

# DECRA 2012 Proposal: Fusion categories and topological quantum field theory

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## **Administrative Summary**

### **A2 Proposal Title**

Fusion categories and topological quantum field theory.

### **A4 Summary of proposal**

A fusion category is a mathematical structure modelling interacting particle systems. I will study fusion categories, and develop the first explicit description of all low dimensional examples. My work has major implications for the development of topological quantum field theory. I will prove new results about 3 dimensional topological quantum field theory, with numerous applications to topology. These mathematical theories successfully model fractional quantum Hall materials, which are the substrate for the proposed topological quantum computing architecture. My project contributes to the development of operator algebra theory, and the new mathematical fields of quantum algebra and quantum topology and has applications to physical models.

### **A5 Summary of Project for Public Release**

This project will involve mathematical research of the highest international calibre on fusion categories and topological field theory. Progress in these fields will lead to advances in computing (e.g. substrates for quantum computers), condensed matter physics, and the mathematical fields of operator algebra, quantum algebra, and quantum topology.

# Research Opportunity and Performance Evidence

## C1 Career and opportunities for research over the last 5 years

i. Over the last 5 years I have enjoyed unparalleled research opportunities in academia and industry.

**UC Berkeley, Ph.D. May 2007.** During my Ph.D. I worked with Fields Medallist Vaughan Jones. I received the Herb Alexander Prize for Best Thesis in Pure Mathematics, for my thesis on “A Diagrammatic Category for the Representation Theory of  $U_q(\mathfrak{sl}_n)$ ”.

**Microsoft Station Q, April 2007-June 2009.** I accelerated the completion of my Ph.D. work because Microsoft offered me a postdoctoral research-only position at ‘Station Q’, situated at UC Santa Barbara. This research groups spans pure mathematics, mathematical physics and condensed matter physics, and works on projects focused on the development of topological quantum computing based on physical substrates supporting topological phases of matter. It is directed by Fields Medallist Michael Freedman. I established significant collaborations with mathematicians working in industry. I work on an ongoing basis with members of Station Q, and anticipate further future development of applications for my work and inspiration for relevant new research problems.

**Miller Institute for Basic Research, July 2009-June 2012.** I am currently a Miller Fellow at UC Berkeley. This prestigious 3 year research-only postdoctoral fellowship has supported my ongoing research. I have been able to focus on developing innovative new ideas. This fellowship has allowed me to travel to conferences, and develop a strong network of disciplinary and interdisciplinary collaborations

**Australian National University, July 2012—** Next year I will begin a continuing teaching/research position as Senior Lecturer in the Mathematical Sciences Institute of the Australian National University.

ii. I have been fortunate to have had no interruptions to my career path. I have written 11 papers in the 4 years since my Ph.D., many of which appear in journals with an ERA ranking of A\*. (See §C2 for more details.) As a result of my work in quantum topology, fusion categories and subfactors I have been invited to speak at many international conferences (e.g. in Sweden, India, France and Japan), and have been a colloquium speaker at Caltech, the University of Oregon, and UC Berkeley.

My research to date has involved a wide network of international collaborators in both academia and industry, including two Fields Medallists. (See §D.1.2.5.)

I am the organiser of several international conferences, including a workshop on fusion categories and subfactors, (sponsored by the American Institute of Mathematics, to be held in March 2012), a conference on subfactors, (sponsored by the Defence Advanced Research Projects Agency, to be held in July 2011), and a session of a joint AMS/NZMS meeting (in Wellington, 2007). I have refereed over 20 papers in my fields, including some for prestigious journals such as *Geometry & Topology* (ERA A\*).

I am a founder of the website MathOverflow, which provides a forum for questions and answers on research level mathematics. MathOverflow sees over 10000 visits each day, from the strongest undergraduate and postgraduate students, to many of the international leaders in mathematics. In its first year MathOverflow is already revolutionising mathematical research, by initiating collaborations, integrating mathematical communities, and providing access to the highest calibre expertise in specialised fields. (See my article in the *Notices of the American Mathematical Society* [27] for more details.)

## C2 Significant publications

I regularly publish in the highest quality journals, including the *Proceedings of the National Academy of Sciences*, *Acta Mathematica* and *Communications in Mathematical Physics*. Of my nine peer-reviewed papers in ERA ranked journals, five have been in A\*-ranked journals, and a further three in A-ranked journals. I have another two peer-reviewed papers in the excellent new journal of Quantum Topology. Further, three new papers [MW10, MS10, MPPS10] are undergoing peer review, two at A\*-ranked journals and the third at a A-ranked journal.

In 2010 I wrote and submitted 7 papers; this is unusually many for a pure mathematician at any stage of their career. Many of my papers have been on questions preliminary to the proposed research, and indicate the considerable existing momentum behind the project.

