Extended Haagerup exists!

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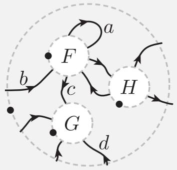
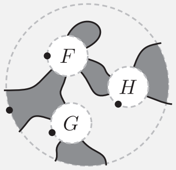
Outline

Subfactors and $\otimes$-categories

- Extremal finite-index II_1 subfactors correspond to unitary spherical $\otimes_A, \otimes_B$-categories with a chosen generator ${}_A X_B$.
  - (objects are bimodules for $A \subset B$, and X is the 'regular' bimodule ${}_A B_B$)
- The 'index' $[A:B]$ of the subfactor is $(\dim_q X)^2$.
- The 'principal graph' encodes the tensor products $V \otimes_A X$ and $V \otimes_B X^*$.
- The 'even part' is the subcategory of $A-A$ objects (alternatively of $B-B$ objects).
- The double of the even part is a modular tensor category.
  - (it doesn't matter which even part you take, because they're Morita equivalent)
  - (these MTCs may be 'exotic', that is, don't come from quantum groups [Hong, Rowell, Wang])

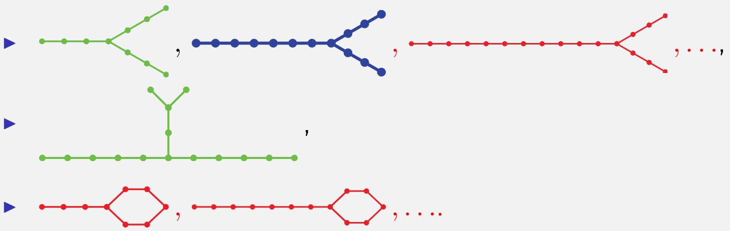
Planar algebras: pictures for subfactors

spherical tensor categories	subfactor planar algebras
	
oriented edges an edge label for each object many Hom-spaces	unoriented edges, with two shadings edges only labelled by X just $\mathcal{P}_{k,\pm}$

- To produce the planar algebra, restrict your objects to $X \otimes X^* \otimes X \otimes \cdots$.
- To recover the $\otimes_A, \otimes_B$-category, form the idempotent completion.

Haagerup's list

- In 1994 Haagerup classified possible principal graphs for subfactors with index less than $3 + \sqrt{3}$:



- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- Today, we construct the missing example ('extended Haagerup'), and complete the classification.

- Asaeda & Haagerup's constructions find a biunitary flat connection, using Ocneanu's paragroup formalism.
- Ikeda (1998) did numerical calculations suggesting that the extended Haagerup subfactor existed "up to 10^{-9}".
- Our techniques are very different. Although we made heavy use of a computer while searching for a solution, verifying the answer only requires a computer to multiply large matrices, using exact arithmetic.

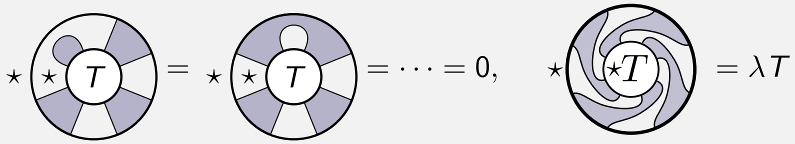
Project

Now that we know about planar algebras, we can give skein theory constructions of subfactors.

- Jones' work on *annular* and *quadratic* tangles gives you information about the generators and relations to expect.
- Peters [math/0902.1294] has implemented this approach to give a new construction of the Haagerup subfactor. Our work follows that model closely.

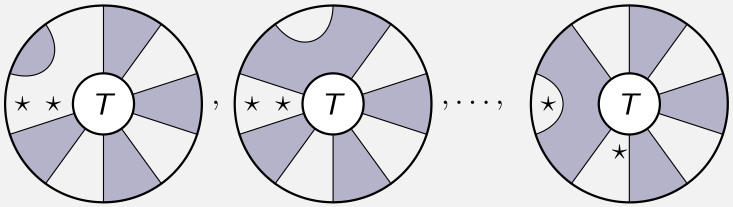
Annular Temperley-Lieb

Every planar algebra $\mathcal{P}$ is a representation of the *annular Temperley-Lieb* category. Every representation breaks up as a sum of representations generated by *lowest weight vectors* $T_{k,\lambda} \in \mathcal{P}_k$ satisfying



for some $k \geq 0$ and $\lambda \in \sqrt[k]{1}$. Thus the planar algebra is generated by elements of this form.

