

Small fusion categories and subfactors, III

Scott Morrison

<http://tqft.net/>

joint work with Vaughan Jones, Emily Peters and Noah Snyder

UC Berkeley / Miller Institute

Fusion categories and Applications, Baylor, October 18 2009

<http://tqft.net/waco2>

Outline

Question

What is the spectrum of possible dimensions for fusion objects?

Dimensions less than 2 are of the form $2 \cos\left(\frac{\pi}{k+3}\right)$.

Theorem (modulo some corners) (Jones-Morrison-Peters-Snyder)

Any object in a fusion category with dimension at least 2 has dimension at least $\sqrt{\frac{5+\sqrt{21}}{2}} \sim 2.1889$.

Conjecture

This lower bound is realised by a fusion category with fusion graph



Principal graphs

Fusion graphs

The fusion graph for a fusion category \mathcal{C} with self-dual object X encodes $-\otimes X$ in the fusion ring.

Principal graphs for subfactors

The principal graph pair for a subfactor $N \subset M$ has vertices for the $N - N$, $N - M$, $M - N$ and $M - M$ bimodules, and encodes the tensor products $-\otimes_M M M_N$ and $-\otimes_N N M_M$.

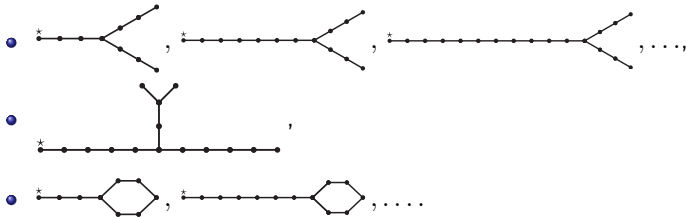
The graph norm is the quantum dimension of the generating object. In the subfactor case, this is $\sqrt{[N : M]}$.

Graph norm increases under inclusions.

Outline

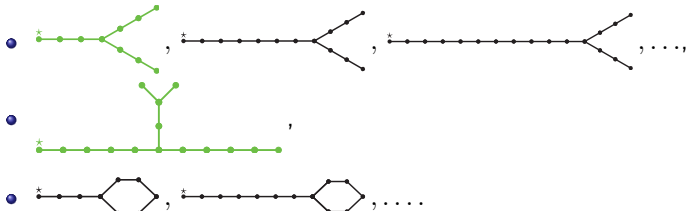
Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index less than $3 + \sqrt{3}$:



Haagerup's list

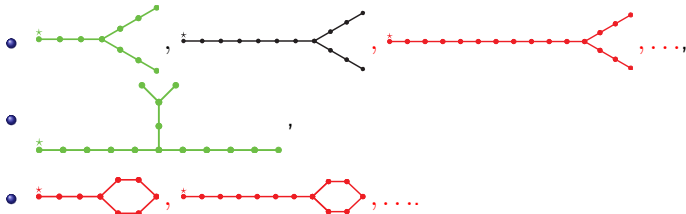
- In 1993 Haagerup classified possible principal graphs for subfactors with index less than $3 + \sqrt{3}$:



- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.

Haagerup's list

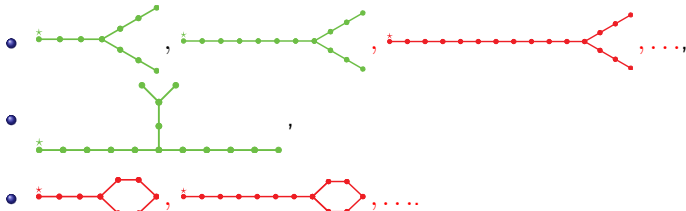
- In 1993 Haagerup classified possible principal graphs for subfactors with index less than $3 + \sqrt{3}$:



- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.

Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index less than $3 + \sqrt{3}$:



- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- Earlier this year with Stephen Bigelow and Emily Peters, we constructed the last missing case. [arXiv:0909.4099](https://arxiv.org/abs/0909.4099)

Could these be alternating parts?

