Khovanov homology is functorial in S^3 as well as in B^3.

Scott Morrison
scott@math.berkeley.edu
joint work with Kevin Walker

Microsoft Station Q

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http://tqft.net/banff-S3

1. The problem
   - Khovanov homology for links in S^3
   - Is Khovanov homology still functorial?
   - The movie move through ∞

2. The solution
   - A lemma: the diagonal map
   - The braid closure case
   - The no-crossings cases

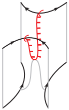
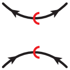
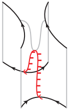
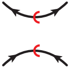
Khovanov homology for links in S^3

- We want to define Khovanov homology for links and cobordisms in S^3, not just in B^3.
- I'm only going to tell you how to do this for links that *happen* to avoid $\infty \in S^3$, and even *happen* to have a generic projection to the equatorial $D^2 \subset S^3$.
- Perhaps you already know how to define the Khovanov homology of such a link – just follow the usual prescription (Khovanov, Bar-Natan, etc.), starting with the generic projection, and ending up with a complex.

That was a really short talk!

A quick review of Khovanov homology

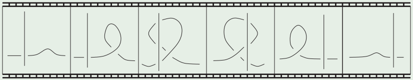
Khovanov homology associates to a tangle diagram a certain complex $(\mathrm{Kh}(T), d^2 = 0)$ in the category with objects 'tangle resolutions' and morphisms 'abstract cobordisms' (modulo some relations).

- $\mathrm{Kh}(T)$ is the 'direct sum' of all the resolutions of T.
- For each pair of resolutions which differ at exactly one crossing, there is a differential
  - for a +ve crossing  , $d =$  :  $\to$ 
  - for a −ve crossing  , $d =$  :  $\to$ 

Cobordisms give chain maps, via movies

Write a cobordism of tangles as a movie.

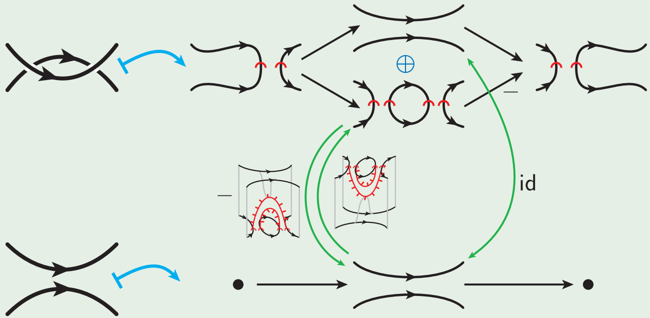
Example



At each step, we see a Reidemeister move, or a Morse move ('birth', 'death' and 'saddle').

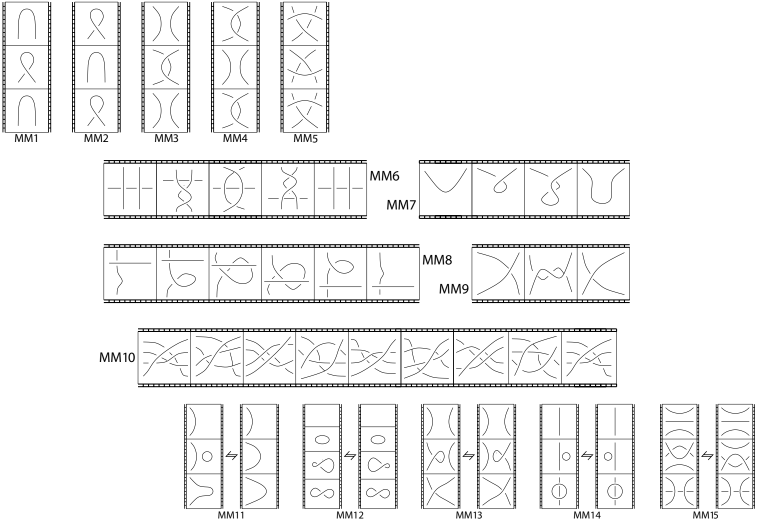
'Functoriality' means that Khovanov homology associates a chain map to each elementary movie.

