Classifying subfactors up to index 5

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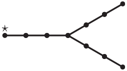
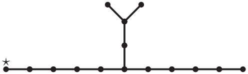
http://tqft.net
joint work with Jones, Penneys, Peters, Snyder, Tener

DARPA kickoff, UCLA, October 8 2010
http://tqft.net/UCLA-2010

Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index less than $3+\sqrt{3}$:
  - , , , . . .,
  - ,
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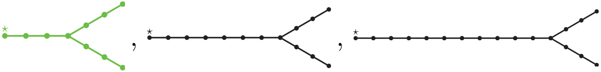
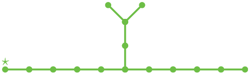
  -  , . . . ,
  -  ,
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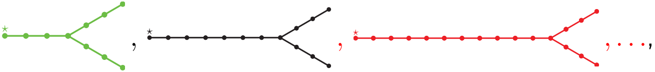
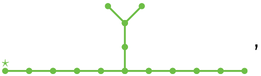
  - 
  - 
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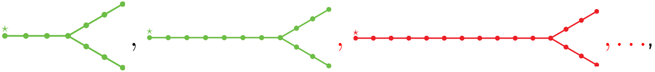
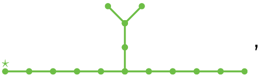
  - 
  - 
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- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
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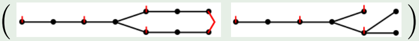
- Last year we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. `arXiv:0909.4099`

Classification statements

We work with principal graph pairs, which describe the simple bimodules for the subfactor, along with their tensor products with the generating bimodule, and which bimodules are dual.

Example (The Haagerup subfactor's principal graph pair)



The pair must satisfy an associativity test:

$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such pairs with index below some number L up to any rank or depth, obtaining a collection of allowed vines and weeds.

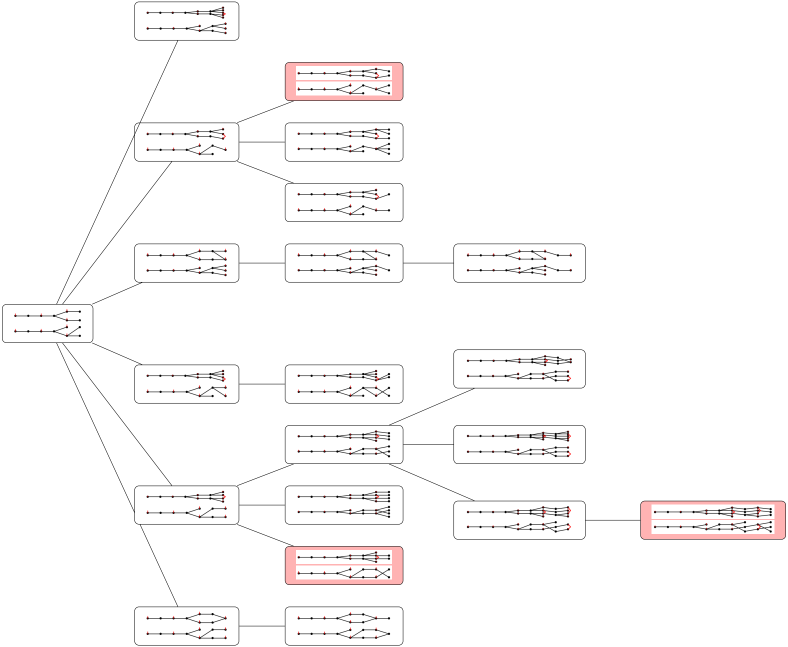
Definition

A vine represents an integer family of principal graphs, obtained by translating the vine.

Definition

A weed represents an infinite family, obtained by either translating or extending arbitrarily on the right.

If the weeds run out, the enumeration is complete. This happens in favourable cases (e.g. Haagerup's theorem up to index $3 + \sqrt{3}$), but generally we stop with some surviving weeds, and have to rule these out 'by hand'.



The classification up to index 5

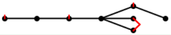
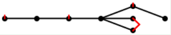
Theorem (Morrison-Snyder, part $\mathbb{I}$, arXiv:1007.1730)

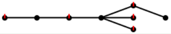
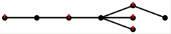
Every (finite depth) II_1 subfactor with index less than 5 sits inside one of 54 families of vines (see below), or 5 families of weeds:

$\mathcal{C} = ($  ,  $)$,

$\mathcal{F} = ($  ,  $)$,

$\mathcal{B} = ($  ,  $)$,

$\mathcal{Q} = ($  ,  $)$,

$\mathcal{Q}' = ($  ,  $)$.

