}}$. Since the former is known to be unitary, this coincidence answers a question of Rowell.

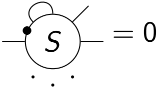
What is D_{2n}?

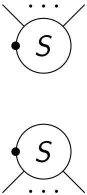
- Start with the braided category $TL_{q+q^{-1}}$ for $q = e^{\frac{\pi i}{4n-2}}$.
- "Modularize," by setting $f^{(4n-3)} \cong \mathbf{1}$
- This quotient isn't braided, instead it's a spin-modular category.
- But the even part $\frac{1}{2}D_{2n}$ is modular.
- $\frac{1}{2}D_{2n}$ is the modularization of $U_q(\mathfrak{so}_3) = \frac{1}{2}TL_{q+q^{-1}}$.
- D_{2n} is related to the Dynkin diagram D_{2n}, but it is not related to SO_{4n}.

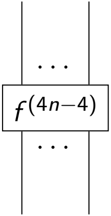
Skein theory description of D_{2n}

Fix $q = \exp(\frac{\pi i}{4n-2})$. Let $\mathcal{PA}(S)$ be the planar algebra generated by a single "box" S with $4n-4$ strands, modulo the following relations.

- A closed circle is equal to $[2]_q = (q+q^{-1}) = 2\cos(\frac{\pi}{4n-2})$ times the empty diagram.
- Rotation relation: 

- Capping relation: 

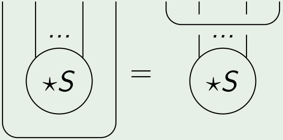
- Two S relation: 

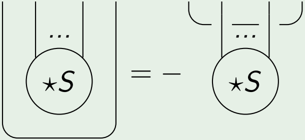
$= [2n-1]_q \cdot$ 

D_{2n} is not braided. However it has a "braiding up to sign."

Theorem

You can isotope a strand above an S box, but isotoping a strand below an S box introduces a factor of -1.

1. 

2. 

Since diagrams in $\frac{1}{2}D_{2n}$ have an even number of strands, $\frac{1}{2}D_{2n}$ is braided.

Theorem

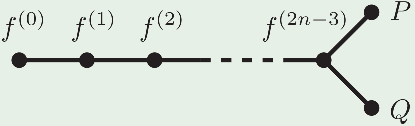
The planar algebra D_{2n} is semi-simple, with minimal projections $f^{(k)}$ for $k = 0, \ldots, 2n-3$ along with P and Q defined by

$$P = \frac{1}{2}\left(f^{(2n-2)} + S\right)$$

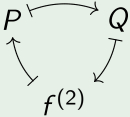
and

$$Q = \frac{1}{2}\left(f^{(2n-2)} - S\right).$$

The principal graph is the Dynkin diagram D_{2n}.



Coincidences involving D_{2n}

- $\frac{1}{2}D_4 \cong \mathbb{Z}/3$ (but an unusual braiding!) with $P \mapsto \chi_{e^{\frac{2\pi i}{3}}}$.
- $\frac{1}{2}D_6 \cong U_{e^{-\frac{2\pi i}{10}}}(\mathfrak{sl}_2 \times \mathfrak{sl}_2)^{unimodularize}$ with $P \mapsto V_{(1)} \boxtimes V_{(0)}$.
- $\frac{1}{2}D_8 \cong U_{e^{-\frac{2\pi i}{14}}}(\mathfrak{sl}_4)^{unimodularize}$ with $P \mapsto V_{(100)}$.
- $\frac{1}{2}D_{10}$ has an automorphism: 

.
- $\frac{1}{2}D_{14} \cong U_{e^{2\pi i \frac{23}{26}}}(\mathfrak{g}_2)$ with $P \mapsto V_{(10)}$.

(Technical point: on the righthand side we've changed the pivotal structure and modularized, furthermore we've been very careful about choosing the right root of q in the definition of the braiding.)

Outline

In D_{2n} we have the following decomposition for $P \otimes P$ (Izumi).

