Fusion categories and subfactors

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What is a fusion category?

Definition
A fusion category is a linear semisimple $\otimes$-category with duals, with finitely many simple objects.

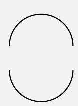
Example
The representation theory of any finite group is a fusion category.

Example
The representation category of $U_q(\mathfrak{g})$ at a root of unity is not semisimple, but after quotient by the 'negligible ideal' it becomes a fusion category.

These are great examples to keep in mind — but are there fusion categories that don't 'essentially' come from one of these two sources?

'With duals' means

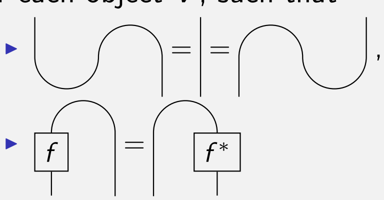
- There's a functor $* : \mathcal{C} \to \mathcal{C}$ that reverses tensor products, and $** = \mathbf{1}_{\mathcal{C}}$.
- There are maps



$: V^* \otimes V \to \mathbf{1}$ (pairing)

$: \mathbf{1} \to V \otimes V^*$ (copairing)

for each object V, such that



More examples

Example (Quantum $SU(2)$ at a root of unity)
The irreps of $SU(2)$ (or of $U_q(\mathfrak{sl}_2)$) are indexed by natural numbers (the 'highest weight', which is one less than the dimension), and the tensor product rule is

$$V_a \otimes V_b = \bigoplus_{c=|a-b|}^{a+b} V_c.$$

When q is a root of unity, the tensor category has a 'negligible ideal'. We need to quotient by this to get a semisimple category. Only finitely many objects survive; when q is a $4n-4$th root of unity, it's the first n objects. The 'truncated tensor product' is

$$V_a \otimes V_b = \bigoplus_{c=|a-b|}^{\min\{a+b,2n-1-a-b\}} V_c.$$

Representations of finite groups always form a symmetric tensor category; representations of quantum groups form a braided tensor category. Both of these are special situations, however.

Example

The category of G-graded vector spaces, with G a finite group, is neither symmetric nor braided when G is nonabelian.

Dimensions

Definition

A dimension function on a fusion category is a homomorphism from the Grothendieck ring to $\mathbb{C}$.

Definition

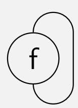
The principal graph for an object $X \in \mathcal{C}$ has vertices $Obj(\mathcal{C})$, and an edge from Y to Z for each summand of Z in $Y \otimes X$.

Definition

The Frobenius-Perron dimension of X is the largest eigenvalue of the principal graph for X.

The Frobenius-Perron dimension is always an algebraic integer.

In a $\otimes$-category with duals, endomorphisms have a trace:

$\mathrm{tr}(f) =$ 

$$= p_{X^*} \circ (f \otimes \mathbf{1}_{X^*}) \circ c_X$$

where $p_{X^*} : X \otimes X^* \to \mathbf{1}$ is the duality pairing, and $c_X : \mathbf{1} \to X \otimes X^*$ is the copairing.

Definition

The categorical dimension of an object is $tr(\mathbf{1}_X)$.

If the fusion category is unitary (there's a $*$-structure on Hom spaces, so $\langle x, y \rangle = \mathrm{tr}(y^* x)$ is positive definite), then the Frobenius-Perron and categorical dimensions agree.

What is a subfactor?

- A subfactor is an inclusion of von Neumann algebras $N \subset M$ each with trivial centre.
- We're interested in II_1 factors (no minimal projections, the identity is finite).
- The index of $N \subset M$ is the von Neumann dimension of M as an N module.

Definition

The "even part of $N \subset M$" is the collection of N–N bimodules generated by ${}_N M_N$. It is a unitary semisimple $\otimes$-category with duals.

The dimension of the object M is the index of $N \subset M$.

When there are finitely many simple objects (so the even part is a fusion category), we say $N \subset M$ is finite depth.

Algebra objects in fusion categories

In the even part of $N \subset M$, there is a special object ${}_N M_N$, which has an algebra structure. From this, we can recover the subfactor.