### C2.iv Refereed journal articles

*Articles with direct relevance to the proposed project are marked with a \*, and ERA rankings for each journal are shown in square brackets at the end of the line.*

- [1] **Scott Morrison** and Kevin Walker, *Higher categories, colimits and the blob complex*, Proceedings of the National Academy of Sciences, May 2 2011, Available at [DOI:10.1073/pnas.1018168108](https://doi.org/10.1073/pnas.1018168108). [A\*]
- \* [2] Frank Calegari, **Scott Morrison** and Noah Snyder, *Cyclotomic integers, fusion categories, and subfactors*, Communications in Mathematical Physics Vol. 303, Issue 3 (2011), pp. 845–896. Available at [arXiv:1004.0665](https://arxiv.org/abs/1004.0665). [A\*]
- \* [3] **Scott Morrison**, Emily Peters and Noah Snyder, *Knot polynomial identities and quantum group coincidences*, Quantum Topology Vol. 2 (2011) pp. 101–156. Available at [arXiv:1003.0022](https://arxiv.org/abs/1003.0022). [new journal - unranked]
- \* [4] **Scott Morrison** and Noah Snyder, *Non-cyclotomic fusion categories*, to appear in Transactions of the American Mathematical Society (accepted 3 Nov 2010), available at [arXiv:1002.0168](https://arxiv.org/abs/1002.0168). [A\*]
- \* [5] Stephen Bigelow, **Scott Morrison**, Emily Peters and Noah Snyder, *Constructing the extended Haagerup planar algebra*, to appear in Acta Mathematica (accepted 30 June 2010), available at [arXiv:0909.4099](https://arxiv.org/abs/0909.4099). [A\*]
- [6] **Scott Morrison**, *The braid group surjects onto  $G_2$  tensor space*, Pacific Journal of Mathematics, Vol. 249 (2011), No. 1, 189–198. Available at [arXiv:0907.0256](https://arxiv.org/abs/0907.0256). [A]
- [7] Michael Freedman, Robert Gompf, **Scott Morrison** and Kevin Walker, *Man and machine thinking about the smooth 4-d Poincaré conjecture*, Quantum Topology, Vol. 1, Issue 2 (2010), pp. 171–208. Available at [arXiv:0906.5177](https://arxiv.org/abs/0906.5177). [new journal - unranked]
- \* [8] **Scott Morrison**, Emily Peters and Noah Snyder, *Skein theory for the  $\mathcal{D}_{2n}$  planar algebras*, Journal of Pure and Applied Algebra, Vol. 214, No. 2 (2010) pp. 117–139. Available at [arXiv:0808.0764](https://arxiv.org/abs/0808.0764). [A]
- [9] David Clark, **Scott Morrison** and Kevin Walker, *Fixing the functoriality of Khovanov homology*, Geometry & Topology, 13 (2009) pp. 1499–1582. Available at [arXiv:math.GT/0701339](https://arxiv.org/abs/math.GT/0701339). [A\*]

[10] **Scott Morrison** and Ari Nieh, *On Khovanov's cobordism theory for  $\mathfrak{su}_3$  knot homology*, Journal of Knot Theory and its Ramifications, Vol. 17, No. 9 (2008), pp. 1121–1173. Available at [arXiv:math.GT/0612754](https://arxiv.org/abs/math.GT/0612754). [B]

[11] Dror Bar-Natan and **Scott Morrison**, *The Karoubi Envelope and Lee's Degeneration of Khovanov Homology*, Algebraic & Geometric Topology, 6 (2006) pp. 1459–1469. Available at [arXiv:math.GT/0606542](https://arxiv.org/abs/math.GT/0606542). [A]

## C2.v Conference presentations

*I have given invited presentations at the following international conferences.*

\* [12] *Classification of subfactors up to index 5*. ICM Satellite Conference on Operator Algebras, Chennai, India, August 2010.

\* [13] *Classification of subfactors up to index 5*. Quantum Groups, Clermont-Ferrand, France, September 2010.

[14] *The blob complex*. Low dimensional topology and categorification, Stony Brook, USA, June 2010.

[15] *Khovanov homology*. Link homology, Mathematical Sciences Research Institute, USA, January 2010.

[16] *The blob complex*. TQFT and Link Homology, Hahei, New Zealand, January 2010.

\* [17] *Fusion categories and small index subfactors*. Fusion categories, Waco, USA, October 2009.

\* [18] *Extended Haagerup exists!* Tensor categories, Bloomington, USA, March 2009.

[19] *The Cappell-Shaneson spheres and the  $s$ -invariant*. Knots in Washington, Washington DC, USA, January 2009.

[20] *Blob homology*. Georgia Topology Conference, Athens, USA, May 2008.

