Blob homology

Scott Morrison, Microsoft

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joint work with Kevin Walker.

Outline

- What is blob homology?
- Background
  - Hochschild homology
  - Skein modules from TQFT
- The definition
- Properties

What is blob homology?

- Blob homology is a gadget which takes
  - an n-manifold M
  - "an n-category with duals" $\mathcal{F}$

    (more precisely, a 'system of fields')

  and produces a chain complex $\mathcal{B}_*(M;\mathcal{F})$.

- It is simultaneously a generalisation of
  - Hochschild homology

    When $M=S^1$, $\mathcal{B}_*(S^1;\mathcal{C}) \simeq HC_*(\mathcal{C})$

    (the Hochschild complex)

  - the TQFT skein module

    $$H_0(\mathcal{B}_*(M;\mathcal{F})) = A(M;\mathcal{F})$$

    "pictures from $\mathcal{F}$ on M, modulo local relations"

Background: Hochschild homology

- Given an associative algebra (or 1-category) A, the Hochschild complex $HC_*(A)$ is:

$$HC_k(A) = A^{\otimes(k+1)}$$

$$\begin{aligned}\partial(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = {} & a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_k \\ & - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_k \\ & + (-1)^{k-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} a_k \\ & + (-1)^k a_k a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}\end{aligned}$$

- $HH_0(A)$ = coinvariants of A
  $= A/_{ab=ba}$

- Hochschild homology is "derived coinvariants":
  - HH_* is an exact functor
  - HC_* is a free resolution of coinv(A).

Background: TQFT skein modules

"pictures mod relations"

- First, we need the notions of
    - a system of fields $\mathcal{F}$
    - local relations $\mathcal{U}$.

- Fix a 'top dimension' n. A <u>system of fields</u> $\mathcal{F}$ is a collection of functors

$$\mathcal{F}_{k<n} : \underset{\{\text{diffeomorphisms}\}}{\overset{\circlearrowleft}{\{k\text{-manifolds}\}}} \longrightarrow \text{Set}$$

and $\mathcal{F}_n : \underset{\{\text{diffeos}\}}{\overset{\circlearrowleft}{\{n\text{-manifolds}\}}} \longrightarrow \text{Vect}$

along with
- a restriction map

$$\mathcal{F}_k(M) \xrightarrow{r} \mathcal{F}_{k-1}(\partial M)$$

- identity fields

$$\mathcal{F}_{k-1}(N) \longrightarrow \mathcal{F}_k(N \times I)$$

such that if $M = M_1 \underset{Y}{\cup} M_2$

$$\mathcal{F}_k(M) = \mathcal{F}_k(M_1) \underset{\mathcal{F}_{k-1}(Y)}{\times} \mathcal{F}_k(M_2) \quad \text{(fibered product)}$$

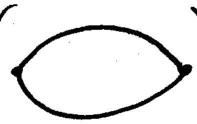
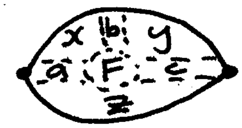
Examples

- Fix a target space X and define

$$\mathcal{F}(M) = \mathrm{Maps}(M \to X)$$

- Fix "an n-category with duals" $\mathcal{C}$, and define

$$\mathcal{F}(M^k) = \left\{ \begin{array}{l} \text{'oriented' handle decompositions of } M, \\ \text{with } j\text{-handles labelled by} \\ (k-j)\text{-morphisms from } \mathcal{C} \end{array} \right\}$$

e.g. $\mathcal{F}\Big($  $\Big) = \Big\{$ 

$, \quad \begin{array}{l} x, y, z \in \mathrm{Obj}(\mathcal{C}) \\ a: z \to x,\ b: x \to y \\ c: z \to y \\ F: c \to a \cdot b \end{array} \Big\}$

$\overset{\shortparallel}{B^2}$

e.g. If $\mathcal{C}$ is the Temperley-Lieb 2-category,

$\mathcal{F}\Big($  $\Big) = k\Big\{$  $, \ldots \Big\}$

Local relations

Given a system of fields $\mathcal{F}$, a family of <u>local relations</u> is a collection of subspaces:

- for each $B \cong B^n$

$$U(B) \subset \mathcal{F}(B)$$

- such that if $B = B_1 \cup B_2$ (all 3 are balls!) and $u \in U(B_1)$, $f \in \mathcal{F}(B_2)$, then

$$u \cdot f \in U(B)$$

"U is an ideal with respect to gluing balls".