Theorem

For every algebra object $\mathcal{A}$ in a unitary fusion category $\mathcal{C}$, there is a finite depth II_1 subfactor $N \subset M$ so

$$(\mathcal{C}, \mathcal{A}) \cong ({}_N mod_N, M).$$

In fact, a finite depth subfactor is equivalent to either

1. a unitary $\otimes$-category $\mathcal{C}$, an algebra object $A \in \mathcal{C}$, and a chosen object X in the category of A-module objects, or
2. a pair of unitary $\otimes$-categories $\mathcal{C}$ and $\mathcal{D}$, a (categorical) Morita equivalence $\mathcal{X}$ between them, and a chosen object $X \in \mathcal{X}$.

Because of this close relationship between unitary fusion categories and subfactors, we often take advantage of both settings to prove theorems.

What's out there?

The dimensions of fusion objects are highly constrained. Jones proved the first result in this direction.

Theorem (Jones, Index for subfactors, '83)

If $1 < \dim V < 2$, *then* $\dim V = 2\cos(\pi/n)$.

Proof.

These are the only real algebraic integers less than 2 which are maximal amongst their conjugates. □

Theorem (Coste-Gannon, '94)

Every dimension is a cyclotomic integer.

Proof.

Entries of the S-matrix of the Drinfeld center are cyclotomic. □

Theorem (Calegari-Morrison-Snyder, CMP '10)

If $2 < \dim V < 76/33$, *then* $\dim V$ *is one of*

$$\frac{\sqrt{7}+\sqrt{3}}{2}, \sqrt{5}, 1+2\cos(2\pi/7), \frac{1+\sqrt{5}}{\sqrt{2}}, \frac{1+\sqrt{13}}{2}$$

Proof.

These are the only real cyclotomic integers less than $76/33$ which are maximal amongst their conjugates. □

Principal graphs

Definition

The principal graph for a subfactor $N \subset M$ has vertices for the $N-N$, $N-M$, $M-N$ and $M-M$ bimodules, and an edge between Y and Z for each copy of Z appearing inside $Y \otimes X$. (Here $X = {}_N M_M$ or ${}_M M_N$ as appropriate.)

The principal graph has two connected components, the left N-modules and the left M-modules.

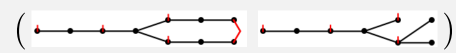
The graph norm is equal to the square root of the index of the subfactor (at least when the subfactor is finite depth, or is amenable).

Graph norm increases under inclusions.

Classification statements

We also remember which bimodules are dual to each other.

Example (The Haagerup subfactor's principal graph)



The principal graph must satisfy an associativity test:

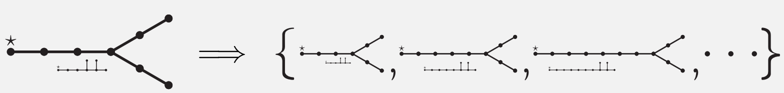
$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such pairs of graphs with index below some number L up to any rank or depth, obtaining a collection of allowed vines and weeds.

Definition

A vine represents an integer family of principal graphs, obtained by translating the vine.

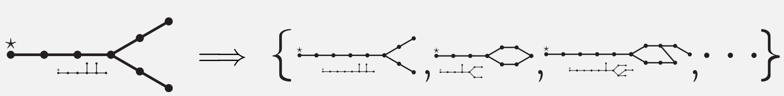
Example

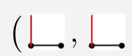


Definition

A weed represents an infinite family, obtained by either translating or extending arbitrarily on the right.

Example



The weed  trivially represents all possible principal graphs.

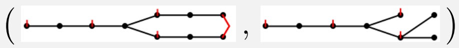
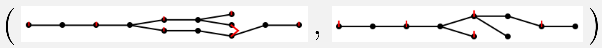
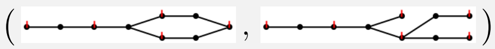
We can always convert a weed into a vine, at the expense of finding all possible depth 1 extensions of the weed (which stay below the index limit, and satisfying the associativity condition) and adding these as new weeds.

If the weeds run out, the enumeration is complete. This happens in favourable cases, but generally we stop with some surviving weeds, and have to rule these out 'by hand'.