Theorem (M-Penneys-Peters-Snyder, part $\mathbb{III}$, arXiv:1007.2240)

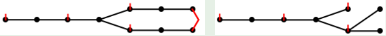
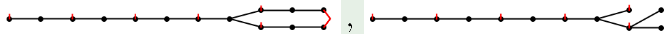
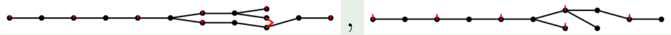
Using quadratic tangles techniques, there are no subfactors in the families $\mathcal{C}$ or $\mathcal{F}$.

Theorem (Calegari-Morrison-Snyder, `arXiv:1004.0665`)

In any family of vines, there are at most finitely many subfactors, and there is an effective bound.

Corollary (Penneys-Tener, part $\mathbb{IV}$, conjecture/work in progress)

There are only four possible principal graphs of subfactors coming from the 54 families

- ()
- ()
- ()
- ().

Recent results

Theorem (Morrison-Penneys-Peters-Snyder, part $\mathbb{V}$, Tuesday)

There are no subfactors coming from the weed

$\mathcal{B} = ($  ,  $)$

Proof.

A connection on the principal graph only exists at a certain index (one for each supertransitivity), but the only graphs with exactly that index are certain infinite graphs which are easily ruled out. □

Work in progress, Wednesday

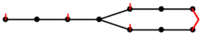
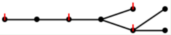
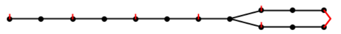
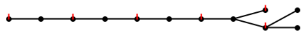
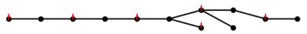
Also by a connection argument (inspired by Izumi), it seems likely that the only subfactor coming from the weeds $\mathcal{Q}$ or $\mathcal{Q}'$ is 3311.

( , )

We're thus very close to completing the classification up to index 5:

Conjecture

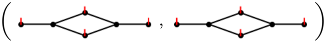
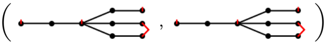
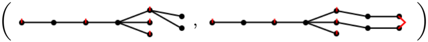
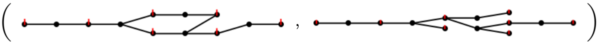
There are exactly ten subfactors other than Temperley-Lieb with index between 4 *and* 5*.*

- ( ),
- ( , ),
- ( , ),
- *The 3311 GHJ planar algebra (MR999799), with index* $3+\sqrt{3}$ ( , ),
- *Izumi's self-dual 2221 planar algebra (MR1832764), with index* $\frac{5+\sqrt{21}}{2}$ ( , )

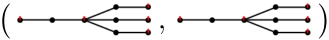
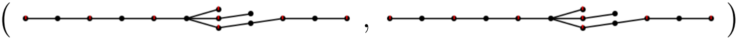
along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.

Index exactly 5

There are 5 principal graphs that come from group-subgroup subfactors, and these are known to be unique.

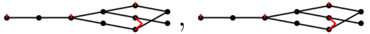
-  $1 \subset \mathbb{Z}/5\mathbb{Z}$
-  $\mathbb{Z}/2\mathbb{Z} \subset D_{10}$
-  $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$
-  $A_4 \subset A_5$
-  $S_4 \subset S_5$

We still have a few other possibilities to rule out

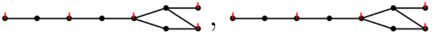
- 
- 
- 

To index $2\tau^2 \sim 5.23607$ and beyond

Beyond index 5, complete classification is still daunting. We can still fish for examples (only supertransitivity > 1)! Some are already known, but most appear to be new. There aren't yet guarantees that any of these exist, however.

- ()
  (from $SU_q(3)$ at a root of unity, index ~ 5.04892)

At index $2\tau^2 \sim 5.23607$

- ()
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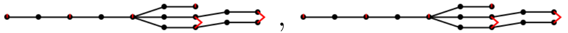
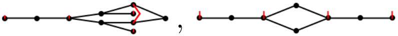
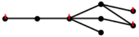
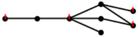
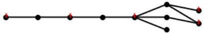
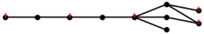
- ()
- ()

- ( , )
  (“Haagerup +1” at index $\frac{7+\sqrt{13}}{2} \sim 5.30278$)
- ( , ) at
  $\frac{1}{2}\left(4+\sqrt{5}+\sqrt{15+6\sqrt{5}}\right) \sim 5.78339$
- ( , ) at $3+2\sqrt{2} \sim 5.82843$

And at index 6

- ( , )
- ( , )

and several more!

Summary and prospects

The classification of subfactors up to index 5 is almost finished.

We can look further out; there are several new examples, but it's sparser than anyone expected. New methods using connections may allow complete classifications to higher indices.

Our techniques also apply to fusion categories. Fusion categories with objects of dimension $2\cos(\pi/n)$ have been used in topological quantum computing. We expect to obtain strong new classification results for dimension slightly above 2.