If T is a lowest weight vector in $\mathcal{P}_5$, then we write $\mathcal{ATL}_{+1}(T)$ for the "annular consequences" of T in $\mathcal{P}_6$.



Lemma

If $\delta > 2$, the annular consequences $\mathcal{ATL}_{+k}(T)$ of a lowest weight vector T are all linearly independent.

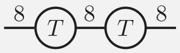
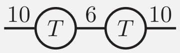
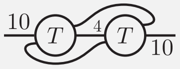
Quadratic tangles

Further, by counting the dimensions of the $\mathcal{ATL}$ representations, we can expect relations between certain *quadratic tangles*.

Example (Extended Haagerup)

$$\mathcal{P} \cong V_{0,\delta} \oplus V_{8,-1} \oplus V_{10,\bullet} \oplus \cdots$$

If T generates $V_{8,-1}$, then

-  $\in P_8$ is a linear combination of T and $\mathcal{TL}$,
- there no lowest weight vector in P_9 , so  must be a linear combination of $\mathcal{ATL}_{+1}(T)$ and $\mathcal{TL}$,
- there's only one lowest weight vector in P_{10}, so some linear combination of  and  must lie in $\mathcal{ATL}_{+2}(T) \oplus \mathcal{TL}$.

Existence

Question

Consider a planar algebra $\mathcal{P}$ generated by an element T satisfying relations like these. Is it the extended Haagerup planar algebra?

- In fact, if we can show that $\mathcal{P}$ is a *subfactor planar algebra* with the correct index (the largest root of $x^3 - 8x^2 + 17x - 5$, $\sim 4.3772...$), then Haagerup's classification guarantees it must have the desired principal graph.
- It's probably also possible to compute the principal graph directly from the skein theory.

When is $\mathcal{P}$ a subfactor planar algebra?

- Do we know the relations are consistent? (i.e., is $\mathcal{P} \neq 0$?)
- Is the planar algebra unitary?
- Is $\dim \mathcal{P}_0 = 1$? That is, can we evaluate every closed diagram using the relations?

To answer the first two questions, we can try to find an element T inside a larger unitary planar algebra. Fortunately, there's an obvious place to look:

Theorem

Every subfactor planar algebra $\mathcal{P}$ is a subalgebra of the graph planar algebra of the principal graph $\Gamma(\mathcal{P})$.

The *evaluation problem* requires some clever skein theory.

The graph planar algebra

Definition

The graph planar algebra $GPA(\Gamma)$ of a bipartite graph Γ has

spaces	$GPA(\Gamma)_k = \mathbb{C}\{\textit{length } 2k \textit{ loops on } \Gamma\}$ *and*
$\mathcal{TL}$ action	*described in Jones', "The planar algebra of a bipartite graph", in terms of the Perron-Frobenius eigenvector.*

The graph planar algebra is always *unitary* and *spherical*. It is not a subfactor planar algebra since

$$\dim(GPA(\Gamma)_0) = \sharp\text{even vertices} > 1.$$

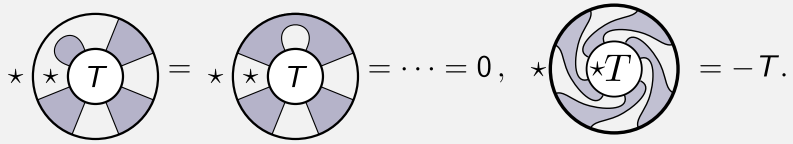
Example

$\dim(GPA(\Gamma(\mathcal{H}))_4) = 375$, $\dim(GPA(\Gamma(\mathcal{EH}))_8) = 148475$

Construction

We now start solving equations in $GPA(\Gamma(\mathcal{EH}))_{16}$.

- Considerations from Jones' Quadratic Tangles tells us the rotational eigenvalue of the generator must be -1.
- We cut down $GPA(\Gamma(\mathcal{EH}))_{16}$ to a 19 dimensional space using the linear relations



- The generator must satisfy $T^2 = f^{(16)}$ — this gives 148475 quadratic equations!
- We look at specially chosen subset of these and find a solution T by ad-hoc methods.

At this point you can forget everything I told you about finding T!

- We can write down an explicit description of T as an element of the $148475d$ vector space. ▸ See the generator!
- It's tedious but straightforward to check by hand that T is a lowest weight vector with rotational eigenvalue -1.
- Certainly the subalgebra generated by T is both nontrivial and unitary.
- We still need to see that all closed diagrams can be evaluated, that is, $\mathcal{P}(T)_0 = \mathbb{C}$.

