Topological field theory & the blob complex

joint w/ Kevin Walker.

- "The blob complex" arXiv:1009.5025
- "Higher categories, colimits and the blob complex"
  http://tqft.net/blobs-pnas
- Course notes online http://web.me.com/teichner/Math/Blob-Complex

2:30

Today I'm going to give you an (idiosyncratic!) overview of topological field theory,

and describe a generalization of the whole framework which we call "the blob complex".

I'll explain the construction and its relationship to other work, and at the end outline some current and anticipated applications.

We have two papers out ~~on~~ this, and there's a course being taught on the blob complex, by ~~Peter~~ Teichner, this semester, which has some notes online.

What is an n-dimensional TFT?

1. ~~a vector space $A(X)$ for each n-manifold X~~
2. a k-category $A(Y)$ for each $(n-k)$-manifold Y
   (a "fully extended" TFT)
3. ~~modules for manifolds with boundaries~~
   e.g. for X an n-manifold

   $$c \in \mathrm{Obj}(A(\partial X)) \rightsquigarrow A(X;c) \text{ a vector space}$$

   $$f \in \mathrm{Hom}(c \to d) \rightsquigarrow A(f) : A(X;c) \longrightarrow A(X;d)$$

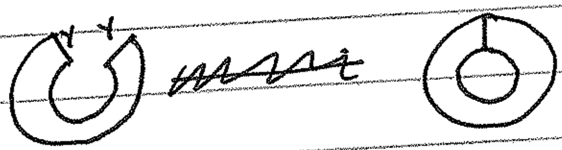
4. $A(X \sqcup Y) \cong A(X) \otimes A(Y)$

Note that this is all relative to a choice of definition of "n-categories".

This won't be a complete characterization of a TFT, just the essential properties.

5) Given X with $Y \sqcup Y \subset \partial X$, a map

$$A(X) \longrightarrow A(X \cup_Y \circlearrowleft)$$



descending to an isomorphism

$$A(X) \otimes_{A(Y)} \circlearrowleft \xrightarrow{\cong} A(X \cup_Y \circlearrowleft).$$

9:22

$A(X)$ is an $A(Y)$ module in two different ways; this self-tensor product identifies the two actions.

Cobordism hypothesis (CH)

"Every extended TFT is determined by its value on a point."

This is a criteria for a good definition of n-categories.

- Recently proved by Lurie for "(∞,n)-categories":
  a TFT for each "fully dualizable" (∞,n)-category.
- Our definition of a "disklike n-category" makes the CH almost trivial

Next (2) Constructing a TFT from a disklike n-category
(3) The "blob complex", whose 0th homology is the TFT invariant.

14:46

We take this to an extreme conclusion;
our definition ... makes the CH almost trivial.
Another way to say this is that our work is complementary to Lurie's: If you have an "algebraically presented" n-category, you have to have his ~~th~~ results before you can see that it ~~gives~~ actually gives an instance of our axioms

~~Start over~~

Disklike n-categories

- for each $(n-1)$-sphere S, a set $\mathcal{C}(S)$ of "boundary conditions";

- for any ~~common~~ submanifold of ~~two~~ spheres
  $$Y \hookrightarrow S$$
  the equivalence relation of ~~"boundary conditions~~
  "boundary conditions agreeing on Y";

- for each n-ball X and boundary condition $c \in \mathcal{C}(\partial X)$,
  a vector space $\mathcal{C}(X;c)$ of "fields on X";

The equivalence relations should further satisfy some sheaf-like axioms:

- if you agree on Y, you agree on submanifolds of Y
- if you agree on each set of an open cover, you agree everywhere

- For each pair of balls X and X' with ~~common submanifold~~
$$X \hookleftarrow Y \hookrightarrow X'$$
such that $X \cup_Y X'$ is a ball, and fields $c \in \mathcal{C}(\partial X)$, $c' \in \mathcal{C}(\partial X')$ which agree on Y, a map
$$\mathcal{C}(X;c) \otimes \mathcal{C}(X';c') \longrightarrow \mathcal{C}(X \cup_Y X'; c'').$$

Such that:
- homeomorphisms act naturally
- successive gluing of balls is strictly associative

8!

The field c'' here is the unique field that agrees with c on $\partial X \setminus Y$ and agrees with c' on $\partial X' \setminus Y$.

