Classifying subfactors up to index 5

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http://tqft.net
joint work with Jones, Penneys, Peters, Snyder, Tener

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Subfactors

We're interested in inclusions $N \subset M$ of II_1 factors. The index $[M : N]$ is the von Neumann dimension of $L^2(M)$ as an N-module.

Theorem (Jones, Index for subfactors, '83)

If $[M : N] < 4$, then $[M : N] = 4\cos^2(\pi/n)$ for some n.

Question

What indices are possible above 4? What indices are possible when M is the hyperfinite II_1 factor?

Principal graphs

Definition

The principal graph for a subfactor $N \subset M$ has vertices for the $N-N$, $N-M$, $M-N$ and $M-M$ bimodules, and an edge between Y and Z for each copy of Z appearing inside $Y \otimes X$. (Here $X = {}_N M_M$ or ${}_M M_N$ as appropriate.)

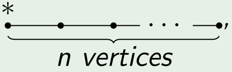
The principal graph has two connected components, the left N-modules and the left M-modules.

The graph norm is equal to the square root of the index of the subfactor (at least when the subfactor is finite depth, or is amenable).

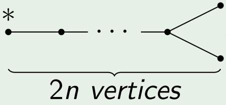
Graph norm increases under inclusions.

Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)

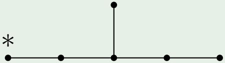
The subfactors with index less than 4 are exactly

A_n 

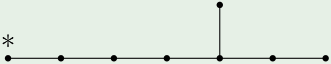
, $n \geq 2$ — *index* $4\cos^2(\frac{\pi}{n+1})$

D_{2n} 

, $n \geq 2$ — *index* $4\cos^2(\frac{\pi}{4n-2})$

E_6 *and* $\overline{E_6}$ 

index $4\cos^2(\frac{\pi}{12})$

E_8 *and* $\overline{E_8}$ 

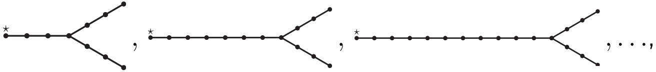
index $4\cos^2(\frac{\pi}{30})$

Index exactly 4

There's a similar classification in terms of extended Dynkin diagrams at index exactly 4. Here the principal graph is no longer a complete invariant, even up to complex conjugation.

At every index ≥ 4, there's the 'Temperley-Lieb' subfactor, with principal graph A_∞

Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index less than $3+\sqrt{3}$:
    - 
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    - 

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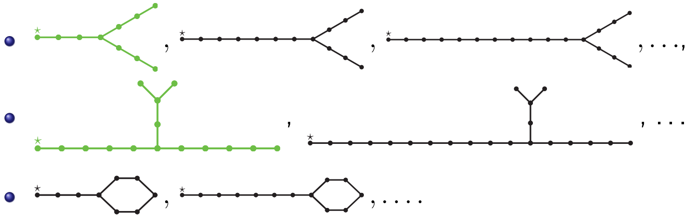

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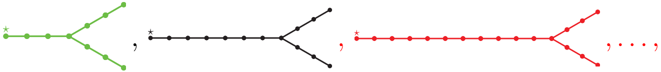
  - 
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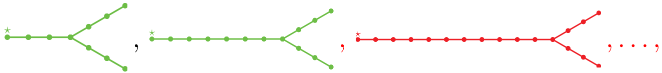
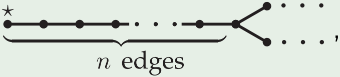
  - 
  - 
  - 
- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- Haagerup (unpublished), Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- Last year we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. `arXiv:0909.4099`

Can we construct a subfactor with a given principal graph?

The principal graph determines the dimensions of invariant spaces.

Example

If $\Gamma =$ 

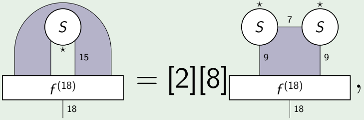
, then

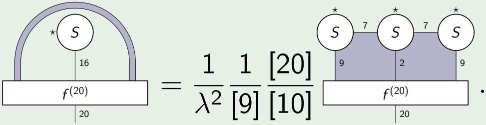
$$\dim \operatorname{Inv}(X^{\otimes 2k}) = \begin{cases} C_k & \text{for } k \leq n \\ C_k + 1 & \text{for } k = n+1. \end{cases}$$

This gives clues for generators and relations for the representation theory.

Theorem (Bigelow-Morrison-Peters-Snyder, `arXiv:0909.4099`)

If a subfactor with principal graph  *exists, its planar algebra is generated by an 8-box S, which is a lowest weight vector with eigenvalue -1 and relations*

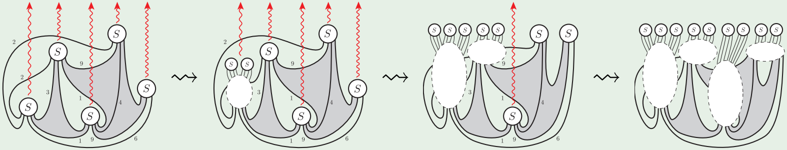
1. $S^2 = \lambda^2 f^{(8)}$ (*here* $\lambda = \sqrt{-\frac{1}{5} + 2\,\mathrm{Re}\sqrt[3]{\frac{117-65i\sqrt{3}}{2250}}}$),
2. 

