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Numerical Evidence for Flatness of Haagerup's Connections

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Abstract. We give a numerical evidence of flatness of Haagerup's connections for subfactors with indices $\frac{5+\sqrt{17}}{2}$ and $4.377\cdots$ by numerical experiments on computer.

1. Introduction

Since V. F. R. Jones introduced the index theory of subfactors in [J1], the theory of operator algebras have been achieving a remarkable development, having relations with several fields of mathematics and physics. It has the connections with low dimensional topology, solvable lattice model theory, conformal field theory and quantum groups that are especially remarkable (See [EK1], [J2]). The following was the first breakthrough in the theory of Jones.

For a subfactor $N \subset M$, the index value $[M : N]$ belongs to the set

$$\left\{4\cos^2\frac{\pi}{n}|n = 3, 4, 5\cdots\right\} \cup [4, \infty]$$

and all the values in this set are realized.

The classification of subfactors is one of the most important and interesting in theory of operator algebras. Using the original *paragroup* theory in 1987, A. Ocneanu announced a complete classification of approximately finite dimensional (AFD) type II_1 subfactors with index less than four as follows:

The (dual) principal graph of a subfactor with index less than four is one of the Dynkin diagrams A_n, D_{2n}, E_6, E_8, and all of these are realized.

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With an analytic side of the classification theory completed by S. Popa [P], a proof of the above classification theorem is reduced to a combinatorial problem on paragroups. Complete proofs of the above classification theorem are now found in [BN], [I1], [I2], [K], [SV].

Paragroups are in a bijective correspondence to subfactors of the AFD type II_1 factor with finite index and finite depth, thanks to the theorem of Popa mentioned above, so the classification of such subfactors is reduced to the classification of paragroups. Classification of paragroups is very important from not only the viewpoint of subfactor theory, but also that of low-dimensional topology or quantum group theory.

After the above classification of paragroups up to index four, Popa [P] gave a classification of subfactors with index equal to four. (See also [IK].) U. Haagerup then tried to classify paragroups with indices slightly beyond four, and found a list of candidates of possible paragroups with indices in the range $(4, 3+\sqrt{3}]$ in [H]. He has proved that any principal graphs of a subfactor in this index range appears in his list of graphs and one of the graphs, $n = 3$ in case (2), does indeed appear as a principal graph, but he was unable to decide whether the other graphs appear as principal graphs or not. (M. Izumi gave a proof for the case $n = 3$ in (2) by a different method.) Recently, D. Bisch showed that the graphs in case (4) in Haagerup's list cannot appear as principal graphs, but the problem is open for the other graphs in the list. Our aim in this paper is to give numerical tests for graphs in this list.

It is fairly easy to determine the biunitary connection on the graphs in the Haagerup's list, and then all we need is a verification of the flatness as in [O2], [K] (or theorem 1 of this paper), which is a computation of finitely many terms and fits for numerical computations. So we will make numerical computations for a biunitary connections on graphs in the Haagerup's list to see whether the flatness holds or not. Our results show that the flatness holds up to a very small numerical error, so this strongly suggests that the connections are indeed flat and the graphs do appear as principal graphs of subfactors, though these do not give rigorous proofs.

In Section 2, we will review some basics, especially the definition of flatness, which are needed in the latter sections, and the Haagerup's graph we will consider in this paper. In Section 3, we will calculate the biunitary con-

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nection on one of these graphs. In Section 4, we will explain the procedures of computations. And finally in Section 5, we will give the conclusions of computations and some comments.

The author is indebted very much to Prof. Y. Kawahigashi and Prof. M. Izumi for many pieces of advice and appreciates their kindness.

2. Basics

In this section, we review the basics which are used in this paper. Suppose that graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are finite and bipartite, V_0, V_1 are the set of vertices in $\mathcal{G}_0$, respectively, V_2, V_3 are the set of vertices in $\mathcal{G}_2$, respectively. And we assume that V_1, V_2 are the set of vertices in $\mathcal{G}_1$, respectively, V_3, V_0 are the set of vertices in $\mathcal{G}_3$, respectively, too.

