Small index subfactors

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What is a subfactor?

- A subfactor is an inclusion of von Neumann algebras $N \subset M$ each with trivial centre.
- We're interested in II_1 factors (no minimal projections, the identity is finite).
- The index of $N \subset M$ is the von Neumann dimension of M as an N module.

We will focus on the $N - N$, $N - M$, $M - M$ and $M - N$ bimodules, generated by ${}_N M_M$ and ${}_M M_N$.

When there are finitely many simple bimodules, we say $N \subset M$ is finite depth.

Principal graphs

Definition

The principal graph for a subfactor $N \subset M$ has vertices for the $N-N$, $N-M$, $M-N$ and $M-M$ bimodules, and an edge between Y and Z for each copy of Z appearing inside $Y \otimes X$. (Here $X = {}_N M_M$ or ${}_M M_N$ as appropriate.)

The principal graph has two connected components, the left N-modules and the left M-modules.

The graph norm (=largest eigenvalue of the adjacency matrix) is equal to the square root of the index of the subfactor (as long as the subfactor is finite depth, or amenable).

What's out there?

The index of a finite depth subfactor is highly constrained. Jones proved the first result in this direction.

Theorem (Jones, Index for subfactors, '83)

If $1 < [N : M] < 4$, *then* $[N : M] = 4\cos(\pi/n)^2$.

Highly ahistorically, I'll emphasize applications of arithmetic to the theory of subfactors.

Proof.

The square root of the index is the largest eigenvalue of the principal graph. The only real algebraic integers less than 2 which are maximal amongst their conjugates are the numbers $2\cos(\pi/n)$. □

Theorem (Coste-Gannon, '94)

The dimension of an object in a fusion category is a cyclotomic integer.

Proof.

Entries of the S-matrix of the Drinfeld center are cyclotomic. □

Corollary

The index of a finite depth subfactor is a cyclotomic integer.

Proof.

The collection of $N - N$ bimodules is a fusion category, and the dimension of M there is just the index $[N : M]$. □

Using these results, we can get purely arithmetic constraints on the index of a finite depth subfactor.

Example (Calegari-Morrison-Snyder, CMP '10)

If $N \subset M$ is finite depth, and $[N : M] \in (4, 4 + 10/33)$, then

$$[N : M] = 3 + 2\cos(2\pi/7) \text{ or } (5 + \sqrt{13})/2.$$

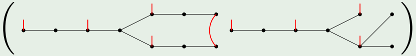
Proof.

We know $[N : M]$ is cyclotomic, and $\sqrt{[N : M]}$ is a real algebraic integer which is maximal amongst its Galois conjugates. Thus all Galois conjugates of $[N : M] - 2$ have norm less than $76/33$, and we have a complete classification of all such cyclotomic integers. □

Enumerating principal graphs

We also remember which bimodules are dual to each other.

Example (The Haagerup subfactor's principal graph)



The principal graph must satisfy an associativity test:

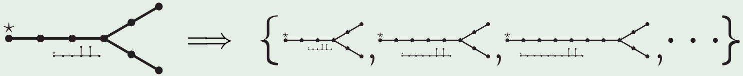
$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such pairs of graphs with index below some number L up to any rank or depth, obtaining a collection of allowed vines and weeds.

Definition

A vine represents an integer family of principal graphs, obtained by translating the vine.

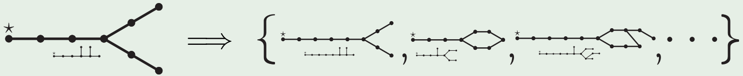
Example

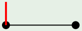
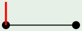


Definition

A weed represents an infinite family, obtained by either translating or extending arbitrarily on the right.

Example



The weed ( , ) trivially represents all possible principal graphs (of irreducible subfactors).

We can always convert a weed into a vine, at the expense of finding all possible depth 1 extensions of the weed (which stay below the index limit, and satisfying the associativity condition) and adding these as new weeds.