- If n is even: $P \otimes P \cong Q \oplus \bigoplus_{l=0}^{\frac{n-4}{2}} f^{(4l+2)}$
- If n is odd: $P \otimes P \cong P \oplus \bigoplus_{l=0}^{\frac{n-3}{2}} f^{(4l)}$

In particular,

- In D_4, $P \otimes P \cong Q$
- In D_6, $P \otimes P \cong \mathbf{1} \oplus P$
- In D_8, $P \otimes P \cong f^{(2)} \oplus Q$
- In D_{10}, $P \otimes P \cong \mathbf{1} \oplus f^{(4)} \oplus P$
- In D_{14}, $P \otimes P \cong \mathbf{1} \oplus f^{(4)} \oplus f^{(8)} \oplus P$

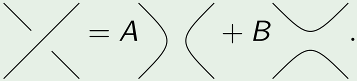
Suppose that X is an object in a unimodal braided semisimple tensor category $\mathcal{C}$.

if $X \otimes X \cong$	then $\mathcal{C}$ takes a braided functor from
A	cyclic group category
$1 \oplus A$	Temperley-Lieb category
$A \oplus B$	HOMFLY skein category
$1 \oplus A \oplus B$	Kauffman or Dubrovnik skein categories
$1 \oplus X \oplus A \oplus B$	$U_q(G_2)$ (if you're lucky...)

Furthermore, the eigenvalues of the braiding determine the parameters in the right column.

Proof.

These results follow from standard skein theory arguments. For example suppose $X \otimes X \cong 1 \oplus A$. Since $\mathrm{End}\,(X \otimes X)$ is 2-dimensional there must be a linear dependence of the form



Following Kauffman, rotate this equation, glue them together and apply Reidemeister 2 to see that $B = A^{-1}$ and $A^2 + A^{-2} = \dim V$. Hence there's a braided functor from Temperley-Lieb.
The other cases are only a little more complicated. □

Technical comments: Notice that TL is related to $U_q(\mathfrak{sl}_2)$ by changing the pivotal structure. Furthermore note that in the above argument A could be either square root of A^2. Finally the functor need not be full (=surjective on morphisms).

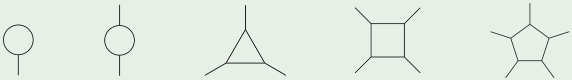
More on the G_2 case.

Definition

A trivalent vertex is a rotationally symmetric map $X \otimes X \to X$. A tree is a trivalent graph without cycles (but allowing disjoint components).

Theorem (Kuperberg)

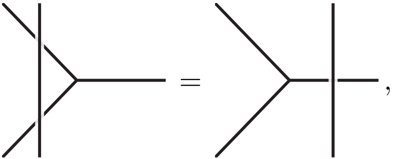
Suppose we have a trivalent vertex in a semisimple pivotal tensor category $\mathcal{C}$, such that the graphs



are each equal to linear combinations of trees. Then there is a faithful pivotal functor $Rep(U_q(G_2)) \to \mathcal{C}$ for some $q \in \mathbb{C}$.

Simplifying Kuperberg's conditions

Generically having a braiding implies a pentagon-bursting relation.



Expand the crossings and there will be exactly one pentagon. This gives an easier way to find a pentagon-bursting relation.

Outline

Level-rank duality

Relates $(\mathfrak{so}_3)_k$ to $(\mathfrak{so}_k)_3$.

Kirby-Melvin symmetry

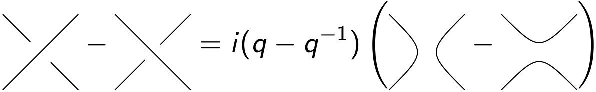
Generalized Kirby-Melvin symmetry relates $A \mapsto A \otimes B$ when $\dim B = 1$.

Coincidences of small Dynkin diagrams

$D_2 = A_1 \times A_1$, $D_3 = A_3$, and D_4 has a 'triality' automorphism.