We can construct the string algebras under these condition, and consider the complex numbers called "the connections" (See [O2] and Chapter 11 of [EK2]) which are expressed as follows:

$$\begin{tikzcd} x \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & y \arrow[d, "\xi_3"] \\ z \arrow[r, "\xi_4"'] & w \end{tikzcd}$$

where $\xi_1 \in \mathcal{G}_3, \xi_2 \in \mathcal{G}_0, \xi_3 \in \mathcal{G}_1, \xi_4 \in \mathcal{G}_2$, or $\xi_1 \in \mathcal{G}_1, \xi_2 \in \mathcal{G}_0, \xi_3 \in \mathcal{G}_3, \xi_4 \in \mathcal{G}_2$, or $\xi_1 \in \mathcal{G}_3, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}_1, \xi_4 \in \mathcal{G}_0$, or $\xi_1 \in \mathcal{G}_1, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}_3, \xi_4 \in \mathcal{G}_0$, and x, y, z, w are the corresponding vertices. We require the connection to satisfy the following two conditions.

Unitarity:

$$\sum_{y,\xi_2,\xi_3} \begin{tikzcd} x \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & y \arrow[d, "\xi_3"] \\ z \arrow[r, "\xi_4"'] & w \end{tikzcd} \quad \overline{\begin{tikzcd} x \arrow[r, "\xi_2"] \arrow[d, "\xi_1'"'] & y \arrow[d, "\xi_3"] \\ z' \arrow[r, "\xi_4'"'] & w \end{tikzcd}} = \delta_{\xi_1,\xi_1'}\delta_{\xi_4,\xi_4'}\delta z, z'$$

and

$$\sum_{z,\xi_1,\xi_4} \begin{tikzcd} x \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & y \arrow[d, "\xi_3"] \\ z \arrow[r, "\xi_4"'] & w \end{tikzcd} \quad \overline{\begin{tikzcd} x \arrow[r, "\xi_2'"] \arrow[d, "\xi_1"'] & y' \arrow[d, "\xi_3'"] \\ z \arrow[r, "\xi_4"'] & w \end{tikzcd}} = \delta_{\xi_2,\xi_2'}\delta_{\xi_3,\xi_3'}\delta y, y'.$$

The renormalization rule:

$$\begin{tikzcd} x \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & y \arrow[d, "\xi_3"] \\ z \arrow[r, "\xi_4"'] & w \end{tikzcd} = \sqrt{\frac{\mu(x)\mu(w)}{\mu(y)\mu(z)}} \quad \overline{\begin{tikzcd} z \arrow[r, "\xi_4"] \arrow[d, "\widetilde{\xi_1}"'] & w \arrow[d, "\widetilde{\xi_3}"] \\ x \arrow[r, "\xi_2"'] & y \end{tikzcd}}$$

$$= \sqrt{\frac{\mu(x)\mu(w)}{\mu(y)\mu(z)}} \quad \overline{\begin{tikzcd} y \arrow[r, "\widetilde{\xi_2}"] \arrow[d, "\xi_3"'] & x \arrow[d, "\xi_1"] \\ w \arrow[r, "\widetilde{\xi_4}"'] & z \end{tikzcd}}$$

where $\widetilde{\xi_j}$ is the reverse edge of ξ_j and $\mu(\cdot)$ is an entry of the Perron-Frobenius eigenvector of the adjacency matrix of each graph (See [GHJ]). Such a connection is called "biunitary connection". We need the following theorem (See [O2]).

THEOREM 1. *The following conditions are equivalent:*
(1) *In the string algebra double sequence, any two elements $x \in A_{\infty,0}$, the vertical string algebra, and $y \in A_{0,\infty}$, the horizontal string algebra, commute.*

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where $C_{\rho,\sigma} \in C$ depe

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where $C_{\rho,\sigma} \in C$ depe

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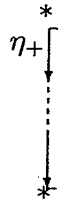
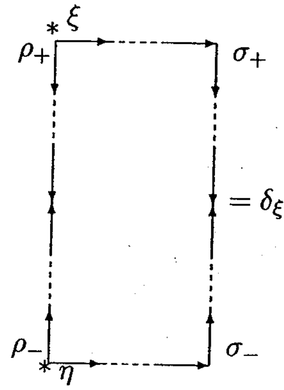


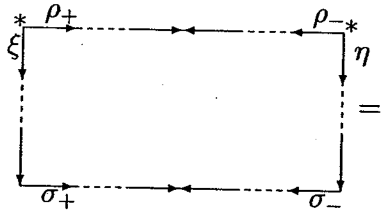
(2) *For each vertical string* $\rho = (\rho_+, \rho_-), \sigma = (\sigma_+, \sigma_-) \in A_{k,0}$, *we get*



$$= \delta_{\xi,\eta} C_{\rho,\sigma}$$

where $C_{\rho,\sigma} \in C$ *depends only on* ρ, σ.