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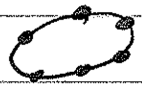
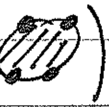
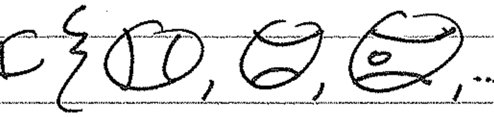
Examples

n=1 $\mathcal{C}(\bullet \quad \bullet) = \{$pairs (a,b) of objects in your favorite $*$-category$\}$

$\mathcal{C}(\overset{a}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\bullet}) = \hom(a,b)$

n=2 $\mathcal{C}(\bigcirc) = \{$  $\}$

$\mathcal{C}($  $) = \mathbb{C}\{$  $, \dots\} /$ isotopy & $\bigcirc = 2$

This is the representation category of $SU(2)$.

3 minutes

n=3 $\mathcal{C}(S^2) = \{$embedded circles and an alternating shading$\}$

$\mathcal{C}(B^3; c) = \mathbb{C}\{$contact structures with dividing curves $c\} /$ overtwisted discs.

Why do we need a $*$-category here?
Recall homeomorphisms should act, including orientation revers[e]ing ones, so we need a map $\hom(a,b) \to \hom(b,a)$.
There are other versions of disklike n-categories where the disks have more structure (e.g. orientations, framings, etc), and these correspond to other familiar types of categories.

$n=4$ $\mathcal{C}(S^3) = \{\text{embedded links}\}$

$\mathcal{C}(B^4; L) = Kh(L)$ (the Khovanov homology of L)

<u>arbitrary n</u>

$\mathcal{C}(S^{n-1}) = \{\text{maps } S^{n-1} \text{ to } T\}$

$\mathcal{C}(B^n; c) = \mathbb{C}\left\{\begin{array}{c}\text{homotopy classes } B^n \to T \\ \text{extending } c\end{array}\right\}$

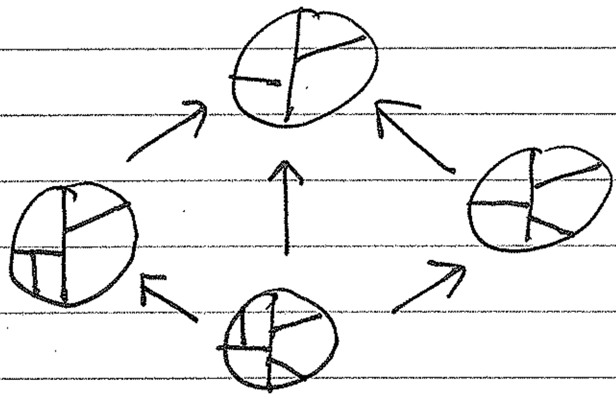
This is the "fundamental n-groupoid" of T.

7:30

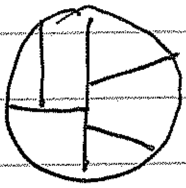
From each of these

To construct a TFT from an n-category,
we need a vector space for every n-manifold,
not just n-balls.

The poset $\mathcal{D}(M)$ of ball decompositions of M has
- elements
  a collection of balls $\{B_i\}$ so $M = \bigcup B_i$
  and $B_i \cap B_j$ is a (possibly ~~non~~empty) $(n-1)$-manifold
- arrows for each way to glue up a subset of the balls to form a larger ball.



A disklike n-category $\mathcal{C}$ provides a functor $\mathcal{D}(M) \to \mathrm{Vect}$



$$\longrightarrow \bigoplus_{\substack{\text{compatible}\\ \text{boundary}\\ \text{conditions } c}} \bigotimes_{\substack{\text{balls}\\ B}} \mathcal{C}(B; c|_B)$$

Gluing maps are sent to gluing maps.

<u>Definition</u> The TFT invariant $A(M;\mathcal{C})$ is the colimit of this functor

$$A(M;\mathcal{C}) = \bigoplus_{\substack{\text{ball } \cancel{\text{decomposi}}\\ \text{decompositions}\\ \beta}} \bigoplus_{c \text{ on } \beta} \bigotimes_{\substack{\text{balls}\\ B\in\beta}} \mathcal{C}(B;c) \Big/ \text{gluing maps}$$

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We define a disklike k-category $A(X^{n-k};\mathcal{C})$ by

$$A(X^{n-k};\mathcal{C})(B^k) = A(X \times B;\mathcal{C})$$

<u>Theorem</u> This gives an extended TFT.