[21] *Lasagna composition for Khovanov homology*. Joint NZMS/AMS meeting, Wellington, New Zealand, December 2007.

[22] *Lasagna composition for Khovanov homology*. Quantum Topology in Hanoi, Vietnam, August 2007.

[23] *Lasagna composition for Khovanov homology*. Link homology, Oporto, Portugal, July 2007.

[24] *Functoriality and duality for Khovanov homology*. Categorification, Kyoto, Japan, May 2007.

[25] *Fixing the functoriality of Khovanov homology*. Categorification in Algebra and Topology, Uppsala, Sweden, September 2006.

[26] *Khovanov homology*. Low dimensional topology, Taipa, New Zealand, January 2006.

## C2.vi Other

[27] *MathOverflow*

Anton Geraschenko, **Scott Morrison** and Ravi Vakil. Notices of the American Mathematical Society, Vol. 57 (2010), p. 701.

## C3 Contributions to the research field of this proposal

*Numbered references are to my published work in §C2. Other references are in §D1.4*

My research fields are fusion categories, subfactors, higher category theory and low dimensional topology. I have made significant contributions to each of these fields. This proposal is focused on fusion categories and subfactors, and builds upon my work there, yet will also draw on ideas from my work in higher category theory.

**C3.1 Subfactors** I have demonstrated expertise in the study of subfactors, exemplified by my paper **Constructing the extended Haagerup planar algebra** [5] to appear in *Acta Mathematica* (ERA ranking A\*). This paper answers a long-standing open problem in the classification of finite depth subfactors (c.f. §D1.2.3 below for background). The extended Haagerup subfactor had been suspected to exist for 15 years, but had eluded all efforts to construct it. My paper introduced new techniques in quantum topology and skein theory, and applied these to a problem in operator algebras and subfactors. My solution to this problem completed the classification of subfactors with index less than  $3 + \sqrt{3}$ .

I am leading an international collaboration to extend the classification program up to index 5. We have now established this result, showing that there are exactly 10 finite depth subfactors with index between 4 and 5. This work will be published in a four part series; Parts 1 and 2 have recently been submitted [MS10], [MPPS10], Part 4 [PT10] has been written by graduate students at UC Berkeley following the techniques of my paper [2] and recently submitted, and the manuscript for Part 3 is in preparation. This series includes a total of 7 coauthors, and this broad collaboration will provide part of the international network supporting my proposed future work.

My paper **Cyclotomic integers, fusion categories and subfactors** [2], to appear in *Communications in Mathematical Physics* (ERA ranking A\*), introduces techniques from number theory to study subfactors. The paper provides a new approach to restricting the possible indices of subfactors, and also provides a powerful technical tool in classification results. These new tools seem likely to solve a number of open problems in the classification of subfactors, and will be essential for the proposed project.

**C3.2 Fusion categories** In my paper **Non-cyclotomic fusion categories** [4], to appear *Transactions of the American Mathematical Society*, I showed that the fusion categories coming from the Haagerup subfactor [AH99] and the newly constructed extended Haagerup subfactor [5] are exotic, in the sense that they can not be defined over any cyclotomic field. This answered a conjecture in literature [ENO05] in the negative; these were the first known non-cyclotomic fusion categories. My number theoretic work [2] mentioned above also had applications to the possible dimensions of objects in fusion categories.

**C3.3 Higher category theory and low dimensional topology** I have a second major research project, on higher category theory and generalisations of Hochschild homology to arbitrary manifolds. My paper [1] describing the first results obtained in this project will appear in the *Proceedings of the National Academy of Sciences*. A second paper [MW10] on this work has been submitted to *Geometry & Topology*, another A\* ranked journal. The deeper understanding of general topological quantum field theories obtained through this work will inform and motivate the work described in the project.

## D1.1 Project Title: “Fusion categories and topological quantum field theory.”

### D1.2 Project

**D1.2.1 Aims** I will study small fusion categories, with a view towards applications to topological quantum field theory and topological quantum computing. This will involve developing new mathematical tools, discovering new examples of fusion categories, and classifying all possible examples satisfying certain constraints. Fusion categories are closely related to subfactors, so I will also be studying small index subfactors. My previous intensive work on subfactors provides the foundation for developing analogues of the tools used there which are appropriate for fusion categories

The primary goals of the project are:

- To classify all fusion categories with an object of dimension at most  $\sqrt{5}$ .
- To classify all subfactors with index less than  $3 + \sqrt{5}$ .
- To discover new examples of fusion categories and subfactors, possibly outside of the ranges above.
- To investigate the 3-dimensional topological quantum field theories associated to existing and new examples, and their application to condensed matter physics and topological quantum computing.