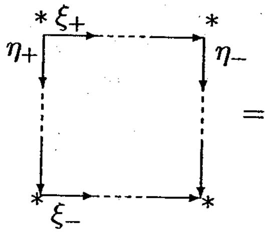
(2)′ *For each horizontal string* $\rho = (\rho_+, \rho_-), \sigma = (\sigma_+, \sigma_-) \in A_{0,k}$, *we get*



$$= \delta_{\xi,\eta} C_{\rho,\sigma}$$

where $C_{\rho,\sigma} \in C$ *depends only on* ρ, σ.

(3) *For any horizontal paths* ξ_+, ξ_- *and vertical paths* η_+, η_- *with all the sources and ranges equal to* $*$, *we get*



$$= \delta_{\xi_+,\xi_-} \delta_{\eta_+,\eta_-}.$$

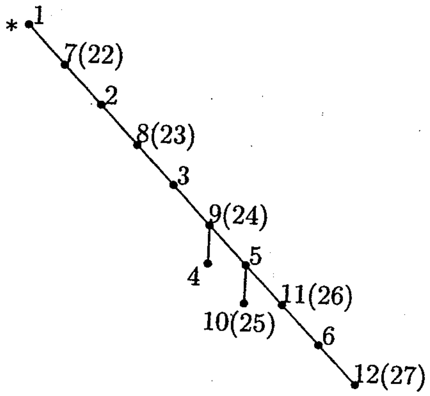


Fig. 1. principal graph in case (2) of Haagerup's list

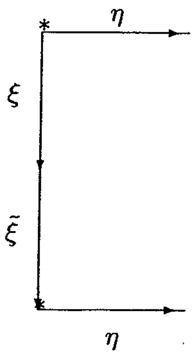
(3)′ For any horizontal path ξ and vertical path η with all the sources and ranges equal to $*$, we get

$$\begin{tikzcd} * \arrow[r, dashed, "\xi"] \arrow[d, dashed, "\eta"'] & * \arrow[d, dashed, "\eta"] \\ * \arrow[r, dashed, "\xi"'] & * \end{tikzcd} = 1.$$

The proof of the above theorem is in [K] or [EK2]. The connection is said to be *flat* if one of the above conditions holds.

We get the graph with index $\frac{5+\sqrt{17}}{2}$ in case (3) in Haagerup's list as in Fig. 1 (See [H]) and will try to check the flatness of this graph in later sections. In order to see the connection on this graph is flat, we have only to see that the vertical string

$$(1\to 22\to 2\to 23\to 3\to 24\to 4, 1\to 22\to 2\to 23\to 3\to 24\to 4)$$



(a) 12×12 c

1

commutes with the hc

$$(1\to 7\to 2\to 8$$

$$(1\to 7\to 2\to 8$$

because the only othe
of Haagerup. And to
the diagrams in Fig.

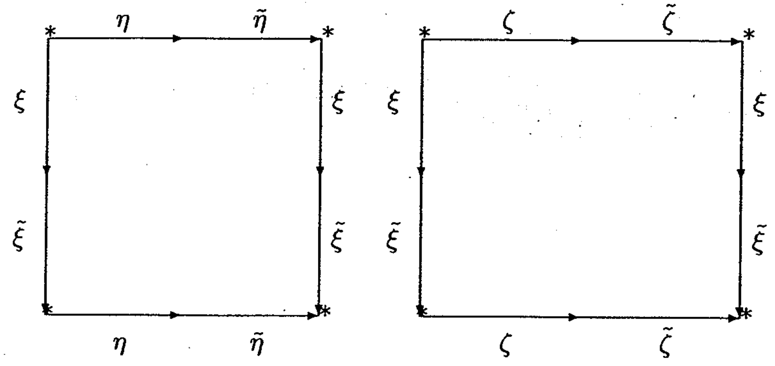
$$\xi = (1\to 22\to 2$$

and horizontal paths

$$\eta = (1\to 7\to 2$$

$$\zeta = (1\to 7\to 2$$

and $\tilde{\xi}$, $\tilde{\eta}$ and $\tilde{\zeta}$ are th
1, by (3)′ of theorem