Example



Is this well-defined? Two movie presentations of a cobordism differ by 'movie moves'. Are the chain maps on either side of a movie move the same?

The 15 movie moves



Is Khovanov homology still functorial in S^3?

Back to S^3.

- Firstly, don't worry overmuch about the restrictions I've placed on the links; some abstract nonsense lets us relax these to arbitrary links in S^3.
- The difficulty comes with cobordisms.
  - We want a functor associating

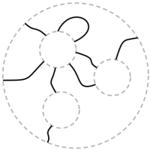
$$\left\{ \begin{array}{c} \Sigma : L_1 \longrightarrow L_2 \\ \text{cobordisms in } S^3 \times I \end{array} \right\} \xrightarrow{\mathsf{Kh}} \left\{ \begin{array}{c} \mathsf{Kh}\left(\Sigma\right) : \mathsf{Kh}\left(L_1\right) \longrightarrow \mathsf{Kh}\left(L_2\right) \\ \text{chain maps} \end{array} \right\}$$

  - When the cobordism is an isotopy, the chain map should be a homotopy equivalence .
  - A pair of cobordisms which are themselves isotopic must give *chain homotopic* chain maps!

Motivation: ‘operadic composition’

- Take a 4-ball, and drill out some internal 4-balls.
- Decorate each boundary S^3 with a link.
- Fill in the remaining 4-ball with some surface, which bounds all of the links.

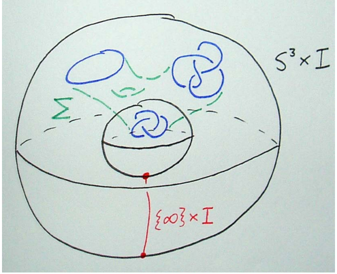
We’d like a map from the tensor product of the interal Khovanov homologies to the external Khovanov homology! This picture is like a ‘planar algebra’, but doubling all the dimensions.



$$: V_7 \otimes V_3 \otimes V_2 \to V_4$$

Defining chain maps by perturbing away from ∞

By a small perturbation, a cobordism in $\Sigma \subset S^3 \times I$ can be made to miss $\{\infty\} \times I$. This allows us to define the chain map $\mathrm{Kh}(\Sigma)$.



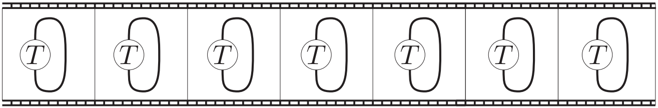
- Perturb it slightly, so it misses ∞, and so its projection to D^2 is a generic movie.
- Follow Khovanov's usual procedure, associating a chain map to each Reidemeister move and Morse move.

Isotopies through ∞.

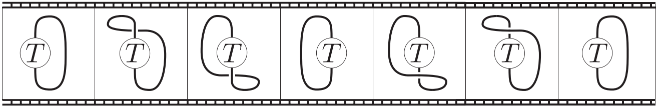
- Even though a cobordism $\Sigma \subset S^3 \times I_1$ generically misses the point at ∞, an isotopy of cobordisms $\Sigma \times I_2 \subset S^3 \times I_1 \times I_2$ does not! $(2 + 1 < 4$, but $3 + 2 \not< 5.)$
- We need to show that two isotopic cobordisms give chain homotopic maps, even if they are isotopic via an isotopy which passes through ∞.
- The movie presentations of two such cobordisms differ by something more complicated than just the usual 15 movie moves!

Look just at the short interval $J \subset I_1$ of time during which an isotopy passes through ∞. At the beginning of the isotopy, we have a cobordism which does nothing during J. At the other end of the isotopy, we have a cobordism which 'passes a strand around the link' during J.

We can write this as an additional 'non-local' movie move:



$$\rightleftharpoons$$

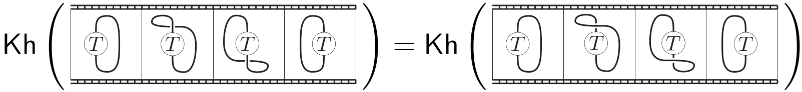


Here T is an arbitrary $1-1$ tangle, and the lower movie is a composition of a great many Reidemeister moves!