Each of these is a significant and important problem in the modern study of operator algebras or quantum algebra. In the course of achieving these goals, I will:

- Develop new tools to construct examples of fusion categories.
- Develop new tools to classify fusion categories and subfactors.
- Generalise existing mathematical techniques specific to the study of subfactors to the general case of multifusion categories.
- Develop, document and distribute computer tools for the study of subfactors and fusion categories.
- Participate in international collaborations, and build connections between Australian mathematics and the international mathematical community.

**D1.2.2 Approach** My approach to these goals will combine several subdisciplines of mathematics: abstract category theory; graph combinatorics; number theory; representation theory; and operator algebras. See §D1.2.3 (Background) for how all of these contribute to answering the problems at hand.

A fusion category is a linear semisimple tensor category with duals and finite many simple objects. The three standard sources of examples are the representation theory of a finite group, the semisimplified representation theory of a quantum universal enveloping algebra at a root of unity, or the bimodules over a von Neumann algebra  $N$  generated by a larger algebra  $M$ . These examples are not exhaustive however, and a major challenge is to understand the structure of an arbitrary fusion category. This challenge incorporates problems from finite group theory, from operator algebras, and from topology. Fusion categories are relevant in topology precisely because they are the algebraic data underlying topological quantum field theories in 3 dimensions. Thus new fusion categories provide new invariants of 3 manifolds, and conversely certain results in topology and homotopy theory have implications for the algebraic structure of fusion categories.

In order to classify fusion categories with at least one object with small dimension, we first enumerate the possible principal graphs. This is a combinatorial problem, and typically certain infinite families of graphs appear. The basic method for such enumerations was introduced in my paper [MS10], in the context of principal graphs for fusion categories. This method can be adapted to the corresponding problem for fusion categories, and seems likely to quickly produce strong restrictions on possible principal graphs. As we proceed to higher dimensions, the combinatorial problem becomes increasingly difficult. I will need to develop new constraints on the principal graphs of fusion categories (generalising, for

example the methods of my paper [MPPS10]), and in particular constraints which can be applied at intermediate steps of the combinatorial enumeration problem. This will require application of ideas from category theory, number theory (e.g. building on [Ost09]), and representation theory. Some of the resulting infinite families can be truncated using my results of [2] and [MPPS10], but more uniform methods will need to be developed.

Once we have effective bounds on the possible principal graphs of fusion categories, for each graph we need to either show that it can not be realised by any fusion category, or to construct and classify all the possible fusion categories with the given graph. One very successful method for subfactors has been to analyse the possible embeddings into the appropriate graph planar algebra. In my paper [5] I showed that there was a unique possible embedding, and then used computer algebra techniques to find an exact solution to the massively overconstrained multivariable polynomial equations which describe the embedding. I plan to adapt this method to the fusion category case. This will require generalising results about the graph planar algebra that are currently only known for subfactors. Further, I propose to construct fusion categories with a specified principal graph via a hybrid method. Previously in the literature smaller exotic examples have been constructed by finding a flat connection on the principal graph [AH99]. However, checking flatness quickly becomes intractable. As an alternative, I propose finding all connections, identifying those which admit at least one nontrivial flat element, and then analysing the planar algebra generated by this flat element.

In order to analyse the topological quantum field theories associated to small fusion categories, I will need to develop algorithmic methods to compute the quantum double, and general methods to analyse the associated annular categories.

**D1.2.3 Background** Fusion categories are a relatively new area of mathematical study. They are intimately related to the theory of subfactors, and at the same time a fusion category is precisely the algebraic datum required to specify a topological quantum field theory in 3-dimensions.

A fusion category is a simple model for the possible interactions of a system of particles. In a fusion category, there are a finite collection of ‘particle types’, also called ‘simple objects’. Given an arrangement of particles on a circle, the fusion category specifies a vector space of possible quantum mechanical states for the system. Further, it specifies a finite collection of possible interactions between particles (e.g. a particle and antiparticle annihilating, two particles fusing into one, possible of some other type, or a single particle fissioning into two), and quantum mechanical amplitudes for any sequence of such interactions. Mathematically, this is axiomatised as a linear semisimple tensor category with duals and finitely many simple objects. The study of fusion categories has its antecedents in physics, especially in the study of conformal field theory, and has been developed further by mathematicians studying topological quantum field theory, especially the Ocneanu-Turaev-Viro theories [TV92]. The landmark paper of Etingof, Nikshych and Ostrik [ENO05] initiated the algebraic study of fusion categories, which has been continued by many authors for example in [Ost03, ENO04, DGNO07, ENO08, Ost09]. Later, the same authors introduced powerful techniques from homotopy theory [PE09].