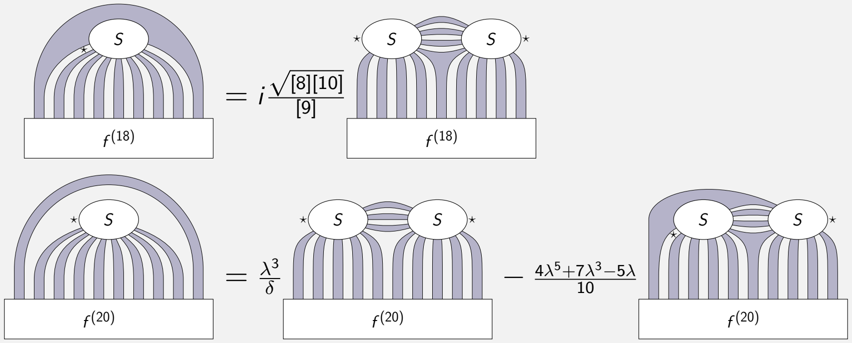
Quadratic relations

We have a candidate generator; let's find the particular quadratic relations it satisfies. First, with the aid of a computer (doing exact arithmetic!) we calculate six moments.

$$
\begin{aligned}
\operatorname{tr}(T^2) &= [9] \sim 24.66097\ldots \\
\operatorname{tr}(T^3) &= 0 \\
\operatorname{tr}(T^4) &= [9] \\
\operatorname{tr}(\rho^{1/2}(T)^2) &= -[9] \\
\operatorname{tr}(\rho^{1/2}(T)^3) &= -i\frac{[18]}{\sqrt{[8][10]}} \sim -15.29004i \\
\operatorname{tr}(\rho^{1/2}(T)^4) &= \frac{1}{5}\left(46\lambda^4 - 2\lambda^2 - 94\right) \sim 34.1409\ldots
\end{aligned}
$$

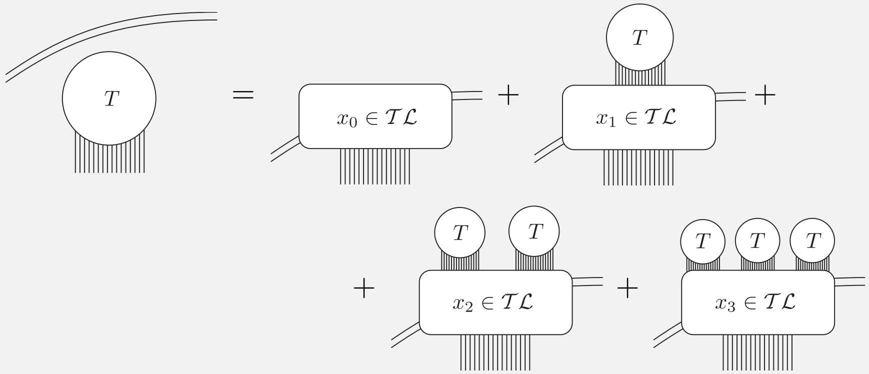
with λ the root of $x^6 + 2x^4 - 3x^2 - 5$ which is approximately $1.12867i$.

Bessel's inequality now gives the desired relations



Note that the first equation only holds with the given shading. For the Haagerup and extended Haagerup graph, when we write  in terms of $\mathcal{ATL}_{+1}(T)$ a certain coefficient is 'unexpectedly' zero.

Now substitute the first equation into the last term of the second, and expand the Jones-Wenzl idempotents.

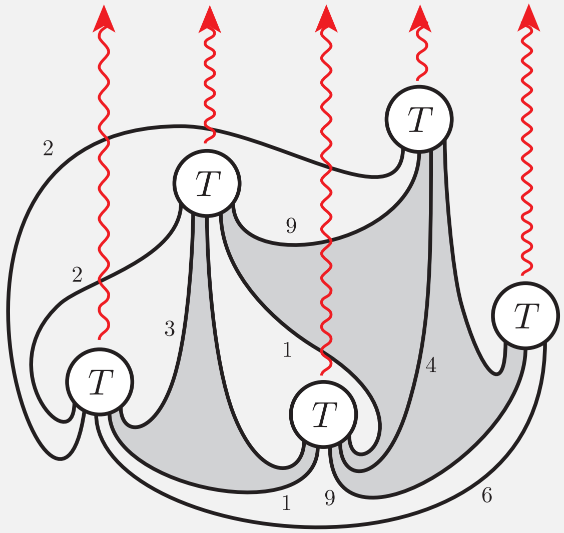


This relation lets us 'pull a generator through a pair of strands'. This increases the number of generators in the diagram, so it isn't immediately obvious how this helps evaluate closed diagrams!