(a) 12 × 12 diagram (b) 12 × 14 diagram

Fig. 2. 12 × 12 and 12 × 14 diagrams

commutes with the horizontal strings

$$(1 \to 7 \to 2 \to 8 \to 3 \to 9 \to 5, 1 \to 7 \to 2 \to 8 \to 3 \to 9 \to 5),$$

$$(1 \to 7 \to 2 \to 8 \to 3 \to 9 \to 5 \to 10, 1 \to 7 \to 2 \to 8 \to 3 \to 9 \to 5 \to 10),$$

because the only other possible graph at this index value is A_∞ in the list of Haagerup. And to check these commutativity, it is enough to verify that the diagrams in Fig. 2, where vertical path

$$\xi = (1 \to 22 \to 2 \to 23 \to 3 \to 24 \to 4)$$

and horizontal paths

$$\eta = (1 \to 7 \to 2 \to 8 \to 3 \to 9 \to 5),$$

$$\zeta = (1 \to 7 \to 2 \to 8 \to 3 \to 9 \to 5 \to 10),$$

and $\tilde\xi$, $\tilde\eta$ and $\tilde\zeta$ are the reverse paths of ξ, η and ζ, respectively, have value 1, by $(3)'$ of theorem 1.

3. Biunitary connection on Haagerup's graph

First we need a biunitary connection on Haagerup's graphs. From Fig. 3, we get the identities as follows:

(3.1)
$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd} = 1$$

where $(x,y,z,w) = (1,7,22,13)$, $(1,7,22,14)$, $(2,7,22,13)$, $(2,8,23,15)$, $(3,8,23,14)$, $(3,9,24,16)$, $(3,9,24,17)$, $(4,9,24,15)$, $(5,9,24,15)$, $(5,11,26,18)$, $(5,11,25,19)$, $(5,10,26,20)$, $(6,11,26,16)$, $(6,11,27,19)$, $(6,12,26,20)$, or $(6,12,27,21)$.

(3.2)
$$\begin{tikzcd} 2 \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & 14 \end{tikzcd} = \begin{array}{c} \\ 22 \\ 23 \end{array} \!\!\begin{array}{cc} 7 & 8 \\ \end{array}\left(\begin{array}{cc} -\frac{\sqrt{17}-3}{4} & \frac{\sqrt{3\sqrt{17}-5}}{2\sqrt{2}} \\ \frac{\sqrt{3\sqrt{17}-5}}{2\sqrt{2}} & \frac{\sqrt{17}-3}{4} \end{array} \right)$$

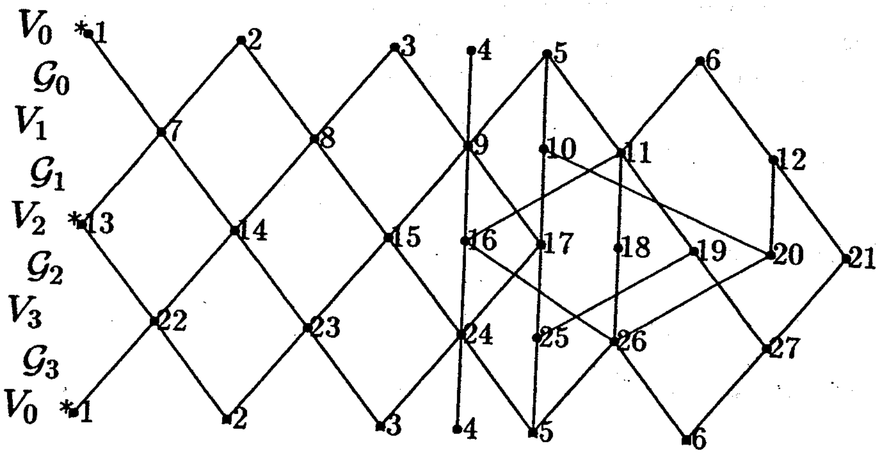


Fig. 3. four graphs in case (3) of Haagerup's list

(3.3)

(3.4)

(3.5)

(3.6)

(3.7)

(3.8)

where $(x,y,$
$(9,3,15,23)$,
$(11,5,16,24$

(3.9)