Fusion categories can be obtained by many different constructions; in particular, the representation theory of any finite group forms a fusion category, as does the (semisimplified) representation theory of a quantum universal enveloping algebra at a root of unity. There also exist fusion categories more exotic than any of these examples [4], and part of the motivation for working on classification problems is to discover the full range of behaviors exhibited by fusion categories. Just as the exceptional Lie algebras were only discovered during the Killing-Cartan classification program, classification work on fusion categories has already uncovered new exotic examples (for example [AH99, Izu01, Pet10] and my work in [4] and [5]). One of the great surprises of recent work in this direction has been just how sparse the new examples are; this seems to make those examples that are found even more interesting. In the proposed project I expect to discover further exotic examples.

The primary mathematical application of fusion categories is in constructing topological quantum field theories (TQFT) in 3-dimensions. A 3-dimensional TQFT associates a vector space to each 2-dimensional manifold, and a numerical invariant to each 3-dimensional manifold, and satisfies certain

locality conditions. These conditions ensure that the invariant to any manifold can be computed by decomposing that manifold into pieces, calculating a certain algebraic invariant of each piece, and then assembling these according to operations determined by the decomposition. A fusion category is the algebraic data associated to the simplest possible piece (a ball), and hence completely characterises the TQFT. Topological quantum field theories in 3-dimensions are especially interesting. They provide part of the mathematical description [MRZ<sup>+</sup>07, DHU<sup>+</sup>08] of the Fractional Quantum Hall Effect (FQHE) [TSG82, Lau83], one of the most important discoveries in condensed matter physics in recent decades. These FQHE materials have been proposed as a substrate for building topological quantum computers [FFN<sup>+</sup>06, DSFN05]. The successful development of a quantum computer will have transformative impact in many industries. In order to understand the possible phases of FQHE materials, we need better understanding of the possible TQFTs in 3-dimensions, and hence better understanding of fusion categories. The fusion categories which correspond to the first few observed FQHE phases are precisely those with objects of very small dimension (less than 2). The lesser understood FQHE phases at higher filling fractions exhibit more complicated behavior [BS08], and to facilitate understanding these phases we would like to understand the correspondingly more complicated fusion categories. Thus it is likely that the proposed classification will be directly relevant to the understanding of more exotic phases of topological matter.

In order to study fusion categories, it is also useful to study subfactors. In particular, the representation theory of a subfactor consists of two fusion categories along with a Morita equivalence between them; studying subfactors allows us to study the relationships between fusion categories.

A subfactor is an inclusion of von Neumann algebras with trivial centres; their study constitutes a noncommutative analogue of Galois theory. The first and most important invariant of a subfactor is its ‘index’, which measures the relative sizes of the algebras. The index of a subfactor is closely analogous to the dimension of an object in a fusion category; hence we want to study subfactors with small index. The ground-breaking theorem of Jones’ [Jon83] says that this index is quantised, and is either of the form  $4 \cos^2(\pi/n)$  or at least 4. The classification of subfactors with index less than 4 was initiated by Ocneanu [Ocn88] and completed by a number of researchers over the following decade [GdlHJ89, Izu91, BN91, Izu94, Pop94, Kaw95, KO02]. Every index above 4 is possible, although in nearly all cases the only possible subfactor is the ‘Temperley-Lieb’ subfactor. Setting these aside, Haagerup [Haa94] made the surprising discovery that the index is again quantised, and gave a complete list of the possible ‘principal graphs’ of subfactors with index between 4 and  $3 + \sqrt{3}$ . Many of these candidate principal graphs were ruled out [Asa07, Bis98, AY09], and subfactors realising two of the principal graphs were constructed [AH99, Izu01], while one mysterious candidate eluded researchers for 15 years.

I recently constructed this subfactor, called the ‘extended Haagerup subfactor’, in a paper published in *Acta Mathematica* [5]. This completed the 15 year old open problem blocking the way to a complete classification of subfactors with index less than  $3 + \sqrt{3}$ .

Since then, I have extended the range of indices for which we have a complete classification. We now know that there are exactly ten subfactors (besides Temperley-Lieb) with index strictly less than 5. This appears in a series of 4 papers; Parts 1 & 2 are my papers [MS10] and [MPPS10], the manuscript for Part 3 is in preparation, and Part 4, [PT10], written by graduate students at UC Berkeley I am mentoring, applies the techniques developed in my work [2]. An immediate next step suggested by this work is to extend the classification to include subfactors with index exactly 5. Reaching an integer index introduces certain new difficulties, particularly because there are a large number of subfactors coming from group theoretic constructions, and proving uniqueness for these is difficult. The techniques established in [2] and [MS10] are not quite sufficient for this, but I anticipate that by applying new number theoretic constraints coming from [Ost09] to subfactors, and generalising my collaborator Izumi’s ideas in [Izu97] (which use the theory of  $III_1$  subfactors and sectors to prove uniqueness theorems for subfactors with a given principal graph), we will be able to achieve this. The proposed project then intends to extend this classification up to index  $3 + \sqrt{5}$ . Reaching this index is particularly interesting and challenging because there are known infinite depth subfactors [BJ97]. The combinatorial methods of [MS10] become weaker at higher indices, simply because of the very rapid growth in the number of potential principal graphs,