Examples

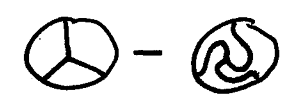
- If $\mathcal{F} = \mathrm{Maps}(\ \to X)$, we can define

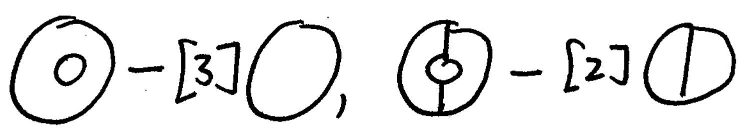
$$U(B) = \left\{ f - g \;\middle|\; \begin{array}{c} f, g : B \to X \\ f \underset{\text{htpy}}{\simeq} g \end{array} \right\}$$

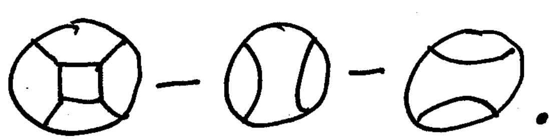
- Kuperberg's A_2 spider:

$\mathcal{F} = k\{\text{oriented trivalent graphs}\}$

$U(B)$ generated by $\{ D_1 - D_2 \mid D_1 \text{ isotopic to } D_2 \}$

e.g. 

and  and

.

TQFT skein modules

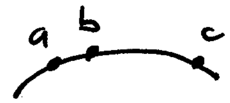
Given a system of fields and local relations $(\mathcal{F}, \mathcal{U})$, for an n-manifold M define the skein module as

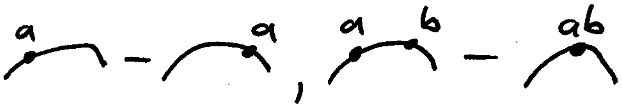
$$A(M;\mathcal{F}) = \mathcal{F}(M) \Big/ \left\{ u \circ f \;\middle|\; \begin{matrix} u \in \mathcal{U}(B) \\ f \in \mathcal{F}(M \setminus B) \end{matrix} \right\}$$

Examples

- With $\mathcal{F} = Maps(\to X)$,

  $$A(M) = [M \longrightarrow X]$$

- With $\mathcal{F}(\frown) = \{$  $\mid a,b,c \in A \}$

  $\mathcal{U}(\frown) = \{$  $\}$

  $$A(S^1) = \mathrm{conv}(A)$$

Properties

- $\mathrm{Diff}(M) \curvearrowright A(M;\mathcal{F})$
- $A(M_1 \sqcup M_2) = A(M_1) \otimes A(M_2)$
- $A(Y \times I)$ is an associative algebra under gluing
- If $Y \subset \partial M$, $A(M)$ is an $A(Y \times I)$ module
- Theorem if $M = M_1 \cup_Y M_2$

  $$A(M) = A(M_1) \otimes_{A(Y \times I)} A(M_2)$$

The definition

Given an n-manifold M and a system of fields $(\mathcal{F}, \mathcal{U})$, define the blob complex $\mathcal{B}_\bullet(M;\mathcal{F})$ as follows:

- $\mathcal{B}_0 = \mathcal{F}(M)$ (arbitrary fields on M, no relations)