What do we need to check for Khovanov homology?

These two movies are isotopic in S^3, through ∞, but they aren't isotopic within B^3. Given a B^3-functorial link invariant, it's extension to S^3 is functorial exactly if it assigns the identity morphism to this movie.

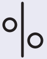
We'll show that the morphisms (chain maps, in our case) associated to the first half and to the time-reversed second half are the same:



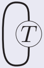
The solution

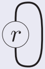
From this point on, we're doing Khovanov homology. We'll show that both chain maps are in fact equal to the 'diagonal map'.

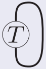
Definition

Each resolution of a $1-1$ tangle T looks like a single vertical strand, with some arrangement of circles (possibly nested, possibly disoriented) on either side, e.g. .

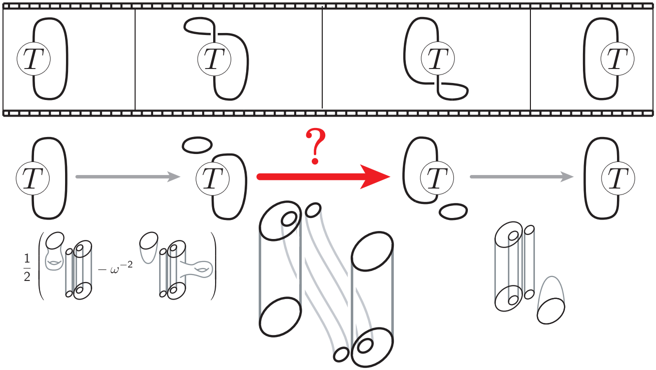
The **diagonal map** takes a resolution 

of 

to the corresponding resolution 

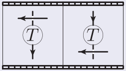
of 

via the obvious abstract cobordism.

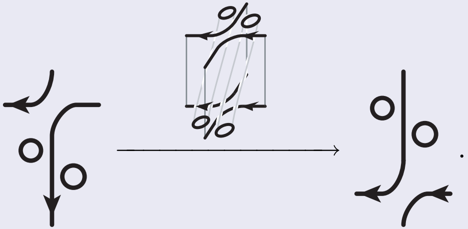


- Some components are 0; we can ignore the disoriented resolution.
- We can write down the other components of $R1$ explicitly.
- If the middle map is 'diagonal', and acts as indicated on resolutions, the composition is exactly what we want.

Lemma

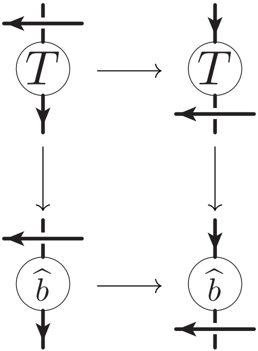
On each resolution  *of T, looking only at the oriented resolution of the outside crossing, the cobordism* 

induces the map



Remark. There's no analogue of this lemma as soon as T has more than 2 endpoints; for example the chain maps for the $R3$ move are already too complicated.

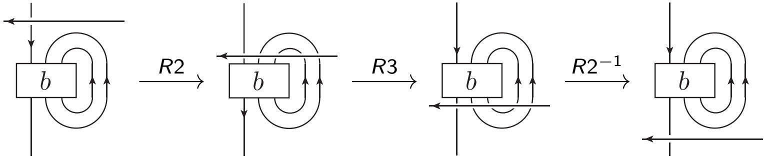
First, we reduce, via an isotopy, to the case that the $1-1$ tangle T is the 'mostly-closure' of a braid. Using functoriality in B^3,



commutes.

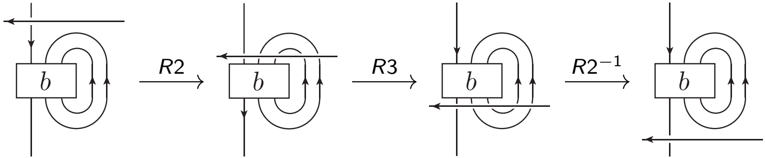
The braid closure case

The cobordism is a sequence of $R2$, then $R3$, then $R2^{-1}$ moves.