Classification results for index less than $3 + \sqrt{3}$

Theorem (Haagerup, '93)

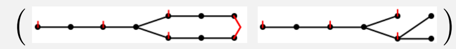
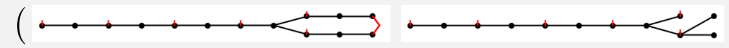
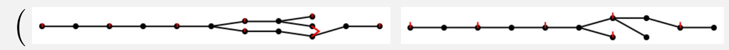
All subfactors other than A_∞ with index in the interval $(4, 3+\sqrt{3})$ are represented by the following vines:

1. 
2. 
3. 

This was a favourable case, where the enumeration ran out of weeds.

Theorem

There are exactly three subfactors in this range, with principal graphs

- 
- 
- 

Proof.

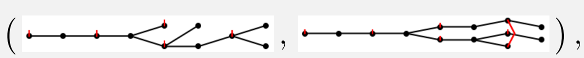
- Asaeda-Haagerup '98 constructed the first two examples.
- An unpublished result of Haagerup's, and results of Bisch '98, Asaeda-Yasuda '07 showed that there are no others except possibly the third example.
- Bigelow-Morrison-Peters-Snyder constructed the 'extended Haagerup' subfactor last year.

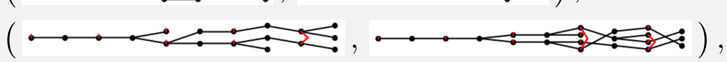
□

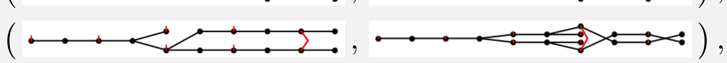
Classification results for index less than 5

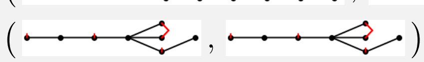
Theorem (Morrison-Snyder `arXiv:1007.1730`)

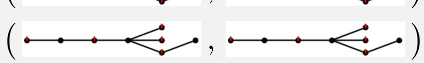
All subfactors other than A_∞ with index between 4 and 5 are represented by 43 families of vines, or the following 5 weeds.

$\mathcal{C} =$ 

$\mathcal{F} =$ 

$\mathcal{B} =$ 

$\mathcal{Q} =$ 

$\mathcal{Q}' =$ 

Eliminating vines

Theorem (Calegari-Morrison-Snyder, CMP '10)

In any vine, only finitely many graphs have a cyclotomic index. With much better bounds, all but finitely many graphs have a multiplicity free eigenvalue which is not cyclotomic.

Either condition is sufficient to eliminate a possible subfactor.

- Penneys-Tenner `arXiv:1010.3797` have recently developed algorithms for efficiently computing these bounds,
- and computed them for the 43 vines in our enumeration.
- They looked at the finitely many cases remaining from the vines, and found obstructions for all but one graph.

All of the weeds have been killed off:

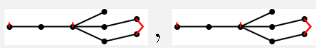
- $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{B}$ by Morrison-Penneys-Peters-Snyder ($\mathcal{C}$, $\mathcal{F}$ in arXiv:1007.2240, $\mathcal{B}$ unpublished),
- $\mathcal{Q}'$ and $\mathcal{Q}$ by Izumi-Jones-Morrison-Snyder (last week!).

Theorem

The only subfactors with index in the interval $[3+\sqrt{3}, 5)$ are

A_∞ *at every index,*

'3311' ()
with index $3+\sqrt{3}$ (Goodman-de la Harpe-Jones, '89),

'2221' ()
with index $\frac{5+\sqrt{21}}{2}$ (Izumi, '01).

These results complete the classification of subfactors with index less than 5.

Indices of hyperfinite subfactors

Question

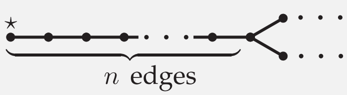
Every index is realised by a subfactor with principal graph A_∞. Which indices are possible for hyperfinite subfactors?

- Popa has shown that the smallest index above 4 that can be realised by a hyperfinite II_1 subfactor is the square of the graph norm of E_{10}, approximately 4.02642.
- The waters here are deep, analytic techniques are required, and little is known.