- $$\mathcal{B}_1 = \bigoplus_{\substack{B \subset M \\ \text{an embedded ball}}} \mathcal{F}(M \setminus B) \otimes \mathcal{U}(B)$$

  with differential $\partial: \mathcal{B}_1 \to \mathcal{B}_0$
  $$f \otimes u \mapsto f \circ u$$

- $\mathcal{B}_2 = \mathcal{B}_2^{\text{disjoint}} \oplus \mathcal{B}_2^{\text{nested}}$

  $$\mathcal{B}_2^{\text{disjoint}} = \bigoplus_{\substack{B_1, B_2 \subset M \\ B_1 \cap B_2 = \phi}} \mathcal{F}(M \setminus (B_1 \cup B_2)) \otimes \mathcal{U}(B_1) \otimes \mathcal{U}(B_2)$$

  with $\partial: \mathcal{B}_2^{\text{disjoint}} \to \mathcal{B}_1$, $f \otimes u_1 \otimes u_2 \mapsto (f \circ u_1) \otimes u_2 - (f \circ u_2) \otimes u_1$

  $$\mathcal{B}_2^{\text{nested}} = \bigoplus_{B_1 \subset B_2 \subset M} \mathcal{F}(M \setminus B_2) \otimes \mathcal{F}(B_2 \setminus B_1) \otimes \mathcal{U}(B_1)$$

  with $\partial: \mathcal{B}_2^{\text{nested}} \to \mathcal{B}_1$, $f \otimes g \otimes u \mapsto f \otimes (g \circ u) - (f \circ g) \otimes u$.