but they also face a fundamental obstacle in the presence of infinite depth subfactors. A classification of subfactors up to and including index  $3 + \sqrt{5}$  will require new techniques. The significant existing understanding of connections on subfactors [Izu94, Kaw95, EK98, Ike98, KS01] will have to be developed further, to provide new obstructions for principal graphs of subfactors. My work in [MPPS10] provides a good example of this direction of research.

**D1.2.4 Significance and National Research Priority** The proposed research will produce significant new mathematical results, and also develop new techniques applicable to future problems and relevant for other researchers.

Successful classification results for fusion categories will help constrain the possible behaviors of new topological states of matter [BKS06, BS08]. These constraints will assist condensed matter physicists studying exotic new materials, such as Fractional Quantum Hall materials [MRZ<sup>+</sup>07, DHU<sup>+</sup>08]. Deeper understanding of these materials, and of the underlying mathematics as described in this proposal, may lead to the successful development of quantum computers via Freedman’s proposal for *topological quantum computing* [FFN<sup>+</sup>06, DSFN05]. I have active collaborations with Microsoft Station Q, which is actively developing exotic topological materials as the substrate for a proposed quantum computing architecture [FKLW03, NSS<sup>+</sup>08, Kit97]. The mathematical results proposed in this project will be of great interest to them, and conversely my research is informed and motivated by the problems faced in these applications.

This project falls within National Research Priority “Frontier technologies and transforming Australian industries: Breakthrough science”.

**D1.2.5 Collaboration** I have active collaborations in academia and industry. Funding for this project would strengthen these, and allow the development of new collaborations. I will allocate part of the project budget to bringing visitors to ANU to work on this project and build ties between international and Australian mathematicians. My existing collaborative relationships with international mathematicians which are most relevant to this project are listed below.

- Prof. Frank Calegari (Northwestern, USA) is an expert in number theory with whom I have collaborated applying unexpected number theoretic techniques to the classification of subfactors and fusion categories.
- Dr. Chris Douglas (Oxford, UK) is an expert in category theory and topological quantum field theory.
- Prof. Masaki Izumi (Kyoto, Japan) is an expert on operator algebras, and in particular a leading figure in the construction of exotic examples of subfactors.
- Prof. Vaughan Jones (UC Berkeley, USA) is a very distinguished operator algebraist whose work is central to the study of subfactors. I will continue to work with Jones developing applications of graph planar algebras and quadratic tangles to the classification of subfactors.
- Dr. Emily Peters (MIT, USA) is a postdoctoral researcher studying planar algebras, whose techniques have inspired recent constructions of exotic subfactors.
- Dr. Noah Snyder (Columbia, USA) is a postdoctoral researcher expert on fusion categories, subfactors and number theory.
- Dr. Kevin Walker (Microsoft Station Q, USA) is an expert in topological quantum field theory and category theory, working for Microsoft on topological quantum computing.

**D1.2.6 Communication of results** The results of the proposed research will be communicated widely through oral presentations at international and national conferences and workshops, and in written form in top-quality journals. Results will also be disseminated through collaborations with a network of international colleagues, as described in the previous section, and through supervising PhD students. The project will additionally involve the coding of custom algorithms for the study of fusion categories

and subfactors, and the resulting software package will be documented and released under an open source license.

### D1.3 Institutional Support

I recently accepted a continuing teaching/research position in the Mathematical Sciences Institute at the ANU, and am particularly excited about participating in the world class research environment there. I chose to come to ANU over competitive offers in Australia and the United States. The Mathematical Sciences Institute (MSI) is providing a substantial start up grant and priority for funding for me to invite collaborators from overseas. Further, the MSI will fund the difference between the salary provided by the DECRA scheme and my salary and loading as a Senior Lecturer (\$47702 in each of 2012 and 2013, and \$54914 in 2014). In addition MSI is guaranteeing scholarships for all suitably qualified research students who are permanent residents and wish to work with me.

The Algebra & Topology and Mathematical Physics research groups at MSI are particularly strong, and both Pure Mathematics and Mathematical Physics at MSI received the highest possible ratings in the recent ERA evaluations. These groups represent priority research areas for MSI. I have had extensive discussions with the Director, Alan Carey, and am confident that MSI provides a supportive and productive environment for the research proposed here that is the best possible in Australia.