Level-Rank duality a la Beliakova-Blanchet

- The BMW (or "Kauffman skein") category depends on parameters q and m.



- It has a manifest symmetry. If

$$q_{n,k} = \begin{cases} \exp\left(\frac{2\pi i}{n+k-2}\right) & \text{when } k \text{ is odd, or} \\ \exp\left(\frac{2\pi i}{2(n+k-2)}\right) & \text{when } k \text{ is even.} \end{cases}$$

then switching between $m = n$, $q = q_{n,k}$ and $m = k$, $q = -q_{n,k}^{-1}$ doesn't change the relations.

- You should think of this symmetry as relating "$U_{q_{n,k}}(O_n)$" and "$U_{-q_{n,k}^{-1}}(O_k)$,"
- except that the usual quantum group is really $U_q(\mathrm{Spin})$.
- To get from O to SO you de-equivariantize, and to get from Spin to SO you restrict to the vector representations.
- Level-rank duality relates de-equivariantizations of the vector reps of the usual $\mathfrak{so}$ quantum groups.

$$BMW_{n,q_{n,k}} = BMW_{k,-q_{n,k}^{-1}}$$

$$BMW_{n,q_{n,k}} \xrightarrow{\det_n \cong \mathbf{1}} U_{q_{n,k}}^{vect}(\mathfrak{so}_n) \qquad BMW_{k,-q_{n,k}^{-1}} \xrightarrow{\det_k \cong \mathbf{1}} U_{-q_{n,k}^{-1}}^{vect}(\mathfrak{so}_k)$$

$$U_{q_{n,k}}^{vect}(\mathfrak{so}_n) \xrightarrow{\det_k \cong \mathbf{1}} U_{q_{n,k}}^{vect}(\mathfrak{so}_n)^{mod} \qquad U_{-q_{n,k}^{-1}}^{vect}(\mathfrak{so}_k) \xrightarrow{\det_n \cong \mathbf{1}} U_{-q_{n,k}^{-1}}^{vect}(\mathfrak{so}_k)^{mod}$$

$$U_{q_{n,k}}^{vect}(\mathfrak{so}_n)^{mod} \cong U_{-q_{n,k}^{-1}}^{vect}(\mathfrak{so}_k)^{mod}$$

Kirby-Melvin symmetry

- Generalized Kirby-Melvin symmetry relates A and $A \otimes B$ (for all A) when $\dim B = 1$.
- Replacing A with $A \otimes B$ modifies the braiding in an understandable way (because the overcrossing and undercrossing labelled by B differ by a scalar).
- In many cases generalized Kirby-Melvin symmetry says that A and $A \otimes B$ have the same image in the "unimodularized" category (where we first change the pivotal structure and then modularize). This is only true if you're very careful about the choice of root of q used in defining the braiding.

Coincidences of Dynkin diagrams

- $D_2 = A_1 \times A_1$, hence $U_q(\mathfrak{so}_4) \cong U_q(\mathfrak{sl}_2) \boxtimes U_q(\mathfrak{sl}_2)$.
- $D_3 = A_3$, hence $U_q(\mathfrak{so}_6) \cong U_q(\mathfrak{sl}_4)$.
- D_4 has a triality automorphism, hence $U_q(\mathfrak{so}_8)$ has a triality automorphism.

Outline

Unitary of $(G_2)_{\frac{1}{3}}$

Rowell showed that $(G_2)_{\frac{1}{3}}$ and $(G_2)_{\frac{2}{3}}$ are pseudo-unitary, and asked if they were unitary.

But $D_{14} \cong (G_2)_{\frac{1}{3}}$, and the former is unitary.

We expect that similar techniques would show that $(G_2)_{\frac{2}{3}}$ agrees with a subcategory of $(\mathrm{Sp}_6)_3$, and thus is unitary.

