Classifying subfactors up to index 5

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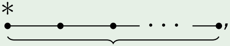
joint work with Jones, Morrison, Penneys, Snyder, Tener

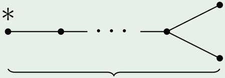
ECOAS, Dartmouth, October 24 2010

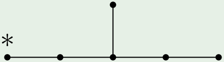
Index less than 4

Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)

The principal graph of a subfactor of index less than 4 is one of

$A_n =$  $, n \geq 2$ — *n vertices* — *index* $4\cos^2(\frac{\pi}{n+1})$

$D_{2n} =$  $, n \geq 2$ — $2n$ *vertices* — *index* $4\cos^2(\frac{\pi}{4n-2})$

$E_6 =$  — *index* $4\cos^2(\frac{\pi}{12}) \approx 3.73$

$E_8 =$  — *index* $4\cos^2(\frac{\pi}{30}) \approx 3.96$

Suppose $N \subset M$ is a subfactor, ie a unital inclusion of type II_1 factors.

Definition

The index of $N \subset M$ is $[M : N] := \dim_N L^2(M)$.

Example

If R is the hyperfinite II_1 factor, and G is a finite group which acts outerly on R, then $R \subset R \rtimes G$ is a subfactor of index $|G|$.

If $H \leq G$, then $R \rtimes H \subset R \rtimes G$ is a subfactor of index $[G : H]$.

Theorem (Jones)

The possible indices for a subfactor are

$$\{4\cos(\frac{\pi}{n})^2 | n \geq 3\} \cup [4, \infty].$$

Let $X = {}_N M_M$ and $\overline{X} = {}_M (M^{op})_N$, and $\otimes = \otimes_N$ or $\otimes_M$ as needed.

Definition

The standard invariant of $N \subset M$ is the (planar) algebra of bimodules generated by X:

$$\begin{array}{ccccccccc} X & , & X \otimes \overline{X} & , & X \otimes \overline{X} \otimes X & , & X \otimes \overline{X} \otimes X \otimes \overline{X} & , & \dots \\ \overline{X} & , & \overline{X} \otimes X & , & \overline{X} \otimes X \otimes \overline{X} & , & \overline{X} \otimes X \otimes \overline{X} \otimes X & , & \dots \end{array}$$

Definition

The principal graph of $N \subset M$ has vertices for (isomorphism classes of) irreducible N-N and N-M bimodules, and an edge from ${}_N Y_N$ to ${}_N Z_M$ if $Z \subset Y \otimes X$ (iff $Y \subset Z \otimes \overline{X}$).

Ditto for the dual principal graph, with M-M and M-N bimodules.

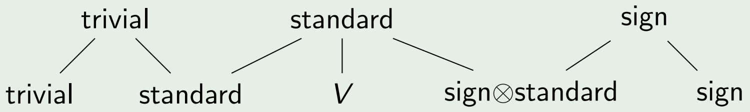
Example: $R \rtimes H \subset R \rtimes G$

Again, let G be a finite group with subgroup H, and act outerly on R. Consider $N = R \rtimes H \subset R \rtimes G = M$.

The irreducible M-M bimodules are of the form $R \otimes V$ where V is an irreducible G representation. The irreducible M-N bimodules are of the form $R \otimes W$ where W is an H irrep.

The dual principal graph of $N \subset M$ is the induction-restriction graph for irreps of H and G.

Example ($S_3 \leq S_4$)

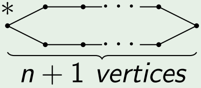
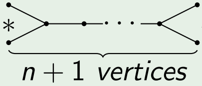


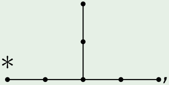
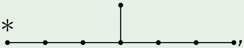
(The principal graph is an induction-restriction graph too, for H and various subgroups of H.)

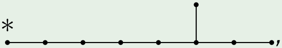
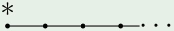
Index 4

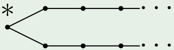
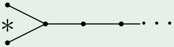
Theorem (Popa)

The principal graphs of a subfactor of index 4 are extended Dynkin diagram:

$A_n^{(1)} =$  , $n \geq 1$, $D_n^{(1)} =$  , $n \geq 3$,

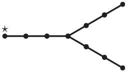
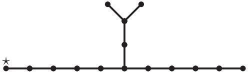
$E_6^{(1)} =$  , $E_7^{(1)} =$  ,

$E_8^{(1)} =$  , $A_\infty =$  ,

$A_\infty^{(1)} =$  , $D_\infty =$ 

There are multiple subfactors for some of these principal graphs (eg, $n-1$ non-isomorphic hyperfinite subfactors for $D_n^{(1)}$).

Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index between 4 and $3 + \sqrt{3} \approx 4.73$:
  - , , , $\ldots$, $(\approx 4.30,\ 4.37,\ 4.38, \ldots)$
  - , (≈ 4.56)
  - , , $\ldots$ $(\approx 4.62,\ 4.66, \ldots)$.

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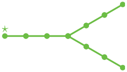
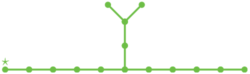
  - , , , …, $(\approx 4.30,\ 4.37,\ 4.38, \ldots)$
  - , (≈ 4.56)
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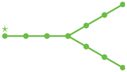
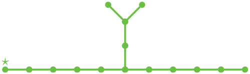
  - , , , ..., (≈ 4.30, 4.37, $4.38, \ldots$)
  - , (≈ 4.56)
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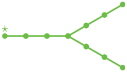
  - , , , . . . ,

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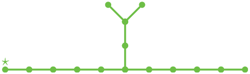
  - , (≈ 4.56)
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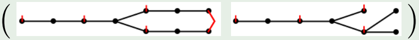
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- Last year we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. `arXiv:0909.4099`

Extending the classification

We work with principal graph pairs, meaning both principal and dual principal graphs, and information on which bimodules are dual.

Example (The Haagerup subfactor's principal graph pair)



The pair must satisfy an associativity test:

$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such pairs with index below some number L up to a given rank or depth, obtaining a collection of allowed <u>vines</u> and <u>weeds</u>.

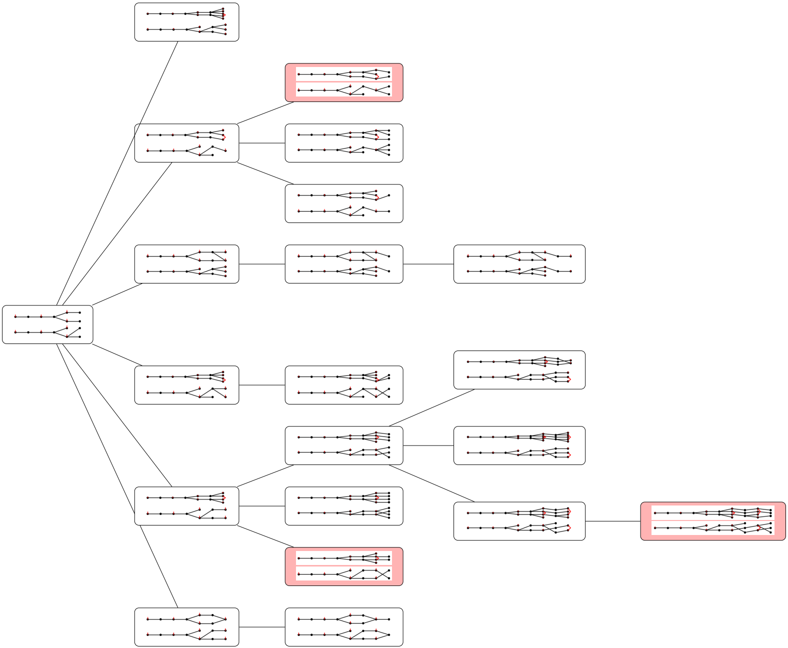
Definition

A vine represents an integer family of principal graphs, obtained by translating the vine.

Definition

A weed represents an infinite family, obtained by either translating or extending arbitrarily on the right.

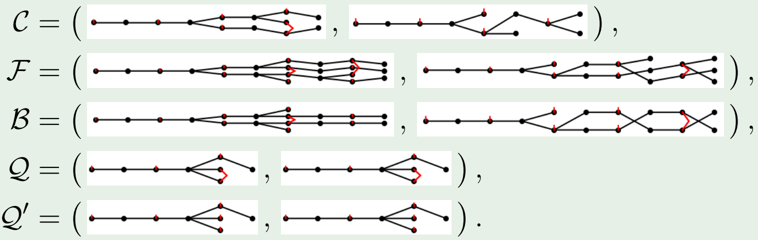
We can hope that as we keep extending the depth, a weed will turn into a set of vines. If all the weeds disappear, the enumeration is complete. This happens in favorable cases (e.g. Haagerup's theorem up to index $3 + \sqrt{3}$), but generally we stop with some surviving weeds, and have to rule these out 'by hand'.



The classification up to index 5

Theorem (Morrison-Snyder, part I, arXiv:1007.1730)

Every (finite depth) II_1 subfactor with index less than 5 sits inside one of 54 families of vines (see below), or 5 families of weeds:



Theorem (Morrison-Penneys-P-Snyder, part II, arXiv:1007.2240)

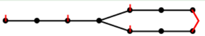
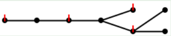
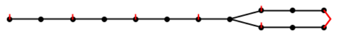
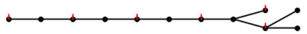
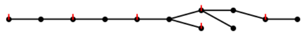
Using quadratic tangles techniques, there are no subfactors in the families $\mathcal{C}$ or $\mathcal{F}$.