3. 

Otherwise, these relations must be inconsistent.

Proof.

All these relations must hold (sizes of invariant spaces, traces of projections, a quadratic tangles argument).
They suffice to evaluate any closed diagram via the "jellyfish algorithm".



Thus this is the representation theory of some subfactor with the right index. The classification results ensure that it has the desired principal graph. □

How do we check that relations are consistent? Just as every finite group sits inside some S_n, we have the following.

Theorem (Jones-Penneys `arXiv:1007.317`, Morrison-Walker)

Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

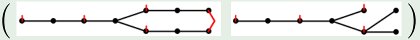
Theorem (BMPS '09)

There is an element of the GPA satisfying the desired relations, so the subfactor  *exists!*

Classification statements

We work with principal graph pairs, which describe the simple bimodules for the subfactor, along with their tensor products with the generating bimodule, and which bimodules are dual.

Example (The Haagerup subfactor's principal graph pair)



The pair must satisfy an associativity test:

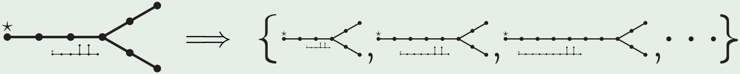
$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such pairs with index below some number L up to any rank or depth, obtaining a collection of allowed vines and weeds.

Definition

A vine represents an integer family of principal graphs, obtained by <u>translating</u> the vine.

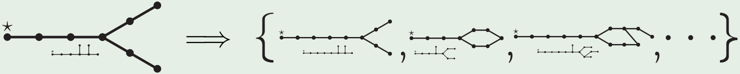
Example



Definition

A weed represents an infinite family, obtained by either translating or <u>extending</u> arbitrarily on the right.

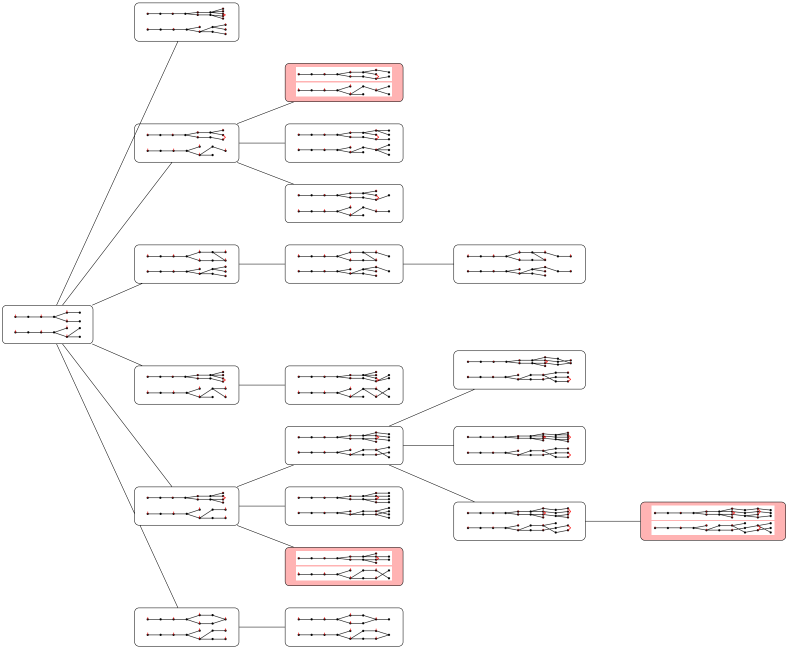
Example



The weed (, ) trivially represents all possible principal graphs.

We can always convert a weed into a vine, at the expense of finding all possible depth 1 extensions of the weed (which stay below the index limit, and satisfying the associativity condition) and adding these as new weeds.

If the weeds run out, the enumeration is complete. This happens in favourable cases (e.g. Haagerup's theorem up to index $3+\sqrt{3}$), but generally we stop with some surviving weeds, and have to rule these out 'by hand'.



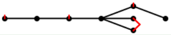
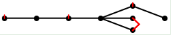
Theorem (Morrison-Snyder, part $\mathbb{I}$, `arXiv:1007.1730`)

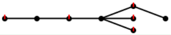
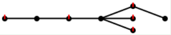
Every (finite depth) II_1 subfactor with index less than 5 sits inside one of 54 families of vines, or 5 families of weeds:

$\mathcal{C} = ($  ,  $)$,

$\mathcal{F} = ($  ,  $)$,

$\mathcal{B} = ($  ,  $)$,

$\mathcal{Q} = ($  ,  $)$,

$\mathcal{Q}' = ($  ,  $)$.

Triple point obstructions

Theorem (M-Penneys-Peters-Snyder, part $\mathbb{III}$, arXiv:1007.2240)

There are no subfactors in the families

$\mathcal{C} = ($  ,  $)$ *or*

$\mathcal{F} = ($  ,  $)$

(except possibly for infinite depth subfactors at certain indices above 5*).*

Sketch.