(3.3)
$$\begin{tikzcd} 3 \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & 15 \end{tikzcd} = \left(\begin{array}{ccc} & 8 & 9 \\ 23 & -\sqrt{33-8\sqrt{17}} & \sqrt{8\sqrt{17}-32} \\ 24 & \sqrt{8\sqrt{17}-32} & \sqrt{33-8\sqrt{17}} \end{array}\right)$$

(3.4)
$$\begin{tikzcd} 4 \arrow[r] \arrow[d] & 9 \arrow[d] \\ 24 \arrow[r] & 16 \end{tikzcd} = -1$$

(3.5)
$$\begin{tikzcd} 4 \arrow[r] \arrow[d] & 9 \arrow[d] \\ 24 \arrow[r] & 17 \end{tikzcd} = 1$$

(3.6)
$$\begin{tikzcd} 5 \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & 16 \end{tikzcd} = \left(\begin{array}{ccc} & 9 & 11 \\ 24 & \frac{\sqrt{17}-1}{8} & \frac{\sqrt{23+\sqrt{17}}}{4\sqrt{2}} \\ 26 & \frac{\sqrt{23+\sqrt{17}}}{4\sqrt{2}} & \frac{1-\sqrt{17}}{8} \end{array}\right)$$

(3.7)
$$\begin{tikzcd} 5 \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & 17 \end{tikzcd} = \left(\begin{array}{ccc} & 9 & 10 \\ 24 & -\frac{\sqrt{13-3\sqrt{17}}}{\sqrt{2}} & \frac{\sqrt{3\sqrt{17}-11}}{\sqrt{2}} \\ 25 & \frac{\sqrt{3\sqrt{17}-11}}{\sqrt{2}} & \frac{\sqrt{13-3\sqrt{17}}}{\sqrt{2}} \end{array}\right)$$

(3.8)
$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd} = 1$$

where $(x, y, z, w) = (6, 11, 26, 18)$, $(7, 2, 14, 23)$, $(8, 2, 14, 22)$, $(8, 3, 15, 24)$, $(9, 3, 15, 23)$, $(9, 5, 17, 25)$, $(9, 5, 16, 26)$, $(10, 5, 17, 24)$, $(10, 5, 17, 25)$, $(11, 5, 16, 24)$, or $(11, 5, 16, 24)$.

(3.9)
$$\begin{tikzcd} 10 \arrow[r] \arrow[d] & 5 \arrow[d] \\ 20 \arrow[r] & 26 \end{tikzcd} = 1$$

(3.10)
$$\begin{tikzcd} 11 \arrow[r] \arrow[d] & 5 \arrow[d] \\ 19 \arrow[r] & 25 \end{tikzcd} = 1$$

(3.11)
$$\begin{tikzcd} 7 \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & 22 \end{tikzcd} = \begin{array}{c} \\ 13 \\ 14 \end{array} \begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{pmatrix} \frac{\sqrt{5-\sqrt{17}}}{2} & \frac{\sqrt{\sqrt{17}-1}}{2} \\ \frac{\sqrt{\sqrt{17}-1}}{2} & -\frac{\sqrt{5-\sqrt{17}}}{2} \end{pmatrix} \end{array}$$

(3.12)
$$\begin{tikzcd} 8 \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & 23 \end{tikzcd} = \begin{array}{c|cc} & 2 & 3 \\ \hline 14 & \frac{\sqrt{31-3\sqrt{17}}}{8} & \frac{\sqrt{33+7\sqrt{17}}}{8} \\ 15 & \frac{\sqrt{33+7\sqrt{17}}}{8} & -\frac{\sqrt{31-7\sqrt{17}}}{8} \end{array}$$

(3.13)
$$\begin{tikzcd} 9 \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & 24 \end{tikzcd} = \begin{array}{c|ccc} & 3 & 4 & 5 \\ \hline 15 & \frac{\sqrt{71-17\sqrt{17}}}{8\sqrt{2}} & \frac{\sqrt{1+\sqrt{17}}}{4} & \frac{\sqrt{49+9\sqrt{17}}}{8\sqrt{2}} \\ 16 & \frac{\sqrt{13+5\sqrt{17}}}{8} & -\frac{\sqrt{7-\sqrt{17}}}{2\sqrt{2}} & \frac{\sqrt{3\sqrt{17}-5}}{8} \\ 17 & \frac{\sqrt{31+7\sqrt{17}}}{8\sqrt{2}} & \frac{\sqrt{1+\sqrt{17}}}{4} & -\frac{\sqrt{89-15\sqrt{17}}}{8\sqrt{2}} \end{array}$$