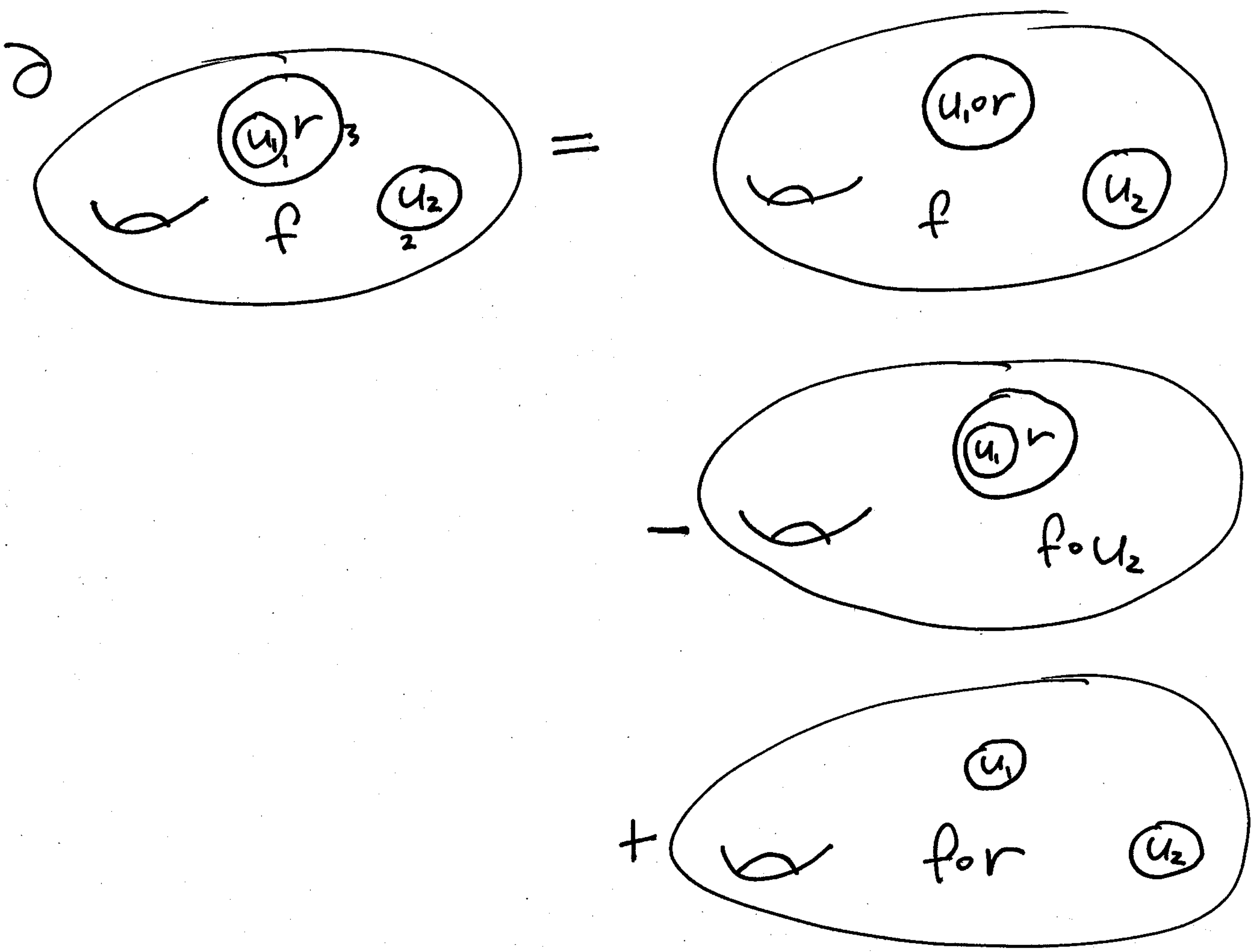
and in general,

$$\mathcal{B}_k = \bigoplus_{\substack{k \text{ properly}\\ \text{embedded balls}\\ \{B_i\}}} \mathcal{F}\left(\begin{array}{c}\text{complement of the}\\ \text{innermost balls}\end{array}\right) \otimes \bigotimes_{\substack{j \text{ an}\\ \text{innermost}\\ \text{ball}}} \mathcal{U}(B_j)$$

and $\partial = \sum_{i=1}^{k} (-1)^{i+1} \cdot$ "forget the i-th ball"

(Two corrections: $\mathcal{B}_2^{\text{nested}}$ allows $B_2 \subset B_1$ as well as $B_1 \subset B_2$, and we identify permutations of balls, with signs.)

Example



Properties of blob homology.

1. Functoriality for diffeomorphisms

   Fixing $\mathcal{F}$, $B_*(-;\mathcal{F})$ is a functor

$$\begin{matrix} \{n\text{-manifolds}\} & & \{\text{chain complexes}\} \\ \circlearrowleft & \longrightarrow & \circlearrowleft \\ \{\text{diffeomorphisms}\} & & \{\text{chain maps}\} \end{matrix}$$

2. Contractibility.

   The complex $B_*(B^n;\mathcal{F})$ retracts onto the complex with $A(B^n) = \mathcal{F}(B^n)/U(B^n)$ concentrated in degree 0. (In this sense $B_*(B^n)$ is a free resolution of $A(B^n)$.)

3. Disjoint union

$$B_*(M_1 \sqcup M_2) \cong B_*(M_1) \otimes B_*(M_2)$$

4. Gluing map

   Given Y a submanifold of ∂M_1, with $-Y$ a submanifold of ∂M_2,

   there's a map

$$gl_Y : B_*(M_1) \otimes B_*(M_2) \longrightarrow B_*(M_1 \cup_Y M_2)$$

Some theorems about blob homology.

Theorem If $\mathcal{F}$ comes from a 1-category $\mathcal{C}$

$$\left(\text{i.e. } \mathcal{F}(\text{—}) = \left\{ {}_x\!\!-\!\!\overset{a}{\bullet}\!\!-_y\!\!\overset{b}{\bullet}\!\!-_z\!\!-\!\!\overset{c}{\bullet}\!\!-_w \;\middle|\; \begin{array}{l} x,y,z,w \in Obj(\mathcal{C}) \\ a: x \to y,\ b: y \to z,\ c: z \to w \end{array} \right\}\right.$$

$$\left. \mathcal{U}(\text{—}) = \left\{ \overset{a}{\bullet}\!\!\text{———} - \text{———}\!\!\overset{a}{\bullet} \right\} \cup \left\{ \overset{a}{\bullet}\overset{b}{\bullet} - \overset{ab}{\bullet} \right\} \right)$$

then $\mathcal{B}_*(S^1; \mathcal{F}) \simeq HC_*(\mathcal{C})$.