If V and W are the objects past the branch point, then

$$|\dim V - \dim W| \leq 1, \qquad \text{(connections)}$$

$$\frac{\dim V}{\dim W} + \frac{\dim W}{\dim V} = \frac{\lambda + \lambda^{-1} + 2}{[m][m+2]} \qquad \text{(quadratic tangles)}$$

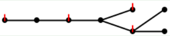
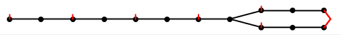
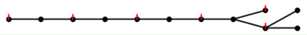
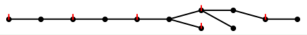
where m is the supertransitivity, and λ is a $m+1$st root of unity. □

Theorem (Calegari-Morrison-Snyder, `arXiv:1004.0665`)

Amongst the infinitely many graphs represented by a vine, there are at most finitely many which are the principal graphs of a subfactor, and there is an effective bound.

Corollary (Penneys-Tener, part $\mathbb{IV}$, `arXiv:1010.3797`)

There are only four possible principal graphs of subfactors coming from the 54 families

- ( )
- ( , )
- ( , )
- ( , ) .

Recent results

Theorem (Morrison-Penneys-Peters-Snyder, unpublished)

There are no subfactors coming from the weed

$\mathcal{B} = ($  ,  $)$

Proof.

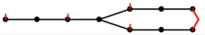
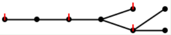
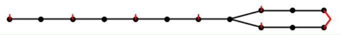
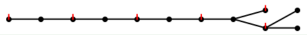
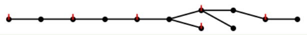
A connection on the principal graph only exists at a certain index (one for each supertransitivity), but the only graphs with exactly that index are certain infinite graphs which are easily ruled out. □

It seems likely that the only subfactor coming from the weeds $\mathcal{Q}$ or $\mathcal{Q}'$ is 3311 ( , ). We're talking to Izumi about this during our visit.

We're thus very close to completing the classification up to index 5:

Conjecture

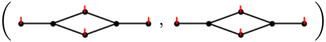
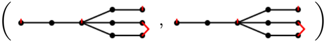
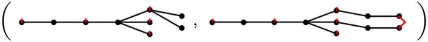
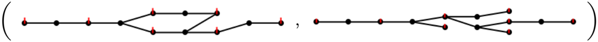
There are exactly ten subfactors other than Temperley-Lieb with index between 4 *and* 5*.*

- ( ),
- ( , ),
- ( , ),
- *The 3311 GHJ planar algebra (MR999799), with index* $3+\sqrt{3}$ ( , ),
- *Izumi's self-dual 2221 planar algebra (MR1832764), with index* $\frac{5+\sqrt{21}}{2}$ ( , )

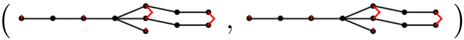
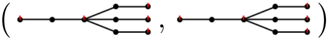
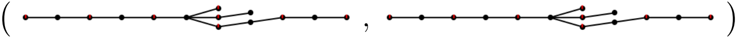
along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.

Index exactly 5

There are 5 principal graphs that come from group-subgroup subfactors, and these are known to be unique, by work of Izumi.

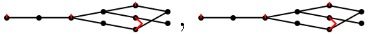
-  $1 \subset \mathbb{Z}/5\mathbb{Z}$
-  $\mathbb{Z}/2\mathbb{Z} \subset D_{10}$
-  $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$
-  $A_4 \subset A_5$
-  $S_4 \subset S_5$

We still have a few other possibilities to rule out

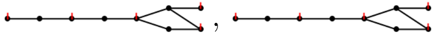
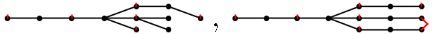
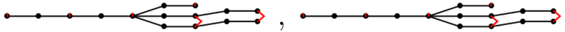
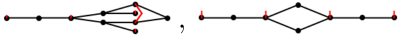
- 
- 
- 

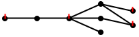
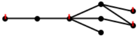
To index $2\tau^2 \sim 5.23607$ and beyond

Beyond index 5, complete classification is still daunting. We can still fish for examples (only supertransitivity > 1)! Some are already known, but most appear to be new. There aren't yet guarantees that any of these exist, however.

- ()
  (from $SU_q(3)$ at a root of unity, index ~ 5.04892)

At index $2\tau^2 \sim 5.23607$

- ()
- ()
- ()
- ()

- ( , )
  ("Haagerup +1" at index $\frac{7+\sqrt{13}}{2} \sim 5.30278$)
- ( , ) at
  $\frac{1}{2}\left(4+\sqrt{5}+\sqrt{15+6\sqrt{5}}\right) \sim 5.78339$

And at index 6

- ( , )
- ( , )

and several more!

Summary and prospects

The classification of subfactors up to index 5 is almost finished.

We can look further out; there are several new examples, but it's sparser than anyone expected. New methods using connections may allow complete classifications to higher indices.

Our techniques also apply to fusion categories. Fusion categories with objects of dimension $2\cos(\pi/n)$ have been used in topological quantum computing. We expect to obtain strong new classification results for dimension slightly above 2.