Theorem (Etingof-Nikshych-Ostrik [arXiv:math.QA/0203060](https://arxiv.org/abs/math/0203060))

Dimensions of objects in fusion categories are cyclotomic integers.

The index of a subfactor is cyclotomic, since it is the sum of the dimensions of objects at depths 0 and 2, all in the even part.

For any subfactor $\dim_q({}_N M_M) = \sqrt{[N \subset M]}$. Thus the square root of the index of an alternating part must also be cyclotomic.

Outline

Classification to a given index

Begin with $\mathcal{F} = \emptyset$ and $\mathcal{W} = \{\bullet \dashrightarrow \bullet\}$.



\mathcal{F} : a finite set of graphs, any translate of which might be a principal graph



\mathcal{W} : a finite set of graphs, any extension of a translate of which might be a principal graph.

Let the weeds grow

Take a graph from \mathcal{W} , move it to \mathcal{F} , and add to \mathcal{W} all its “depth 1 extensions” that remain below the index limit.

\implies we can increase the minimal depth in \mathcal{W} .

Do some gardening

- Graphs in \mathcal{F} (\mathcal{W}) must satisfy the ‘(weak) Ocneanu test’.
- ‘Quadratic tangles’ tests can be applied to both \mathcal{F} and \mathcal{W} , by proving certain inequalities.

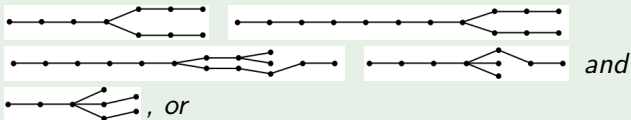
Subfactors with index at most $\frac{5+\sqrt{21}}{2}$

So far, we can't empty \mathcal{W} . Nevertheless, we have a result which is almost strong enough for our purpose!

Theorem (Jones-Morrison-Peters-Snyder)

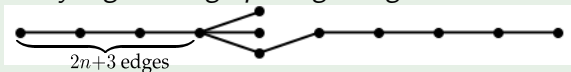
Below index $\frac{5+\sqrt{21}}{2}$, the principal graph must be one of

II



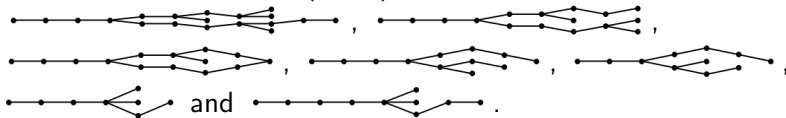
III a very high translate of a certain list of 7 graphs, or

IIII a very high rank graph beginning



What about case III?

We need to worry about (even-)translates of the graphs




Cyclotomicity


Any given translate can be easily ruled out (we've checked ~ 45). It's hard to check cyclotomicity for families (c.f. Asaeda-Yasuda).


Quadratic tangles

We've already ruled out some other families using quadratic tangles. We expect to eventually rule out the rest too.

All 5 graphs in case II have been realized as principal graphs of subfactors:


GHJ  is a GHJ subfactor, with index $3 + \sqrt{3}$.

Xu  can be constructed indirectly from the conformal inclusion $(G_2)_3 \subset (E_6)_1$, with index $\frac{5+\sqrt{21}}{2}$.

Only  can be the alternating part of a fusion category.

Corollary

Either something exists in Cases III and IIII, or the smallest dimension above 2 in a fusion category is $\sqrt{\frac{5+\sqrt{21}}{2}}$.


Is  the alternating part of a fusion category?

We think it is the alternating part arising from a self-dual object: the fusion graph would be the same graph.

Clues

- Xu constructed it via a conformal inclusion.
- Izumi constructed it using the Cuntz algebra.
- The planar generator is at an odd depth.
- The fusion ring passes all of Ostrik's d -number tests.

Open question

Can you find a generators mod relations planar algebra presentation of . Does it require a shading?

Sketch of classification to index 5

- II The same 5 subfactors, H , EH , AH , 3311 and 2221.
- III 17 infinite families of translates, with no small super-transitivity examples.
- IIII One new 'bad seed'



with no small rank examples.

Improving Cases III and IIII will require new 'quadratic tangles' style techniques.