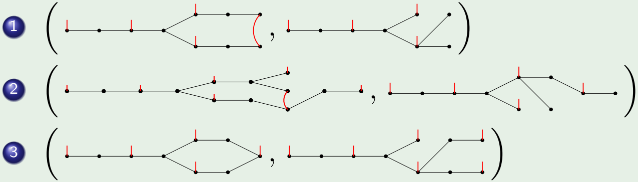
The is a finite problem, since high valence implies large graph norm, and graph norm increases under inclusions.

If the weeds run out, we go home happy. Realistically, we stop with some surviving weeds, and have to rule these out 'by hand'.

Classification results for index less than $3 + \sqrt{3}$

Theorem (Haagerup, '93)

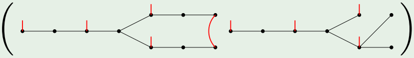
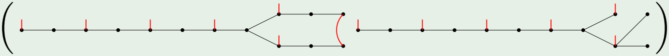
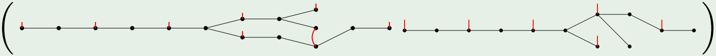
All subfactors other than A_∞ with index in the interval $(4, 3 + \sqrt{3})$ are represented by the following vines:



This was a favourable case, where the enumeration ran out of weeds.

Theorem

There are exactly three subfactors in this range, with principal graphs

- 
- 
- 

Proof.

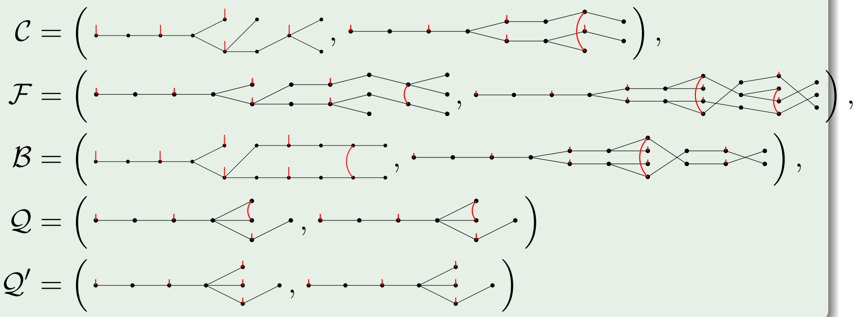
- Asaeda-Haagerup '98 constructed the first two examples.
- An unpublished result of Haagerup's, and results of Bisch '98, Asaeda-Yasuda '07 showed that there are no others except possibly the third example.
- Bigelow-Morrison-Peters-Snyder '09 constructed the 'extended Haagerup' subfactor.

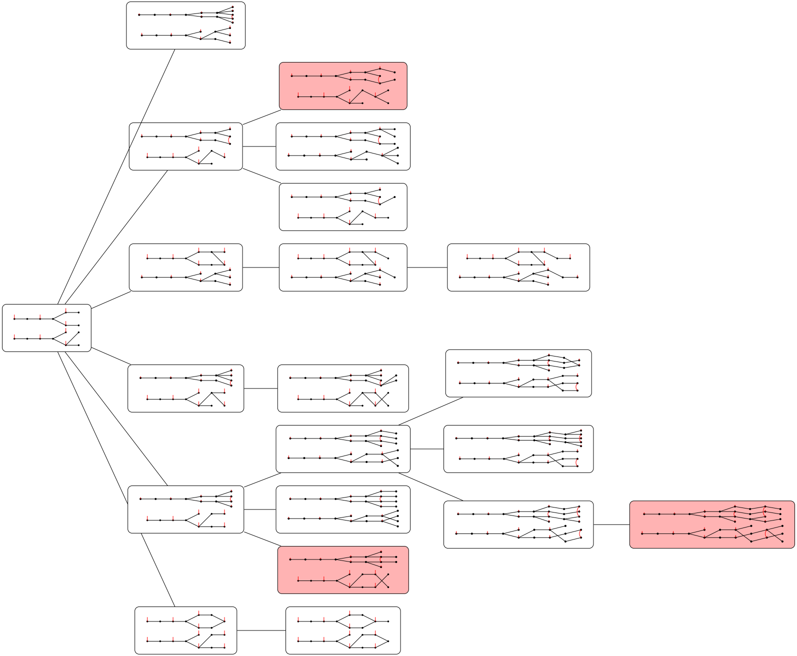
□

Classification results for index less than 5

Theorem (Morrison-Snyder `arXiv:1007.1730`)