(3.14)
$$\begin{tikzcd} 11 \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & 26 \end{tikzcd} = \begin{array}{c|cc} & 5 & 6 \\ \hline 16 & -\frac{\sqrt{\sqrt{17}-3}}{2} & \frac{\sqrt{7-\sqrt{17}}}{2} \\ 18 & \frac{\sqrt{7-\sqrt{17}}}{2} & \frac{\sqrt{\sqrt{17}-3}}{2} \end{array}$$

(3.15)
$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd} = 1$$

where $(x,y,z,w) = (11,6,19,27)$, $(12,6,20,26)$ or $(12,6,21,27)$.

In the first step, we determine (3.1) by the unitarity and gauge choices. Then, using the renormalization rule, we get the (13,1),(13,2) and (14,1)-entries of (3.11), the (14,3) and (15,2)-entries of (3.12), the (15,4), (15,5), (16,3), (17,3)-entries of (3.13), (3.9), (3.10), the (16,6) and (18,5)-entries of (3.14), and (3.15). Considering that we can choose the (15,3)-entry of (3.13)

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4. Procedure for

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tion in later steps or r

0, and $\begin{tikzcd} i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}$ is of "t

Next we compute

(4.2)

as a real number by a gauge choice, and absolute values of the (16,4) and (17,4)-entries are known by unitarity of (3.4) and (3.5) and the renormalization rule, we complete (3.13) by unitarity. Similarly (3.11) is accomplished. Using the renormalization rule again, we get the (22,7)-entry of (3.2), (3.4), (3.5), the (24,9)-entry of (3.6) and (3.7). Then using gauge choices and unitarity again, we complete (3.2), (3.6), (3.7). After repeating these operations, we get the above connection. This connection is unique up to equivalence.

Now, it is possible that we compute the diagrams in Fig. 2 theoretically, but it is hard to do so in reality. So it is necessary for us to contrive: in the next section we will see the procedure of numerical calculations.

4. Procedure for calculations of the diagrams

We need several steps for the calculations on computer as shown below.

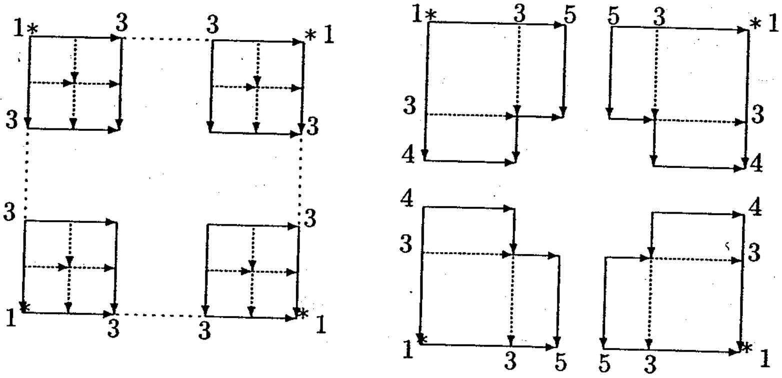
First we make the connection on the graph in Fig. 3 (See section 3) and define the function ϕ_A on the connection as follows:

$$\phi_A\left(\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}\right) = \begin{cases} 0 & \text{if } \left|\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}\right| < \varepsilon \\ 1 & \text{otherwise} \end{cases} \tag{4.1}$$

with sufficiently small $\varepsilon > 0$. We call the above operations "Step A". The function $\phi_A\left(\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}\right)$ determines whether we shall use $\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}$ for computation in later steps or not. We say that $\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}$ is of "type 0" if $\phi_A\left(\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}\right) = 0$, and $\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}$ is of "type 1" if $\phi_A\left(\begin{tikzcd}[sep=small] i \arrow[r] \arrow[d] & j \arrow[d] \\ k \arrow[r] & l \end{tikzcd}\right) = 1$.

Next we compute all kind of the following 2 × 2 diagrams.