Collaborations with several members of the department, including Prof. Amnon Neeman, Prof. Peter Bouwknegt, Prof. Alan Carey and Dr. Bai-Ling Wang on the research outlined in this proposal are highly likely. Indeed there is a lot of interest in MSI in the topics that I plan to work on. A Discovery Early Career Researcher Award will enable the MSI to relieve me of teaching and administrative responsibilities, facilitating the greatest possible research impact.

The ANU has superb library facilities both print and on-line and computing infrastructure of the highest quality. MSI will assist me to establish national and international research networks. They have a research collaboration agreement with the Pacific Institute of Mathematical Sciences that I will be able to take advantage of to strengthen my interaction with North American researchers. MSI plans to also establish a stronger relationship with Vanderbilt University where Fields medallists Vaughan Jones and Alain Connes hold positions. The MSI has hosted Jones on numerous occasions from 1987 onwards and I expect to be a key part of this planned new relationship.

### D1.4 References

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## D2 Strategic Statement by the Administering Organisation

**Strategic alignment.** Scott Morrison has a continuing, Level C teaching and research position in the Department of Mathematics, Mathematical Sciences Institute, Australian National University (ANU). Mathematical Sciences is one of the 21 areas of research strength at the Australian National University (ANU). The excellence of its research in this area has been documented both by reviews initiated by the ANU and by international University rankings. In the 2010 ERA Institution Report, Pure Mathematics (FOR code 0101) and Mathematical Physics (FOR code 0105) at ANU both received a score of 5, the highest possible ranking.

In a review conducted in 2004, each area was subject to rigorous international peer assessment; work of every researcher in each area was ranked by the assessors against the quality of the entire world output in that area. In Mathematical Sciences, more than 37% of the output examined by the review was judged to be in the top 5% of worldwide output, with over 83% in the top 25% of world output. In the 2007 Shanghai Jiao Tong rankings of universities by field, ANU was ranked 38th in the world in natural sciences and mathematics (the only Australian university in the top 100 in this category).

Much of the research in Mathematical Sciences at ANU is conducted in the Mathematical Sciences Institute (MSI). Within MSI there are 7 research groups. The relevant ones for Morrison's interests are Algebra & Topology and Mathematical Physics. Morrison's research areas of category theory and quantum topology places him primarily in the Algebra & Topology group, although he will also be associated to the Mathematical Physics group through his interests in quantum field theory and applications of category theory to condensed matter physics. Upon his arrival at ANU, Morrison will become an integral member of these research groups, and has interests overlapping with those of Prof. Amnon Neeman, Prof. Peter Bouwknegt, Prof. Alan Carey, Dr. Bai-Ling Wang, Dr. Anthony Licata, Dr. Joan Licata and Dr. David Ridout. Along with other new hires made this year, Scott Morrison will contribute to significant new strength at MSI in the interaction of category theory and topology.

Morrison's research record over the past 5 years is outstanding, with 9 journal articles published or in press in the period 2006-2010 of which 5 are in A\* journals (according to the ERA journal ranking).

Scott Morrison works in areas that are highly active internationally but are not yet well represented in Australia. Part of the mission of the MSI is to invigorate Australian mathematics by establishing research groups in new and emerging areas of the mathematical sciences. Our aim is to keep Australian research at the forefront of internationally active areas. Scott Morrison will bring significant new strengths to Australian mathematics, while at the same time developing collaborations with a number of current members of the department.

**Strategic importance.** The granting of a Discovery Early Career Researcher Award to Morrison is of great strategic importance to MSI in particular and to Australian mathematics more generally. Morrison is in an exceptionally productive and creative phase of his mathematical career. He is shortly to begin a standard teaching/research position after completing two prestigious research only postdoctoral positions in the US. A Discovery Early Career Researcher Award would enable him to focus on research, and make the greatest possible impact. He would also be able to begin research supervision activities, by incorporating PhD students in the project proposed here. Morrison's ability to attract PhD students will be aided by the ANU

College of Physical Science guaranteeing PhD scholarships to all qualified applicants, and also by ANU's Bachelor of Philosophy (PhB) degree in Science, which is an elite research-oriented undergraduate degree open to the top 1% of students, and has proven to be a rich source of the most outstanding domestic PhD students. His wide network of international collaborators will enable him to recruit top PhD students from international institutions, and the proposed budget for this proposal includes appropriate funding for this.