All subfactors other than A_∞ with index between 4 and 5 are represented by 43 families of vines, or the following 5 weeds.





Eliminating vines

Theorem (Calegari-Morrison-Snyder, CMP '10)

In any vine, only finitely many graphs have a cyclotomic index. With much better bounds, all but finitely many graphs have a multiplicity free eigenvalue which is not cyclotomic.

Either condition is sufficient to eliminate a possible subfactor.

- Penneys-Tenner `arXiv:1010.3797` developed algorithms for efficiently computing these bounds,
- and computed them for the 43 vines in our enumeration.
- They looked at the finitely many cases remaining from the vines, and found obstructions for all but one graph.

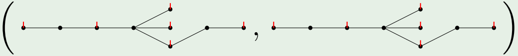
All of the weeds have been killed off:

- $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{B}$ by M-Penneys-Peters-Snyder `arXiv:1007.2240`,
- $\mathcal{Q}'$ and $\mathcal{Q}$ by Izumi-Jones-M-Snyder (in preparation).

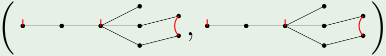
Theorem

The only subfactors with index in the interval $[3+\sqrt{3}, 5)$ *are*

A_∞ *at every index,*

'3311' 

with index $3+\sqrt{3}$ *(Goodman-de la Harpe-Jones, '89)*

'2221' 

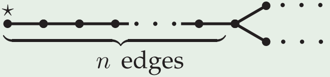
with index $\frac{5+\sqrt{21}}{2}$ *(Izumi, '01).*

These results complete the classification of subfactors with index less than 5.

Can we construct a subfactor with a given principal graph?

The principal graph determines the dimensions of objects and the dimensions of invariant spaces.

Example

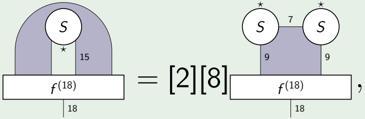
If $\Gamma =$ 

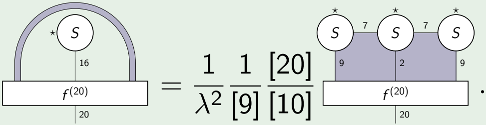
, then

$$\dim \mathrm{Inv}(V^{\otimes 2k}) = \begin{cases} C_k & \text{for } k \leq n \\ C_k + 1 & \text{for } k = n+1. \end{cases}$$

These data strongly constrain generators and relations for the representation theory.

If a subfactor with principal graph  exists, its representation theory is generated by an 8-box S, which is a lowest weight vector with eigenvalue -1 and relations

1. $S^2 = \lambda^2 f^{(8)}$ (here $\lambda^2 = -\frac{1}{5} + 2\,\mathrm{Re}\sqrt[3]{\frac{117-65i\sqrt{3}}{2250}}$),
2. 

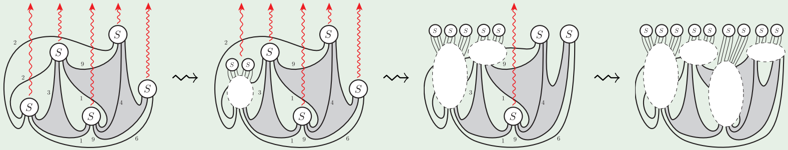
3. 

Theorem (Bigelow-Morrison-Peters-Snyder, Acta Math. '09)

There is at most one subfactor with these generators and relations.