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d] & y_1 \arrow[r] \arrow[d, dotted] & x_2 \arrow[d] \\ z_1 \arrow[r, dotted] \arrow[d] & {} \arrow[r, dotted] \arrow[d, dotted] & z_2 \arrow[d] \\ x_3 \arrow[r] & y_2 \arrow[r] & x_4 \end{tikzcd} \tag{4.2}$$

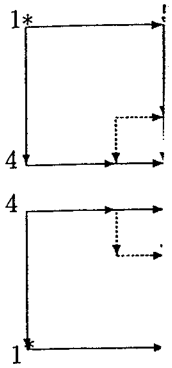


(a) 4 × 4 diagrams (b) diagrams of Step F

Fig. 4. diagrams of Step C, D, E and F

where $x_i \in V_0$, $y_i \in V_1$ and $z_i \in V_3$. Then we make the function ϕ_B on these diagrams in the same way as in Step A. These procedures are called "Step B". And we set up all kinds of 4 × 4 diagrams as in (a) of Fig. 4 from Step B and make the type information function ϕ_C on these diagrams (Step C). In Step D, we compute the 4 × 2 and 2 × 4 diagrams which are put to 4 × 4 diagrams of Step C as in (b) of Fig. 4, and in Step E we calculate the 4 × 6 diagrams from the 4 × 4 diagrams of Step C and the 4 × 2 diagrams of Step D. And we make the all kinds of diagrams in (b) of Fig. 4 (Step F). In each step, we make "type information functions" ϕ_D, ϕ_E and ϕ_F, respectively. Then we make all kinds of 6 × 6 diagrams in (a) of Fig. 5 and the type information function ϕ_G for them in Step G. And then, in Step H, we compute the value of (a) of Fig. 2 using the 6 × 6 diagrams of Step G. Next, in Step X and Y, we make the tables for 6 × 2 diagrams in (b) of Fig. 5 with their type information functions. At last we compute the value of 12 × 14 diagram in (b) of Fig. 2 in Step Z from Step G and Step Y, and finish.

In each step, we omit the operations using the type 0-diagrams for efficiency.



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5. Conclusio

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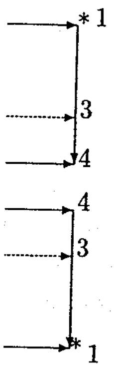

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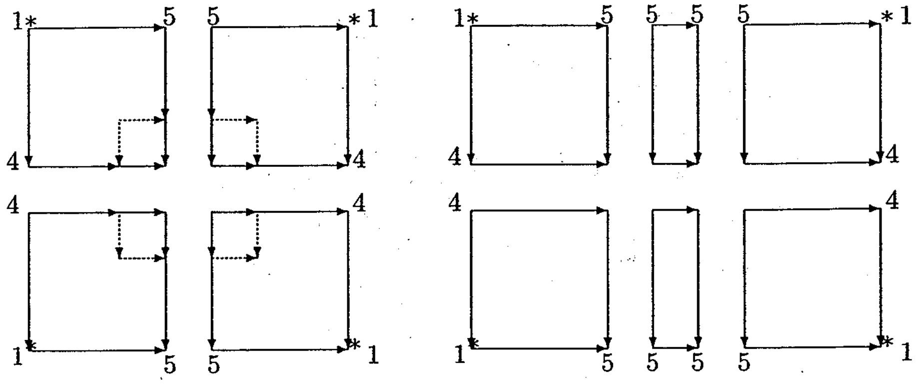

(a) 6 × 6 diagrams (b) diagrams of Step Y

Fig. 5. diagrams of Step G, X and Y

5. Conclusion of Computations and Comments

By the counter setting in the C-program, we know that the number of arithmetic operations is about 4.4 millions. So we have a conclusion that the error is less than 10^{-9}. In table 1, we examine the minimum of the absolute values of the type 1-diagrams, the maximum of the absolute values of the type 0-diagrams, in each steps with $\varepsilon = 1.0 \times 10^{-12}$.