Morrison has exceptional collaborative links with top mathematics researchers and institutions overseas. The award of a Discovery Early Career Researcher Award to Morrison would greatly strengthen mathematical research within Australia through his capacity to establish networks and collaborations for his students and for other Australian mathematicians. International collaborations are a priority in MSI with a significant budget devoted to sustaining them. An indication of Morrison's capacity in this area include his early postdoctoral career in the USA, at UC Berkeley and in industry at Microsoft Station Q where he retains strong links. In addition his list of collaborators is stellar and includes Fields medallists Michael Freedman and Vaughan Jones, Robert Gompf (Texas), Frank Calegari (Northwestern), Stephen Bigelow (UC Santa Barbara), Kevin Walker (Microsoft) amongst others. More recently, he has forged collaborative links with mathematicians in Japan (where his field is particularly strong), especially with Masaki Izumi in Kyoto, and has accepted several invitations there to visit for collaborations and giving lectures.

**Succession planning.** The Mathematical Sciences Institute is engaging in a generational change with coordinated appointment of younger mathematicians representing a broad cross section of modern mathematical research areas. The Mathematical Sciences Institute is therefore actively involved in succession planning in which younger people are needed to take up leadership roles both at ANU and in a national context. At age 31 Morrison is in an age group not well represented in Australian mathematics in general or in the MSI in particular. At the same time, he has a strong research track record and an international reputation. The MSI is anticipating moving Scott Morrison into a leadership role at an early stage given the large number of retirements anticipated in the next few years. The award of a Discovery Early Career Researcher Award would support his development and preparation for research leadership roles in MSI and nationally.

# F1 Justification of funding requested from the ARC

## Personnel

I am requesting the standard Discovery Early Career Research Award salary contribution of \$85,000. I have recently accepted a continuing Level C position at the Mathematical Sciences Institute at ANU; the MSI will fund the difference between the DECRA salary and my standard salary plus loading. I am a researcher with a strong international reputation, evidenced by regular invitations to speak internationally, and a proven track record of research success, supported by my publication record in top journals (see §C). Support from the DECRA scheme will allow me to concentrate my efforts on the proposed research project, due to reduced teaching and administrative obligations at the MSI.

## Equipment

The execution of this project will require specialised computing equipment and software. In particular, I will need a computer with the capacity to run advanced computer algebra packages, which will require fast processors and more internal memory than a standard desktop computer. Parts of the project will require intensive computation. I am budgetting \$8,000 in the first year to purchase a customized computer, and \$3,000 in each subsequent year for hardware upgrades. I will also need to purchase licenses for certain specialised mathematical software, which I have budgetted at \$2,000 in the first year and \$1,000 in each of the two subsequent years. This budget for specialised computer equipment will help ensure the successful completion of the difficult combinatorial searches required for the project.

## Travel

I am requesting support for travel and accommodation costs associated with my international collaborations. This project involves collaborations with a significant number of international mathematicians, the United States, Japan and the United Kingdom (see §D1.2.5) and I will need to travel internationally to work with these collaborators. I will need to travel to conferences to disseminate the results of the project and maintain my international standing in the field.

I anticipate making 3 international trips each year, and spending a total of 5 to 8 weeks per year overseas. The cost of this is difficult to estimate, especially given the recent extreme volatility in currency exchange rates. However, the cost of accommodation is estimated as \$1,200–\$1,800 per week (using the relevant ATO ruling for reasonable travel expenses, and \$150 per night accommodation), while an international airfare is around \$2,000 per trip (c.f. recent quotes to San Francisco at \$2,430 on Virgin Blue, Osaka at \$1,400 on Air China, Paris at \$1,620 on Qantas). The cost of this travel is therefore between \$15,000 and \$21,000 per year. I have budgetted \$15,000 in the first year, anticipating less travel while I concentrate on establishing deeper local research ties, and \$20,000 in subsequent years. My research field is internationally active, and many of the relevant conferences are in the United States, Japan and France. I am regularly invited to present research at the highest calibre conferences (c.f. §C2.v). In order to be able to continue the collaborations necessary for this project, and to disseminate the research outputs of this project, I will need to continue attending these international conferences.

Reciprocally, my collaborators will also travel to ANU in order to work on this project. I have budgetted \$15,000–\$16,000 each year to support their travel costs (using quotes of \$2,500 for airfares and ATO rates for travel in Canberra). During 2012 I will ask Kevin Walker and Noah Snyder to visit ANU, each for a period of 3 weeks, to work on this project with me. In particular, Dr. Snyder's visit will enable the development of classification of subfactor principal graphs in the range between 5 and  $3 + \sqrt{5}$  (c.f. the main goals listed in §D1.2.1), while travel support for Dr. Walker's visit will allow work on applications of the classification project to topological quantum field theory. During 2013 I will ask Emily Peters, Vaughan Jones and Masaki Izumi to travel to ANU, to work on techniques for classifying subfactors and fusion categories based on connections and quadratic tangles. During 2014 I will plan collaborators' visits based upon the progression of the project work up to that point.