Proof.

These relations suffice to evaluate any closed diagram via the "jellyfish algorithm".



□

Two things could go wrong:

- The relations are inconsistent (i.e. this is a presentation of the 'zero planar algebra').
- The resulting representation theory is not unitary.

How do we check that relations are consistent? Every finite group sits inside some S_n. Analogously, we have

Theorem (Jones-Penneys `arXiv:1007.317`, Morrison-Walker)

Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

Theorem (BMPS '09)

There is an element of the GPA satisfying the desired relations. Unitarity is inherited from the GPA, so the subfactor

exists!

Just as the exceptional Lie groups were discovered via the Killing-Cartan program of classification, the classification of small index subfactors is producing examples of exotic subfactors.

The new exotic subfactors and fusion categories we are finding

- constrain possible structural theorems,
- give counterexamples to conjectures, and
- give interesting new examples of modular data and thus exotic 3-manifold invariants.

Exotic fusion categories

A classical theorem of Brauer shows that the representation theory of any finite group can be defined over a cyclotomic field. (The same holds for quantum groups at roots of unity.) Etingof, Nikshych and Ostrik asked if this is true of every fusion category.

Theorem (Morrison-Snyder, Transactions of the AMS '10)

The even parts of the Haagerup and extend Haagerup subfactors cannot be defined over any cyclotomic field.

Proof.

Using the skein theory, we produce a canonical element of the ground field which is not cyclotomic. □

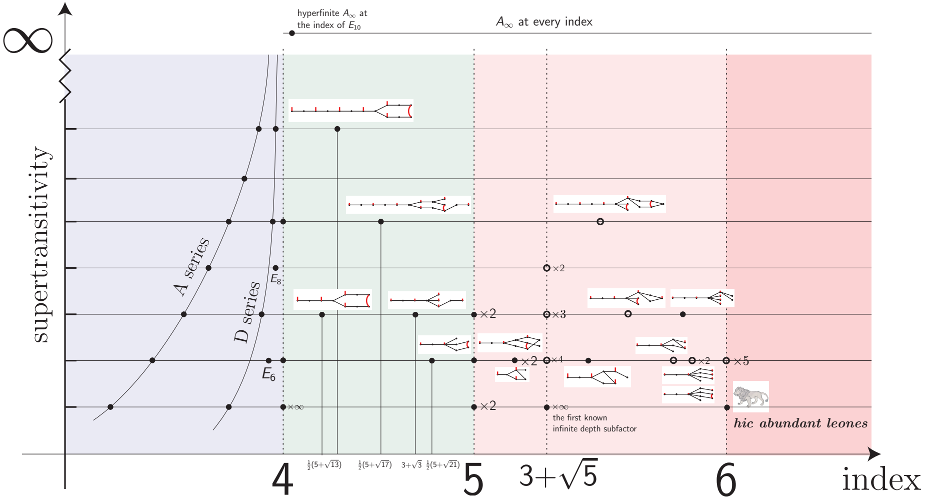
Summary

Recent progress

- Constructed the missing case, extended Haagerup, from earlier classifications.
- Developed uniform new techniques from number theory to reduce classifications to finite problems.
- The classification of subfactors with index less than 5.

There's still plenty more to do:

- The classification beyond 5 looks difficult; perhaps we'll need new techniques.
- Computer searches show that the classification remains sparse, but also find new candidate examples.
- It will be interesting to try to construct these!



hyperfinite A_∞ at the index of E_{10}
A_∞ at every index
supertransitivity
A series
D series
E_8
E_6
×2
×3
×4
×5
×∞
the first known infinite depth subfactor
hic abundant leones
$\frac{1}{2}(5+\sqrt{13})$
$\frac{1}{2}(5+\sqrt{17})$
$3+\sqrt{3}$
$\frac{1}{2}(5+\sqrt{21})$
4
5
$3+\sqrt{5}$
6
index