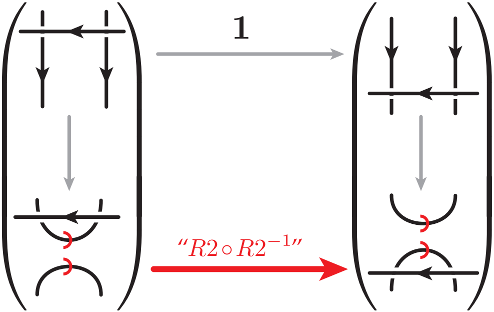


Definition

For a state of the 2nd or 3rd diagram, write the word in '$\mathcal{O}$' and '$\mathcal{D}$' that records whether the crossings with the horizontal strand are *oriented* or *disoriented* resolutions.
Define the **palindromic subspace** as those resolutions in which this word (ignoring its initial letter) is a palindrome.



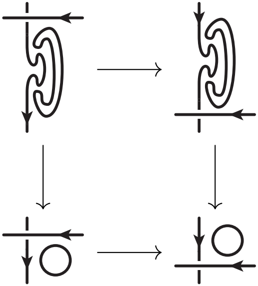
- The initial $R2$s map into the palindromic subspace.
- The final $R2^{-1}$s are supported on the palindromic subspace.
- We can ignore the part of the $R3$ map outside the palindromic subspace!



Let's understand the Reidemeister 3 map.

- The 'up-right' map is zero.
- The 'down-right' map takes us irreversibly away from the palindromic subspace, so we can ignore it.
- The upper horizontal map is the identity.
- The lower horizontal map "is" a composition of an $R2$ and an $R2^{-1}$ move.

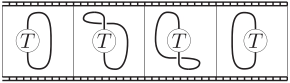
- We can prove the lemma 'one resolution at a time', reducing to the case of crossingless tangles.
- This is easy*, we can put the crossingless tangle in standard form by an isotopy:



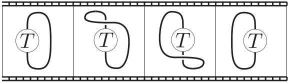
- ***Warning:** If you want to get all the signs right, this argument relies on a not-entirely-worked-out extension of Khovanov homology to *disoriented* tangles.

Summary

- A B^3-functorial knot invariant is S^3-functorial exactly if it assigns the same morphisms to



and



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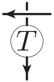
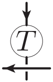
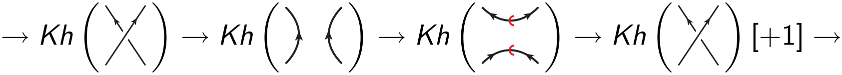
- For Khovanov homology, this follows if the chain map for  $\to$  has the simplest form you might imagine: the 'diagonal map'.
- We saw this was the case for
  - crossingless $1 - 1$ tangles, then
  - 'mostly-closures' of braids, then
  - arbitrary $1 - 1$ tangles.

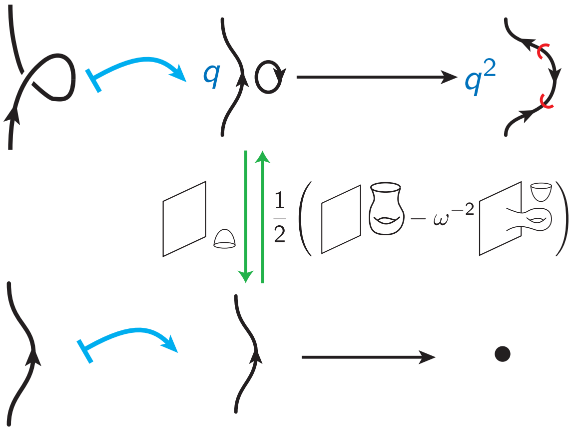
A proof for the future

With an extension to a functorial theory for disoriented tangles in B^3, there's an exact triangle



which commutes with cobordisms. This lets us prove the result by inducting on crossing number! If the two cobordisms are equal at two out of three positions in the exact triangle, the two cobordisms must be equal at the third position.

The Reidemeister 1a move



A Reidemeister 3 move