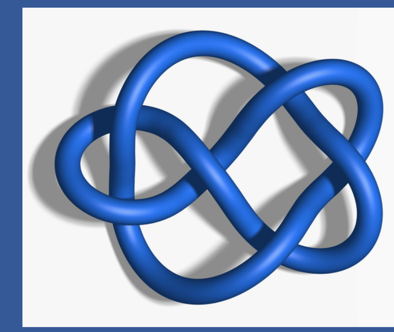
Classifying fusion categories



Miller Symposium
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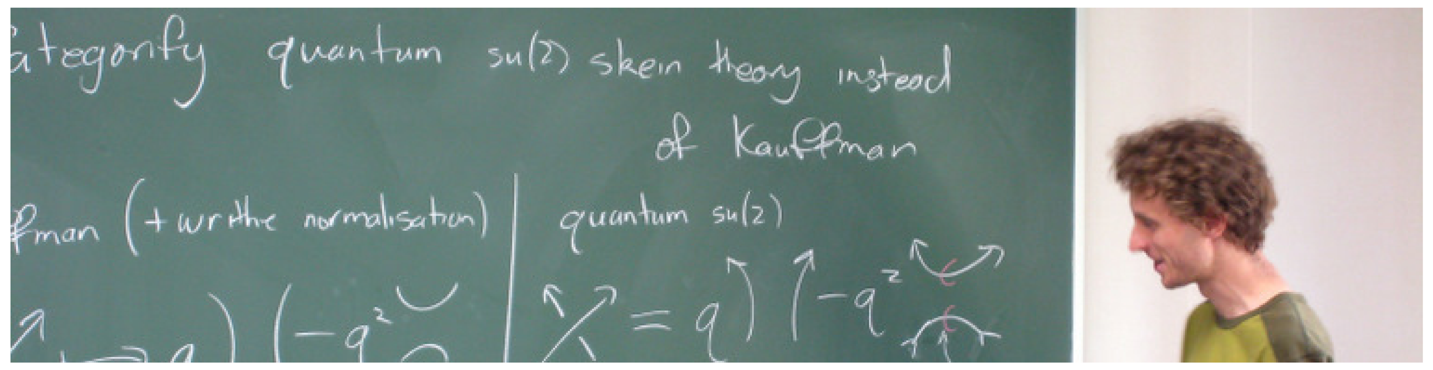
Scott Morrison
Miller Institute / Mathematics

Background



Bio

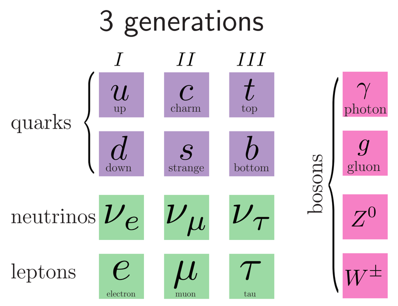
- I'm from .
- Berkeley Ph.D 2007.
- Microsoft Station Q 2007-2009.
- Miller Fellow since mid-2009.
  - Miller host Vaughan Jones.

Research interests

- Higher-dimensional algebra (fusion categories, *n*-categories)
- Quantum field theory and homotopy theory
- Subfactors and von Neumann algebras

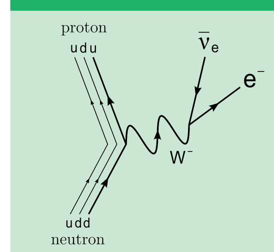
What is a fusion category?

In the **Standard Model** of particle physics, we have
fundamental particles leptons and quarks in 3 generations, and 4 bosons



interactions described by trivalent graphs (∼ Feynman diagrams)

Example (beta decay)

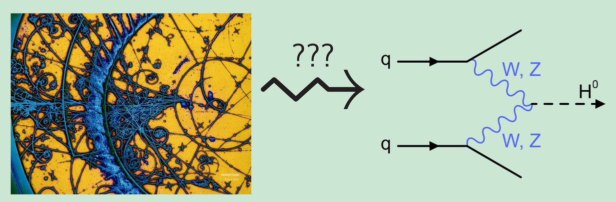


Beta decay transmutes one chemical element into another, e.g.

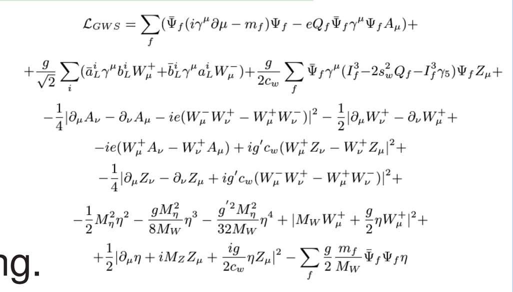
$$^{137}_{55}Cs \to {}^{137}_{56}Ba + e^- + \nu_e$$

Example (Higgs boson at the LHC?)

At high energies, Higgs bosons might be produced.



amplitudes for histories
- quantum mechanical 'probabiliities'
- described by a complicated *Lagrangian*
- depend on position, momentum, energy, etc.