Theorem

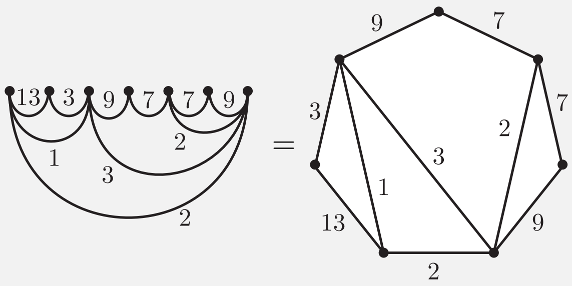
Stephen's jellyfish algorithm shows these relations suffice to evaluate arbitrary closed diagrams.

Begin with arbitrary planar network of Ts.



Now float each generator to the surface, using the relation.

The diagram now looks like a polygon with some diagonals, labelled by the numbers of strands connecting generators.



- Each such polygon has a corner, and the generator there is connected to one of its neighbours by at least 8 edges.
- Use $T^2 = f^{(16)}$ to reduce the number of generators, and recursively evaluate the entire diagram.

Thank you!

To specify the generator T we need to give its value on every one of the 148475 loops of length 16.

$$T(\gamma) = r \cdot \sigma(\gamma) \cdot p_{\widehat{\gamma}} \cdot \frac{1}{d_{\gamma_1}} \cdot \prod_{i=1}^{16} \frac{1}{\sqrt{d_{\gamma_i}}}.$$

- $r = -1843700 + 5847375d^2 - 1614050d^4$
- If γ is a path on the principal graph, produce a sequence $\widehat{\gamma} \in \{0,1,2\}^8$ so that if γ_{2i-1} is in the j-th arm of the principal graph, then $\widehat{\gamma}_i = j$.
- Let $\sigma(\gamma)$ to be -1 raised to the number of times the vertices v_0, w_0, z_i and a_i appear in γ.

We still need to specify $3^8 = 6561$ values for $p_{\widehat{\gamma}}$.

Fix $\lambda = \sqrt{(2 - d^2)}$. Define 21 elements of $\mathbb{Z}[\lambda]$

$$p_{00000001} = -2\lambda^4 - \lambda^2 + 9$$
$$p_{00000101} = 2\lambda^4 + \lambda^2 - 9$$
$$p_{00001001} = \lambda^5 - \lambda^3 - 3\lambda$$
$$p_{00010001} = 2\lambda^4 + \lambda^2 - 9$$
$$p_{00010101} = \lambda^4 - 2\lambda^2 + 1$$
$$p_{00011011} = \lambda^4 - \lambda^2 - 3$$
$$p_{00100111} = \lambda^2 + 1$$
$$p_{00101101} = \lambda^5 - \lambda$$
$$p_{00110111} = -\lambda^5 - 2\lambda^3 - 4\lambda^2 - \lambda - 5$$
$$p_{01010111} = \lambda^4 + \lambda^2$$
$$p_{01110111} = \lambda^4 + 6\lambda^2 + 6$$

$$p_{00000011} = -\lambda^5 - \lambda^3 + 3\lambda$$
$$p_{00000111} = 1$$
$$p_{00001011} = \lambda^3 - 1$$
$$p_{00010011} = \lambda^5 - \lambda^4 + \lambda^2 - 3\lambda + 4$$
$$p_{00010111} = 1 - \lambda^4$$
$$p_{00100101} = 5 - 2\lambda^4$$
$$p_{00101011} = -\lambda^5 - \lambda^3 + \lambda + 1$$
$$p_{00110011} = 2\lambda^5 + 5\lambda^3 + 4\lambda$$
$$p_{01010101} = -4\lambda^4 + 3\lambda^2 + 7$$
$$p_{01011011} = \lambda^4 - 2\lambda^2 - 4$$

Extend these definitions to every p_w for $w \in \{0,1\}^8$ by the rules

$$p_{abcdefgh} = -p_{bcdefgha}$$
$$p_{abcdefgh} = \overline{p}_{ahgfedcb}$$

and

$$p_{00000000} = 0$$
$$p_{abcd1111} = 0.$$

Note that we have to check these rules are well-defined. For example, one can get from $p_{00110011}$ to $p_{01100110}$ either by rotating, or by reversing.

Further extend these definitions to every p_w for $w \in \{0,1,2\}^8$ by the rules

$$p_{x0y} + p_{x1y} + p_{x2y} = 0.$$

◂ Return to the talk...