Small fusion categories and subfactors

Noah Snyder
http://math.columbia.edu/∼nsnyder
joint work with Vaughan Jones, Scott Morrison, and Emily Peters

Columbia University

Fusion categories and Applications, Baylor, October 18 2009
http://tqft.net/waco1

Outline

What does small mean?

We want to find all "small" tensor categories. But what definition of small?

Some notions of small

- Small global dimension.
- Small rank.
- $X^{\otimes n}$ not too complicated for small n.
- One object of small dimension.

If the tensor category is not unitary then you need to think harder about what "small" means for dimensions, so we'll concentrate on unitary tensor categories.

Current progress

Global dimension

Lots of progress in the case where the global dimension is an integer (Etingof, Gelaki, Jordan, Larsen, Nikshych, Ostrik, etc.). But essentially no progress for general global dimension.

Rank

Ostrik solved rank 2. Assuming a braiding or modular structure additional progress made by Ostrik, Rowell-Strong-Wang, Hong-Rowell. Big open question: "are there finitely many?"

$X^{\otimes n}$ not too complicated for small n

Kazhdan and Wenzl did $X \otimes X \cong A \oplus B$ and $X^{\otimes 3}$ not too bad. Assuming a braiding Wenzl and Tuba did the case of $X \otimes X \cong 1 \oplus A \oplus B$, and Snyder did $X \otimes X \cong 1 \oplus X \oplus A \oplus B$.

One small object

See this talk!

Motivating question

What is the smallest possible dimension strictly larger than 2 among all objects in all unitary fusion categories?

Why should you care?

- Source of new examples.
- The analogous question for finite groups, namely finite subgroups of small Lie groups, has lead to very interesting math like the McKay correspondence.
- Small global dimension means all individual representations are small.
- Subfactor results make it easy to make progress.

Outline

Relationship between unitary tensor categories and subfactors

What is a subfactor?

A factor is a von Neumann algebra with trivial center. A subfactor is an inclusion $A \subset B$ of factors. For nice applications to category theory we want this inclusion to be finite index, extremal, and we probably want A and B to be type II_1 factors

But I hate analysis!

Don't worry, the whole theory of subfactors can be translated into algebra or into pictures. The only vestige of analysis is assuming that a certain inner product is positive definite.

Can you say that in algebra?

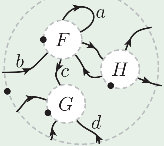
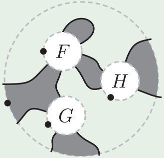
Yes!

Notice that B is an algebra object in the category of $A - A$ bimodules. This suggests several algebraic versions of subfactor theory.

- A Frobenius algebra object in a unitary tensor category.
- A categorical Morita equivalence between two unitary tensor categories.
- A 2-category with exactly 2 objects together with a choice of generating 1-morphism.

Can you say that in pictures?

Yes!

spherical tensor categories	subfactor planar algebras
	
oriented edges, one shading an edge label for each object many Hom-spaces	unoriented edges, two shadings edges only labelled by X just $\mathcal{P}_{k,\pm}$

Going from a unitary tensor category to a subfactor

Given a unitary tensor category and a preferred object X we can get a subfactor by taking the "alternating part."

In terms of diagrams

Restrict your attention to diagrams with edges labelled with only with X and which alternate in and out. Alternating diagrams can always be checkerboard shaded.

In terms of algebra

Inside the tensor category is a Frobenius algebra object, $X \otimes X^*$. The algebra structure is given by contraction. The Frobenius structure comes from the pivotal structure, namely $X \otimes X^* \to X^{**} \otimes X^* \to \mathbf{1}$.

What about fusion categories?

If you start with a unitary fusion category then you get a finite depth subfactor.

Warning!

- Finite depth-ness of the subfactor does not imply fusion-ness for the tensor category.
- Not every finite depth subfactor comes from a fusion category.
- By looking at the even parts (A-A or B-B) you can recover two fusion categories from any subfactor, but this is not inverse to the "alternating part" construction.

Outline

Small subfactors

Global dimension

In subfactor theory this is called the "global index." There's been very little work.

Rank

This has not been studied much in subfactor theory. But see depth.

$X^{\otimes n}$ not too complicated for small n

There has been some work in this direction, most notably papers of Bisch and Jones. Also D. Thurston's work on dominoes.

Small subfactors

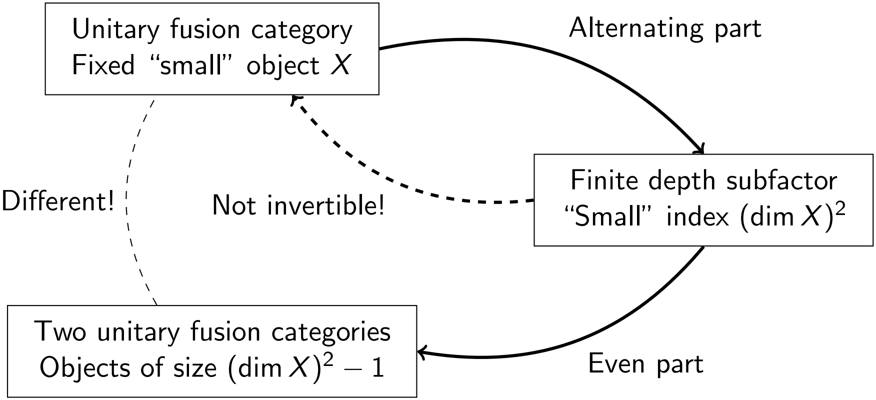
One object of small dimension

The square of the dimension of the chosen object X is called the index. This is where there's been the most work in subfactor theory. See Scott's talk for further details!

Depth

The depth is the smallest n such that $(X \otimes X^*)^{\otimes \frac{n}{2}}$ contains all simples. Finite rank is equivalent to finite depth. Depth 2 always comes from Hopf algebras (Ocneanu-Szymanski).

Summary


Unitary fusion category
Fixed "small" object X
Alternating part
Finite depth subfactor
"Small" index $(\dim X)^2$
Not invertible!
Different!
Even part
Two unitary fusion categories
Objects of size $(\dim X)^2 - 1$

Fusion graphs

Fusion graphs give a visual depiction of the fusion rules.

- Vertices correspond to simple objects.
- Edge(V,W) correspond to $\dim \operatorname{Hom}(X \otimes V, W)$

Example: $U_q(\mathfrak{su}_3)$