(the Hochschild complex for $\mathcal{C}$)

(Proof: extend blob homology to allow marked points labelled by $\mathcal{C}$-bimodules, show this has the same universal properties as the Hochschild complex.)

Theorem $H_0(\mathcal{B}_*(M; \mathcal{F})) = A(M; \mathcal{F})$

(immediate from the definition:
$\text{image}(\partial_1 : \mathcal{B}_1 \to \mathcal{B}_0) = \{ f \circ u \mid u \in \mathcal{U}(B \subset M) \}$)

$$\begin{tikzcd}
 & \text{Hochschild homology} & \\
 & \mathcal{B}_*(M;\mathcal{F}) \arrow[u, squiggly, "M=S^1"'] \arrow[r, squiggly, "*=0"] \arrow[dl, dotted, "\text{e.g. } \mathcal{F}=k[x]"'] & \text{TQFT Skein module } A(M;\mathcal{F}) \\
C_*^{sing}(\Sigma(M)) & &
\end{tikzcd}$$

Theorem

There is an "evaluation map"

$$ev : C_*(\mathrm{Diff}(M)) \otimes B_*(M) \longrightarrow B_*(M)$$

so that

- $ev : C_0(\mathrm{Diff}(M)) \otimes B_*(M) \to B_*(M)$ is just the action of diffeomorphisms
- the following diagram commutes:

$$\begin{tikzcd}
& C_*(\mathrm{Diff}(M_1)) \otimes C_*(\mathrm{Diff}(M_2)) \otimes B_*(M_1) \otimes B_*(M_2) \arrow[dl, "gl_Y^{\mathrm{Diff}} \otimes gl_Y"'] \arrow[dr, "ev_{M_1} \otimes ev_{M_2}"] & \\
C_*(\mathrm{Diff}(M)) \otimes B_*(M) \arrow[dr] & & B_*(M_1) \otimes B_*(M_2) \arrow[dl] \\
& B_*(M) &
\end{tikzcd}$$

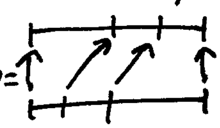
(where $M = M_1 \cup_Y M_2$).

Moreover, up to homotopy, this map is unique.

A_∞-modules, a gluing formula

If Y is an $(n-1)$-manifold, $\mathcal{B}_*(Y \times I)$ is naturally an A_∞-category:

- $m_2 : \mathcal{B}_*(Y \times I) \otimes \mathcal{B}_*(Y \times I) \to \mathcal{B}_*(Y \times I)$
  is just gluing.
- the higher multiplications are defined using
  $$\mathrm{Diff}(I) \subset \mathrm{Diff}(Y \times I)$$
  and the evaluation map.

  (e.g., for m_3, take $\varphi =$  $\in C_1(\mathrm{Diff}(I))$.

  $$m_3(a,b,c) = ev(\varphi, m_2(m_2(a,b),c))$$)

Further, if $Y \subset \partial M$, $\mathcal{B}_*(M)$ is naturally an A_∞-module over $\mathcal{B}_*(Y \times I)$ (using collars).

<u>Theorem</u> If $Y \subset \partial M_1$, $-Y \subset \partial M_2$, then

$$\mathcal{B}_*(M_1 \cup_Y M_2) \cong \mathcal{B}_*(M_1) \overset{A_\infty}{\underset{\mathcal{B}_*(Y \times I)}{\otimes}} \mathcal{B}_*(M_2).$$