Can we construct a subfactor with a given principal graph?

The principal graph determines the dimensions of objects and the dimensions of invariant spaces.

Example

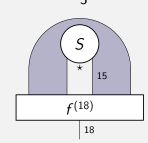
If $\Gamma =$ 

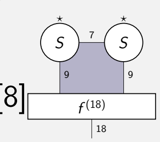
, then

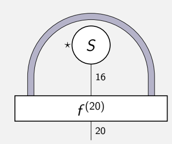
$$\dim \mathrm{Inv}(V^{\otimes 2k}) = \begin{cases} C_k & \text{for } k \leq n \\ C_k + 1 & \text{for } k = n+1. \end{cases}$$

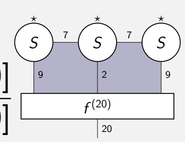
These data strongly constrain generators and relations for the representation theory.

If a subfactor with principal graph  exists, its representation theory is generated by an 8-box S, which is a lowest weight vector with eigenvalue -1 and relations

1. $S^2 = \lambda^2 f^{(8)}$ (here $\lambda^2 = -\frac{1}{5} + 2\,\mathrm{Re}\sqrt[3]{\frac{117-65i\sqrt{3}}{2250}}$),
2. 

$= [2][8]$ 

,
3. 

$= \frac{1}{\lambda^2}\frac{1}{[9]}\frac{[20]}{[10]}$ 

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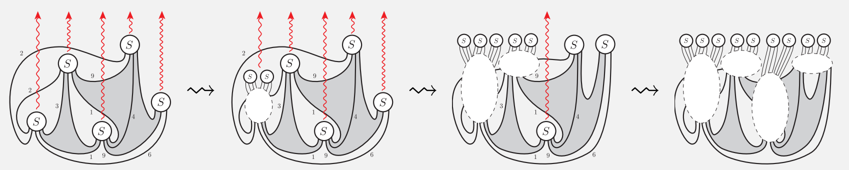
Theorem (Bigelow-Morrison-Peters-Snyder, Acta Math. '09)

There is at most one subfactor with these generators and relations.

Proof.

These relations suffice to evaluate any closed diagram via the "jellyfish algorithm".



□

Two things could go wrong:

- The relations are inconsistent (i.e. this is a presentation of the 'zero subfactor').
- The resulting representation theory is not unitary.

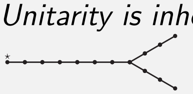
How do we check that relations are consistent? Every finite group sits inside some S_n. Analogously, we have

Theorem (Jones-Penneys `arXiv:1007.317`, Morrison-Walker)

Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

Theorem (BMPS '09)

There is an element of the GPA satisfying the desired relations. Unitarity is inherited from the GPA, so the subfactor



exists!

Just as the exceptional Lie groups were discovered via the Killing-Cartan program of classification, the classification of small index subfactors is producing examples of exotic subfactors.

By contrast, fusion categories with integer dimension objects are conjectured to be 'weakly group-theoretical', essentially definable in terms of group-theoretic data.

The new exotic subfactors and fusion categories we are finding

- constrain possible structural theorems,
- give counterexamples to conjectures, and
- give interesting examples of new modular data and thus exotic 3-manifold invariants.

Exotic fusion categories

A classical theorem of Brauer shows that the representation theory of any finite group can be defined over a cyclotomic field. (The same holds for quantum groups at roots of unity.) Etingof, Nikshych and Ostrik asked if this is true of every fusion category.

Theorem (Morrison-Snyder, Transactions of the AMS '10)

The even parts of the Haagerup and extend Haagerup subfactors cannot be defined over any cyclotomic field.

Proof.

Using the skein theory, we produce a canonical element of the ground field which is not cyclotomic. □

Summary

Recent progress

- Constructed the missing case, extended Haagerup, from earlier classifications.
- Developed uniform new techniques from number theory to reduce classifications to finite problems.
- The classification of subfactors with index less than 5.

There's still plenty more to do:

- The classification beyond 5 looks difficult; perhaps we'll need new techniques.
- Computer searches show that the classification remains sparse, but also find new candidate examples.
- It will be interesting to try to construct these!