Step	Min	Max
B	7.2×10^{-3}	5.5×10^{-16}
C	3.6×10^{-3}	1.0×10^{-15}
D	1.5×10^{-3}	8.6×10^{-16}
E	9.0×10^{-4}	6.9×10^{-16}
F	1.9×10^{-4}	4.5×10^{-16}
G	1.8×10^{-4}	5.4×10^{-16}
X	2.7×10^{-3}	3.9×10^{-16}
Y	1.9×10^{-4}	4.8×10^{-16}

Table 1. Data about Type 0 or 1-diagrams

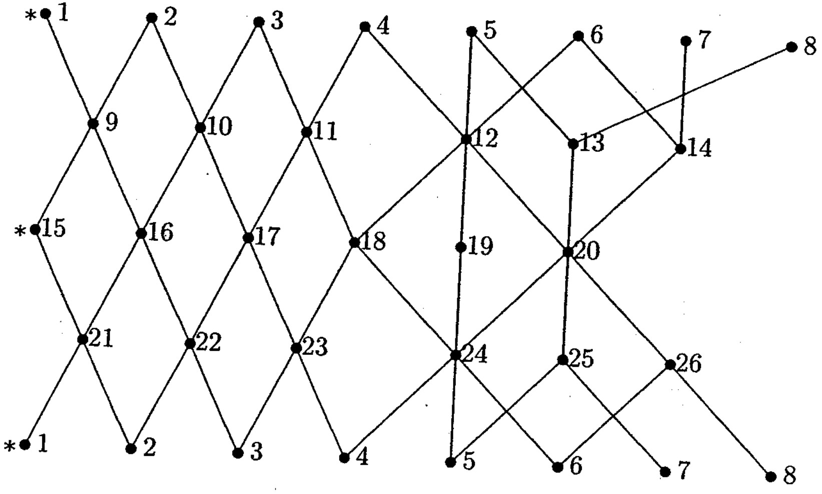

Fig. 6. four graphs of case $n = 7$ in (2) of Haagerup's list

Considering the error order, we conclude that non-zero type 0-diagrams are caused by dealing with fractions and the true minimum of the absolute values of diagrams is Min in Table 1 in each step. We then conclude that the following statement is highly likely from our numerical experiments.

CONCLUSION 1. *The connection on figure* 3 *is flat.*

Now, we try to check the flatness of connections on the graph with index $4.377\cdots$ of case $n = 7$ in (2) of Haagerup's list (See [H]). We get this four graphs as in Fig. 6 and know that we have only to compute the value of the following 16×16 diagram with the same argument of the end of section 2.

$$\begin{tikzcd}
* \arrow[r, "\xi\quad\tilde{\xi}"] \arrow[d, "\eta\quad\tilde{\eta}"'] & * \arrow[d, "\eta\quad\tilde{\eta}"] \\
* \arrow[r, "\xi\quad\tilde{\xi}"'] & *
\end{tikzcd}$$

where horizontal patl

$\xi = (1 \rightarrow 9 \rightarrow 2$

and vertical path

$\eta = (1 \rightarrow 21 \rightarrow 2$

$\tilde{\xi}$ and $\tilde{\eta}$ are the reve
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CONCLUSION 2.

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where horizontal path

$$\xi = (1 \to 9 \to 2 \to 10 \to 3 \to 11 \to 4 \to 12 \to 5)$$

and vertical path

$$\eta = (1 \to 21 \to 2 \to 22 \to 3 \to 23 \to 4 \to 24 \to 5),$$

$\tilde{\xi}$ and $\tilde{\eta}$ are the reverse paths of ξ and η, respectively. This graph has 2 biunitary connections up to equivalence. These connections are isomorphic in the sense of [O1] and give one isomorphism class of subfactors. Using the similar techniques effectively, we can compute the 16×16 diagram in both cases in a very short time, about only 4 min. Thus we get the following numerical conclusion in the same sense as above.

CONCLUSION 2. *Both connections on figure* 6 *are flat.*

In these computations, we know that the errors are less than 10^{-8}.

Finally we see the efficiency from the technical way to put diagrams (See section 4), the information of type 0 and type 1, and so on. For example, in Fig. 3, the operations from Step A to Step C take about 5 sec. with above techniques, but they do about 200 min. without them. Considering that all procedures from Step A to Step Z cost about only 3 min., and the above comparison, we know that we can save the time sharply using our C-program. We expect that our techniques are useful for the other graphs and hope that the above conclusions become motives for rigorous proofs.

REMARK 1. After the submission of this paper, a rigorous proof of the flatness for the case of index $\frac{5+\sqrt{17}}{2}$ has been given in the following paper.

M. Asaeda, & U. Haagerup, Exotic subfactors of finite depth with Jones indices $\frac{5+\sqrt{13}}{2}$ and $\frac{5+\sqrt{17}}{2}$, preprint 1998.

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