Theorem (Calegari-Morrison-Snyder, `arXiv:1004.0665`)

In any family of vines, there are at most finitely many subfactors, and there is an effective bound.

Corollary (Penneys-Tener, part IV, `arXiv:1010.3797`)

There are only four possible principal graphs of subfactors coming from the 54 families

- ( )
- ( , )
- ( , )
- ( , ) .

Recent results

Theorem (Morrison-Penneys-Peters-Snyder, part V, work in progress)

There are no subfactors coming from the weed

$\mathcal{B} = ($  ,  $)$

Proof.

A connection on the principal graph only exists at a certain index (one for each supertransitivity), but the only graphs with exactly that index are certain infinite graphs which are easily ruled out. □

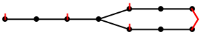
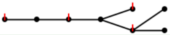
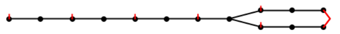
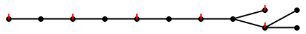
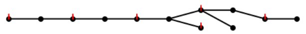
Izumi, work in progress

Also by a connection argument, the only subfactor coming from the weeds $\mathcal{Q}$ or $\mathcal{Q}'$ is 3311. ( , )

We're thus very close to completing the classification up to index 5:

Conjecture

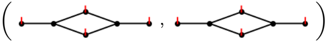
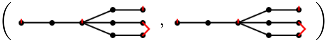
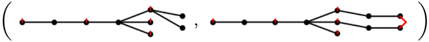
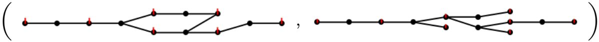
There are exactly ten subfactors other than Temperley-Lieb with index between 4 *and* 5*.*

- ( ),
- ( , ),
- ( , ),
- *The 3311 GHJ subfactor (MR999799), with index* $3+\sqrt{3}$ ( , ),
- *Izumi's self-dual 2221 subfactor (MR1832764), with index* $\frac{5+\sqrt{21}}{2}$ ( , )

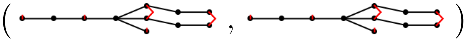
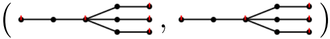
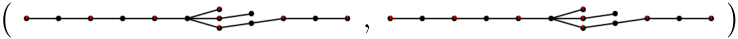
along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.

Index exactly 5

There are 5 principal graphs that come from group-subgroup subfactors, and these are known to be unique.

-  $1 \subset \mathbb{Z}/5\mathbb{Z}$
-  $\mathbb{Z}/2\mathbb{Z} \subset D_{10}$
-  $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$
-  $A_4 \subset A_5$
-  $S_4 \subset S_5$

We still have a few other possibilities to rule out

- 
- 
- 

Index beyond 5

Somewhere between index 5 and index 6, things get wild:

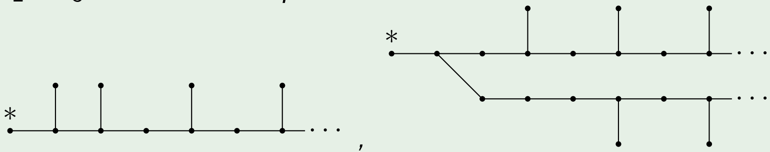
Theorem (Bisch-Nicoara-Popa)

At index 6, there is an infinite one-parameter family of irreducible, hyperfinite subfactors having isomorphic standard invariants.

and

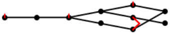
Theorem (Bisch-Jones)

$A_2 * A_3$ is an infinite depth subfactor at index $2\tau^2 \sim 5.23607$.

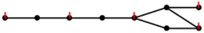
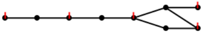
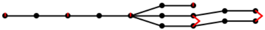
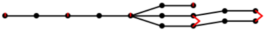


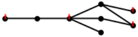
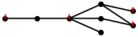
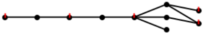
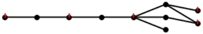
Classification above index 5 looks hard, but we can still fish for examples (only supertransitivity > 1)!

Here are some graphs that we find. (A few are previously known)

- ( , )
  (from $SU_q(3)$ at a root of unity, index ~ 5.04892)

At index $2\tau^2 \sim 5.23607$

- ( , )
- ( , )
- ( , )
- ( , )

- ( , )
  ("Haagerup +1" at index $\frac{7+\sqrt{13}}{2} \sim 5.30278$)
- ( , ) at
  $\frac{1}{2}\left(4+\sqrt{5}+\sqrt{15+6\sqrt{5}}\right) \sim 5.78339$
- ( , ) at $3+2\sqrt{2} \sim 5.82843$

And at index 6

- ( , )
- ( , )

and several more!

The End!