Coincidences of tensor categories

Scott Morrison
http://tqft.net/
joint work with Emily Peters and Noah Snyder

UC Berkeley / Miller Institute for Basic Research

Algebraic structures in knot theory
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slides: http://tqft.net/UCR-identities
article: http://tqft.net/identities

Outline

Quantum knot invariants

Reshetikhin-Turaev define a polynomial knot invariant for every

- quantum group $U_q(\mathfrak{g})$, with $\mathfrak{g}$ a complex simple Lie algebra,
- and irreducible representation V of $U_q(\mathfrak{g})$:

$$\mathcal{J}_{U_q(\mathfrak{g}),V}(K)(q).$$

Example

$\mathcal{J}_{U_q(\mathfrak{sl}_4),\Lambda^2\mathbb{C}^4}\Bigg($  $\Bigg)(q) = q^{16}+q^{12}+q^{10}+q^{-10}+q^{-12}+q^{-16}.$

These invariants generalise the *Jones polynomial* ($SU(2)$, $\mathbb{C}^2$), the *coloured Jones polynomials* ($\mathrm{Sym}^n\,\mathbb{C}^2$), *HOMFLYPT* ($SU(n)$, $\mathbb{C}^n$) and the 2-variable *Kauffman polynomial* ($SO(n)$ or $Sp(2n)$, $V^\natural$).

We can compute these invariants!

A computer can calculate the universal $\mathcal{R}$-matrix acting on any irreducible representation. A braid presentation of the knot tells us a sequence of matrices with entries in $\mathbb{Z}[q, q^{-1}]$ to multiply, and then take trace.

Really!

See my `QuantumGroups`` package, available as part of the `KnotTheory`` package from `http://katlas.org/`.

Example

```
<<KnotTheory`
QuantumKnotInvariant[A3][Irrep[A3][0,1,0]][Knot[4,1]]
```

$$== \; q^{16} + q^{12} + q^{10} + q^{-10} + q^{-12} + q^{-16}$$

Some mysterious identities

Let's search for identities between these polynomials, specialising q to roots of unity.

We find lots of examples!

$$\mathcal{J}_{SU(2),(2)}(K)(\exp(\tfrac{2\pi i}{12})) = 2$$
$$\mathcal{J}_{SU(2),(4)}(K)(\exp(\tfrac{2\pi i}{20})) = 2\mathcal{J}_{SU(2),(1)}(K)(\exp(\tfrac{-2\pi i}{10}))$$
$$\mathcal{J}_{SU(2),(6)}(K)(\exp(\tfrac{2\pi i}{28})) = 2\mathcal{J}_{SU(4),(1,0,0)}(K)(\exp(\tfrac{-2\pi i}{14}))$$
$$\mathcal{J}_{SU(2),(8)}(K)(\exp(\tfrac{2\pi i}{36})) = 2\mathcal{J}_{SO(8),(1,0,0,0)}(K)(-\exp(\tfrac{-2\pi i}{18}))$$
$$\mathcal{J}_{SU(2),(12)}(K)(\exp(\tfrac{2\pi i}{52})) = 2\mathcal{J}_{G_2,V_{(1,0)}}(K)(\exp(\tfrac{2\pi i\cdot 23}{26}))$$

Question

What's going on? Is there some algebraic structure underlying these strange identities between knot polynomials?

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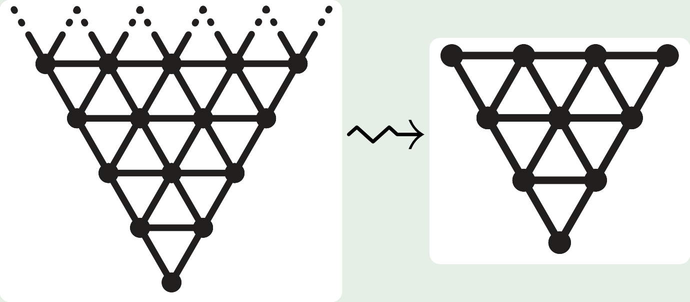
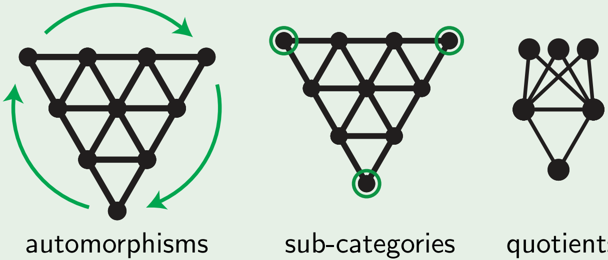
Algebraic structure: modular tensor categories

At a root of unity, the representation theory of a quantum group truncates to a **modular ⊗-category** with finitely many objects.

Example ($SU(3)$ at 'level 3', $q = \exp(\frac{2\pi i}{12})$)



Braided tensor categories are like finite groups



- Not all automorphisms come from 'group-like' sub-categories.
- Not all quotients are 'modular', or even $\otimes$.

These algebraic operations explain identities between the corresponding knot invariants.

We'd like to prove

$$\mathcal{J}_{SU(2),(6)}(K)(\exp(\tfrac{2\pi i}{28})) = 2\mathcal{J}_{SU(4),(1,0,0)}(K)(\exp(\tfrac{-2\pi i}{14})).$$

On right hand side, we look at the modular tensor category $SU(2)$ at $q = \exp(\frac{2\pi i}{28})$. This has 12 objects, so we call it $SU(2)_{11}$ ('$SU(2)$ at level 11').