A **fusion category** is a combinatorial abstraction of this setting.
- throw out all geometry and dynamics! (no position, no momentum)

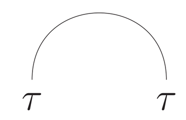
A fusion category has
- finitely many *particle types*
- combinatorial rules describing *particle interactions*
- an *amplitude* for each history

satisfying a *locality condition*: the amplitude for a large history can be computed from the amplitudes for constituent parts.

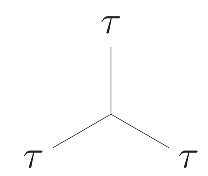
Example: the golden category

Particles
Just one type of particle, called τ.

Interactions
When two τ particles interact, they can either
- annihilate, producing the vacuum

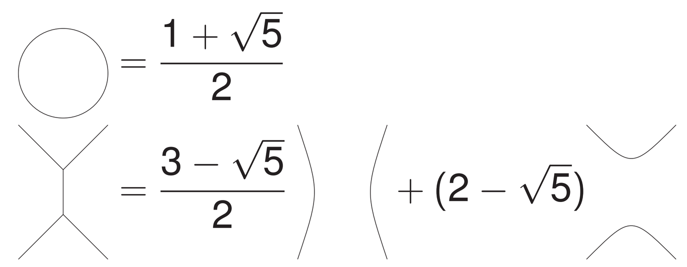


- or combine to form a single τ particle.



We write this symbolically as $\tau \otimes \tau \cong 1 \oplus \tau$, or graphically as 

Amplitudes can be calculated by local rules:

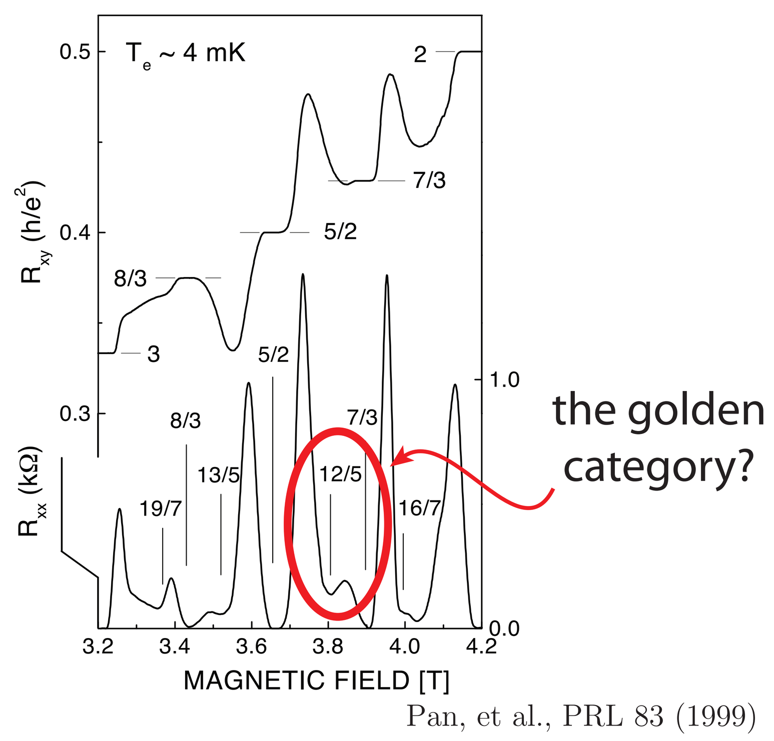


Amazing fact: any way you use these rules, you get a *consistent answer*!

Quantum computing

"Fusion categories are just mathematical toys!" ... or are they?

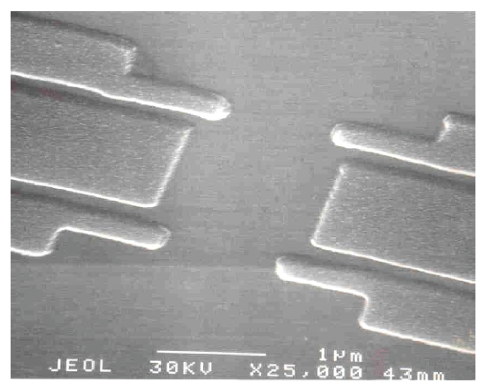
The **Fractional Quantum Hall Effect** (low temperature, high magnetic field, 2d electron gases) seems to be described by certain fusion categories!



Pan, et al., PRL 83 (1999)

The "quasi-particle excitations" of the system satisfy the rules of a fusion category. In the low temperature limit, the combinatorial abstraction reflects real lab benchtop physics!

It may be possible to build a computer using the FQHE: the golden category is **universal for quantum computing**. For now, people are trying to build simpler devices to characterise the systems.