<u>Theorem</u> If Q is an extended TFT,

$$Q(M) = A(M; Q(\bullet)).$$

Now, define $\mathcal{B}_*(M;\mathcal{C})$, the blob complex of M with coefficients in $\mathcal{C}$, to be the homotopy colimit $\mathcal{D}(M) \to \mathrm{Vect}$.

It's immediate that $H_0(\mathcal{B}_*(M;\mathcal{C})) = A(M;\mathcal{C})$.

21 ~~20~~

total 43.

start over.

An explicit model

For M an n-manifold, define "fields on M",

$$\mathcal{F}(M) = \bigoplus_{B \in \mathcal{Q}(M)} \mathcal{C}(B)$$

For M an n-ball, define "null fields"

$$\mathcal{U}(M) = \ker\left(\mathcal{F}(M) \xrightarrow{\text{glue}} \mathcal{C}(M)\right)$$

Now $\mathcal{B}_0(M) = \mathcal{F}(M)$, $\mathcal{B}_1(M) = \bigoplus_{\substack{B \\ \text{a ball} \\ \text{embedded in } M}} \mathcal{U}(B) \otimes \mathcal{F}(M \setminus B)$

and $d : \mathcal{B}_1(M) \to \mathcal{B}_0(M)$

$$u \otimes f \longmapsto uf$$

<u>Lemma</u> $\mathcal{B}_0(M)/d(\mathcal{B}_1(M)) = A(M)$.

We now define $\mathcal{B}_k$ in the essentially unique way so that if M is a ball there is no higher homology.

$$\mathcal{B}_2(M) = \mathcal{B}_2^{\text{nested}}(M) \oplus \mathcal{B}_2^{\text{disjoint}}(M)$$

$$\mathcal{B}_2^{\text{nested}} = \bigoplus_{b_1 \subset b_2} \mathcal{U}(b_1) \otimes \mathcal{F}(b_2 \setminus b_1) \otimes \mathcal{F}(M \setminus b_2)$$

$$d(b_1 \subset b_2, u \otimes r \otimes r') = (b_2, ur \otimes r') - (b_1, u \otimes rr')$$

$$\mathcal{B}_2^{\text{disjoint}} = \bigoplus_{b_1 \sqcup b_2} \mathcal{U}(b_1) \otimes \mathcal{U}(b_2) \otimes \mathcal{F}(M \setminus (b_1 \cup b_2))$$

$$d(b_1 \sqcup b_2, u \otimes u' \otimes r) = (b_2, u' \otimes ur) - (b_1, u \otimes u'r)$$

And generally

$$B^k(M) = \bigoplus_{\substack{k \text{ balls,} \\ \text{pairwise} \\ \text{nested or} \\ \text{disjoint}}} \bigotimes_{\text{regions}} \begin{pmatrix} \mathcal{U}(\text{inner-most balls}) \\ \mathcal{F}(\text{other regions}) \end{pmatrix}$$

and the differential is the sum over ways to forget one ball, gluing fields as appropriate

Theorem $\mathcal{B}_*(S^1;\mathcal{C}) = \mathrm{Hoch}_*(\mathcal{C})$

Theorem There is a chain map
$$C_*\mathrm{Homeo}(M) \otimes \mathcal{B}(M) \to \mathcal{B}(M),$$
and taking H_0 gives the mapping class group action on $A(M)$.

Theorem $\mathcal{B}_*(M; \pi^{\infty}_{\leq n}(T)) = C_*(\mathrm{Maps}(M \to T))$

Theorem The blob complex gives an "extended A_∞ TFT" satisfying a gluing rule with an A_∞ tensor product.

Applications

- A generalization of Deligne's conjecture ("the little discs operad acts on Hochschild cohomology") to higher dimensions, with an easy proof.

- Many interesting higher categories are triangulated (contact topology, Khovanov homology, Heegaard-Floer theory). TFT invariants are not exact, so computations are difficult. The blob complex is "exact w.r.t. boundary conditions."

- In some cases, it is easier to compute all of $\mathcal{B}_*(M;\mathcal{C})$ than just $H_0 = A(M;\mathcal{C})$.

19 minutes...

total 62...
probably okay, given the real thing is always faster.