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$$\mathbb{Z}/2\mathbb{Z} \subset SO(3)_6 \subset SU(2)_{11}$$

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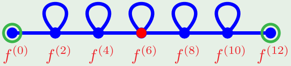
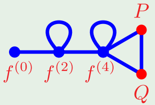
Here's the subcategory $SO(3)_6$.

$SO(3)_6 =$ 

Take the quotient $SO(3)_6/2$; it's a new modular tensor category.

$SO(3)_6/2 =$ 

Quotients of braided $\otimes$-categories are usually called 'de-equivariantisations'.

To match conventions between SU and SO, replace q with q^2. The object (6) splits into two pieces, P and Q, with the same knot invariants.

Corollary

$$\mathcal{J}_{SU(2)_{11},(6)}(K)(\exp(\tfrac{2\pi i}{28})) = 2\mathcal{J}_{SO(3)_6/2,P}(K)(\exp(\tfrac{2\pi i}{14})).$$

Level-rank duality: “$SO(n)_m \cong SO(m)_n$”

Level-rank duality is tricky! The correct statement is

Theorem

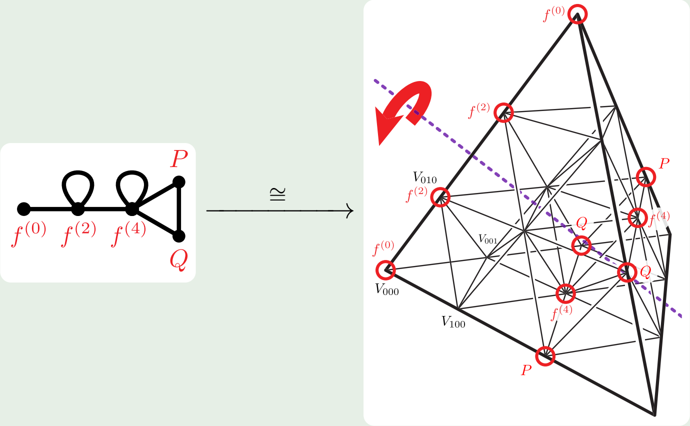
With n odd, q a $4(n+m-2)$-th root of unity,

$$SO(n)_{|q}/2 \cong SO(m)_{|-q^{-1}}/2.$$

Translating to levels, this is $SO(n)_m/2 \cong SO(m)_n/2$, but not at the obvious root of unity!

The quotients are by V_{me_1} and V_{ne_1}, the highest multiples of the standard representation.

Example $((SO(3)_6)/2 \cong (SO(6)_3)/2)$



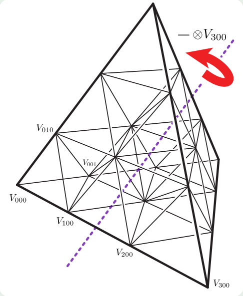
Here we show $SO(6)_3$ as the 'vector' subset of $Spin(6)_4 \cong SU(4)_3$.

Corollary

$$\mathcal{J}_{SO(3)/2,P}(K)(\exp(\tfrac{2\pi i}{14})) = \mathcal{J}_{SO(6)/2,(200)}(K)(-\exp(\tfrac{-2\pi i}{14}))$$

Kirby-Melvin symmetry

'Include up' to all of $SU(4)$. There's a "Kirby-Melvin symmetry" given by $-\otimes(300)$, interchanging (200) and (100).



Kirby-Melvin symmetries aren't quite 'quotients' unless we change the pivotal structure. Knot invariants may change by a sign.

Corollary

$$\mathcal{J}_{SU(4),(200)}(K)(-\exp(\tfrac{-2\pi i}{14}) = -\mathcal{J}_{SU(4),(100)}(K)(-\exp(\tfrac{-2\pi i}{14}).$$

Putting it all together

Theorem

$$\mathcal{J}_{SU(2),(6)}(K)_{|q=\exp(\frac{2\pi i}{28})} = 2\mathcal{J}_{SU(4),(1,0,0)}(K)_{|q=\exp(-\frac{2\pi i}{14})}$$

Proof.

$$\begin{aligned}
\mathcal{J}_{SU(2),(6)}(K)(e^{\frac{2\pi i}{28}}) &= \mathcal{J}_{SO(3)_6,(6)}(K)(e^{\frac{2\pi i}{14}}) && \text{(sub-category)}\\
&= 2\mathcal{J}_{SO(3)_6/2,P}(K)(e^{\frac{2\pi i}{14}}) && \text{(quotient)}\\
&= 2\mathcal{J}_{SO(6)_3/2,2e_3}(K)(-e^{-\frac{2\pi i}{14}}) && \text{(level-rank)}\\
&= 2\mathcal{J}_{SU(4),2e_1}(K)(-e^{-\frac{2\pi i}{14}}) && (D_3 = A_3)\\
&= -2\mathcal{J}_{SU(4),e_1}(K)(-e^{-\frac{2\pi i}{14}}) && \text{(Kirby-Melvin)}\\
&= 2\mathcal{J}_{SU(4),e_1}(K)(e^{-\frac{2\pi i}{14}}) && \text{(parity)}
\end{aligned}$$

Conclusion

It's fun to explain strange identities between knot polynomials by understanding algebraic relationships between the underlying modular tensor categories.

Read our paper http://tqft.net/identities for

- all the coincidences and automorphisms related to $SO(3)_m/2$,
- a nice summary of level-rank duality, especially for $SO(3)$,
- the best description of Kirby-Melvin symmetry in the literature,
- many more pretty pictures!

Thank you!