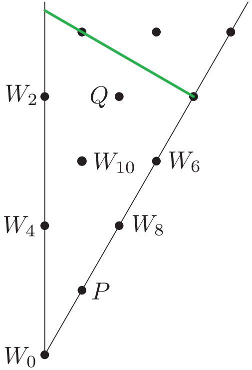


Figure: The positive Weyl chamber for G_2, showing the surviving irreducible representations in the semisimple quotient at $q = e^{2\pi i \frac{23}{26}}$, and the correspondence with the even vertices of $\mathcal{D}_{14}$.

Knot polynomial identities

Knot invariants from P

- The usual Reshetikhin-Turaev technique gives knot invariants coming from P.
- Any closed diagram involving a single S is zero.
- If K is a knot (but not a link!) then

$$\frac{1}{2}\mathcal{J}_{SU(2),(2n-2)}(K)(e^{\frac{2\pi i}{8n-4}}) = \mathcal{J}_{D_{2n},P}(K).$$

- So coincidences involving D_{2n} give identities relating colored Jones polynomials to other knot polynomials.

Theorem

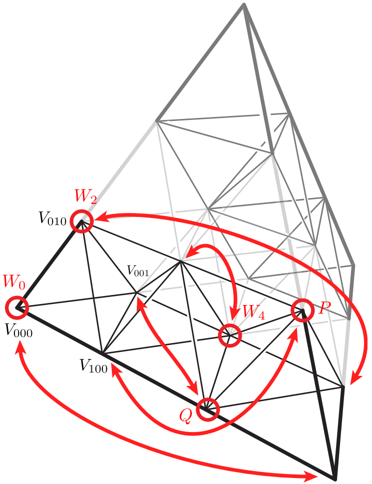
$$\mathcal{J}_{SU(2),(2)}(K)_{|q=e^{\frac{2\pi i}{12}}} = 2\mathcal{J}_{D_4,P}(K) = 2$$

$$\mathcal{J}_{SU(2),(4)}(K)_{|q=e^{\frac{2\pi i}{20}}} = 2\mathcal{J}_{D_6,P}(K) = 2\mathcal{J}_{SU(2),(1)}(K)_{|q=\exp(-\frac{2\pi i}{10})}$$

$$\mathcal{J}_{SU(2),(6)}(K)_{|q=e^{\frac{2\pi i}{28}}} = 2\mathcal{J}_{D_8,P}(K) = 2\,\mathrm{HOMFLYPT}(K)(q^4, q-q^{-1})_{|q=e^{\frac{-2\pi i}{14}}}$$

$$\mathcal{J}_{SU(2),(8)}(K)_{|q=e^{\frac{2\pi i}{36}}} = 2\mathcal{J}_{D_{10},P}(K) = 2\,\mathrm{Kauffman}(K)(-iq^7, i(q-q^{-1}))_{|q=-e^{\frac{-2\pi i}{18}}}$$

The D_8-SL_4 coincidence



Theorem

$$\mathcal{J}_{SU(2),(6)}(K)_{|q=\exp(\frac{2\pi i}{28})} = 2\mathcal{J}_{SU(4),(1,0,0)}(K)_{|q=\exp(-\frac{2\pi i}{14})}$$

Proof.

$$\begin{aligned}
\mathcal{J}_{SU(2),(6)}(K)(e^{\frac{2\pi i}{28}}) &= 2\mathcal{J}_{D_8,P}(K) \\
&= 2\mathcal{J}_{SO(6),2e_3}(K)(-e^{-\frac{2\pi i}{14}}) && \text{(LR)} \\
&= 2\mathcal{J}_{SU(4),2e_1}(K)(-e^{-\frac{2\pi i}{14}}) && (D_3 = A_3) \\
&= -2\mathcal{J}_{SU(4),e_1}(K)(-e^{-\frac{2\pi i}{14}}) && \text{(KM)} \\
&= 2\mathcal{J}_{SU(4),e_1}(K)(e^{-\frac{2\pi i}{14}}) && \text{(parity)}
\end{aligned}$$