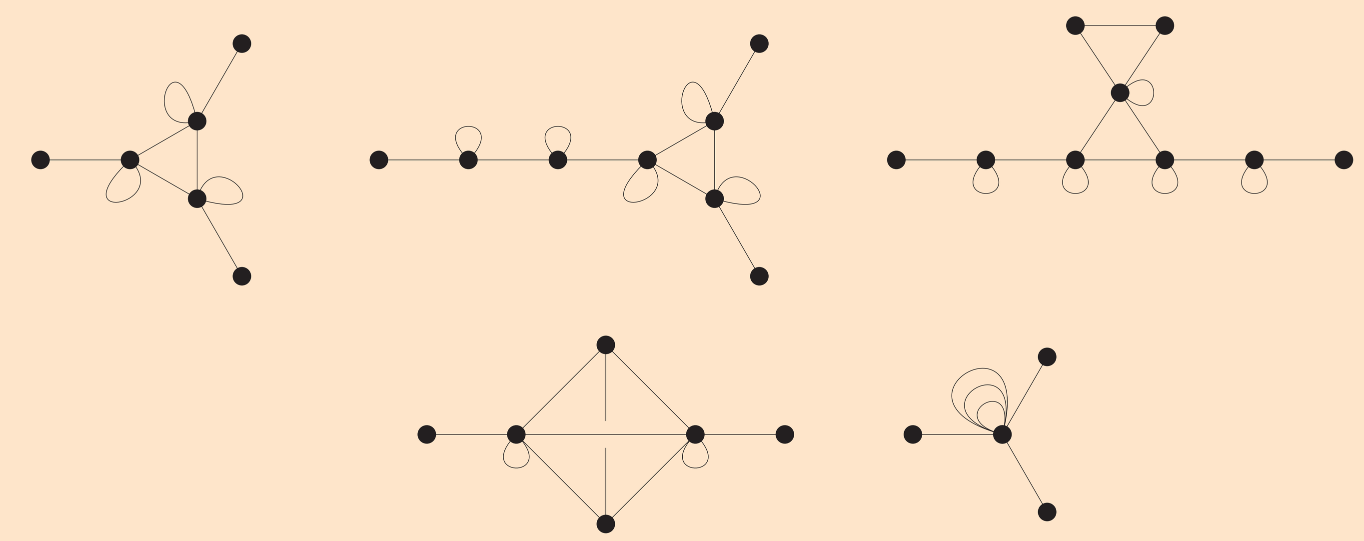


Classification

Classifying all fusion categories is probably too hard.
We'd like to understand all the 'small' ones.

Theorem (many people, 2009-2010, `http://tqft.net/ncgoa2010`)

We can classify all the possible combinatorial interactions for (subfactor) fusion categories with "index less than 5".
- *several well-understood families (including the golden category)*
- *five 'exotic' examples*

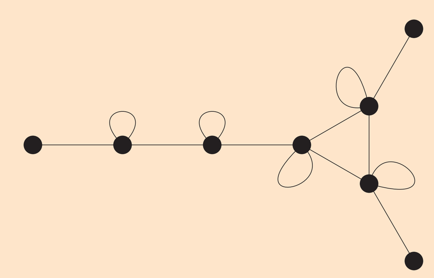


Constructing exotic examples

That result restricts what is possible. Just recently, we discovered the last missing **exotic fusion category** from this classification!

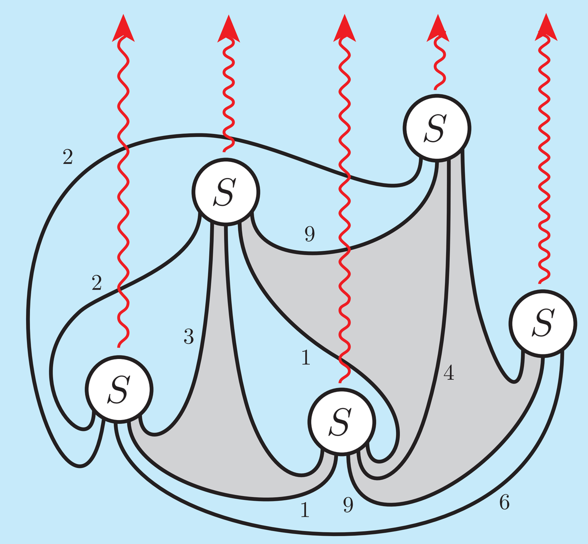
Theorem (Bigelow-Morrison-Peters-Snyder, 2009, arXiv:0909.4099)

There really is a fusion category with particle types:

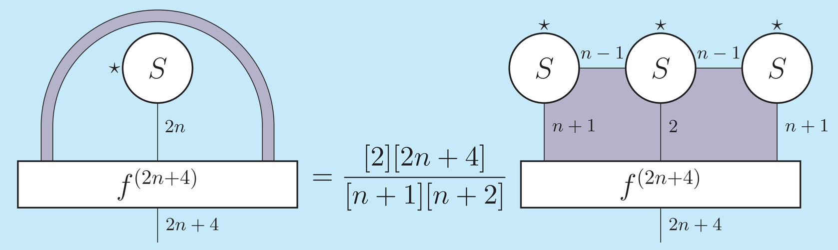


Proof.

Interactions are described by graphs which are 3- or 8-valent,
Amplitudes are defined via the "jellyfish algorithm":



using certain rules, e.g.:



Proving that these rules give consistent answers is **hard**! □