The Cappell-Shaneson spheres and the s-invariant

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Outline

The smooth 4-dimensional Poincaré conjecture

The smooth 4-dimensional Poincaré conjecture is the 'last man standing' in classical geometric topology. It says

Conjecture (SPC4)

A smooth 4-manifold Σ homeomorphic to the 4-sphere, $\Sigma \cong S^4$, is actually diffeomorphic to it, $\Sigma = S^4$.

There's some 'evidence' either way, but I think by now most people think that it's *false*:

Conjecture ($\sim$SPC4)

Somewhere out there, perhaps not far away, there's is a 4-manifold homeomorphic but not diffeomorphic to the 4-sphere.

Evidence for SPC4

- Gromov showed that if $\Sigma \setminus \mathrm{pt}$ is symplectic and standard near the point, then it is symplectomorphic to $T^*\mathbb{R}^2$. Eliashberg used this to show that the Gluck construction on certain knotted 2-spheres $S^2 \hookrightarrow S^4$ doesn't change the smooth structure.
- Gabai's property R, "Only surgery on the unknot in S^3 can yield $S^1 \times S^2$" has generalisations which are equivalent to SPC4.

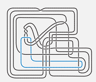
Evidence against SPC4

The old news:

- Donaldson and Seiberg-Witten theory produce multiple smooth structures on closed simply connected 4-manifolds (although these have $H_2 \neq 0$).
- The h-cobordism theorem is broken in dimension 4.
- There are many proposed counterexamples, few of which have been ‘killed’.

The new news: certain *combinatorial invariants* of particular knots provide obstructions to SPC4.

Theorem (Freedman-Gompf-Morrison-Walker)

For example, if s () $\neq 0$*, then SPC4 is false.*

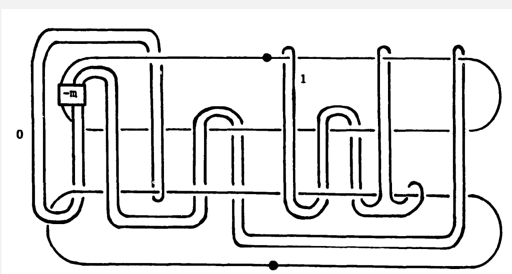
The Cappell-Shaneson spheres

- Consider the 3-torus bundle over S^1 with monodromy $A \in SL(3, \mathbb{Z})$.
- If $\det(I - A) = \pm 1$, surgery on the “zero section” produces a homotopy 4-sphere, denoted W_A.
- (There’s a choice here of a $\pi_1(SO(3)) = \mathbb{Z}_2$ framing for the zero section. One choice is always standard.)
- Conjugation of A in $GL(3, \mathbb{Z})$ doesn’t change W_A. In fact there are finitely many conjugacy classes for each possible trace, and only one when $-4 \leq \mathrm{tr} A \leq 9$.
- We’ll consider a family realising every trace:

$$A_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & m+1 \end{pmatrix}$$

Known results

- W_0 and W_4 naturally cover an exotic $\mathbb{RP}^4$ (C-S 1976).
- Kirby-Akbulut conjectured that W_0 was exotic (1985),
- ... but Gompf later showed it was actually standard!
- Moreover, Gompf gave a handle presentation for each W_n:



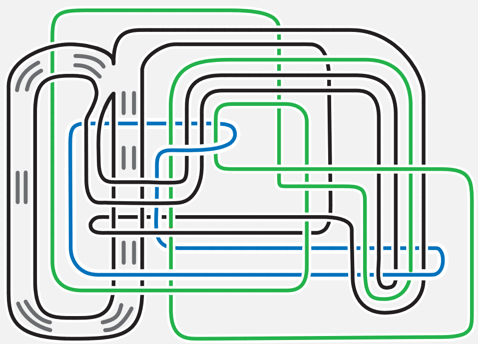
(Unknotted dotted circles indicate 1-handles, knotted circles indicate (framed) attaching curves for 2-handles.)

Localisation

- Sadly, there are no known 4-manifold invariants which can distinguish the Cappell-Shaneson spheres from the standard sphere. (Gauge theory is not good at homotopy spheres.)
- Notice that Gompf’s handle presentation has no 3-handles. The 0-, 1- and 2- handles give a homotopy 4-ball, with S^3 boundary. (Since there are no exotic diffeomorphisms of S^3, there’s only one way to glue on the 4-handle.)
- The meridians of the 2-handles form a two component link in S^3, which must be slice in the Cappell-Shaneson ball.

Theorem (Freedman-Gompf-Morrison-Walker)

If the two component link L_m



is not slice in B^4, the Cappell-Shaneson ball $\dot{W}_m$ must be exotic.

(Here, the blue component is not 'real'; it represents a 2π twist.)

What is Khovanov homology?

- Khovanov homology is an invariant of links. It is a doubly-graded vector space, $Kh^{\bullet,\bullet}(L)$.
- The Khovanov polynomial counts the graded dimensions:

$$Kh(L)(q,t) = \sum_{r,j} q^j t^r \dim Kh^{j,r}(L) \in \mathbb{N}[q^{\pm}, t^{\pm}].$$

- The 'euler characteristic' of Khovanov homology is the Jones polynomial:

$$Kh(L)(q,-1) = J(L)(q).$$

Khovanov homology is functorial

Theorem (Jacobsson/Bar-Natan)

Khovanov homology is projectively functorial. It associates to a cobordism $\Sigma : L_1 \to L_2$ a linear map $Kh^{\bullet,\bullet}(L_1) \to Kh^{\bullet,\bullet}(L_2)$. Isotopy of the cobordism changes the linear map by at worst a sign.

Theorem (Clark-Morrison-Walker)

For a suitable variation of Khovanov homology, this linear map is well-defined on the nose.

The s-invariant gives genus bounds

Other variations of Khovanov homology give more information.

Theorem (Rasmussen)

There is an integer invariant of knots $s(K)$, and

$$|s(K)| \leq g_{slice}(K).$$

Theorem (Morrison-Walker)

There is a family of polynomial invariants $f_k(K) \in \mathbb{N}[q^{\pm}, t^{\pm}]$ and

$$Kh(K)(q,t) = q^{s(K)}(q+q^{-1}) + \sum_{k \geq 2} (1+q^{2k}t) f_k(K)(q,t).$$

A chain of programs (Green/Bar-Natan/Morrison-Shumakovitch) can compute these invariants directly. They are *combinatorial*.

Extracting the s-invariant.

Conjecture (Morrison-Walker/Shumakovitch/Khovanov)

Only f_2 is nonzero, and the s-invariant is determined by the Khovanov polynomial, via

$$q^{s(K)}(q+q^{-1}) = Kh(K)(q, -q^{-4}).$$

- Even without this conjecture, often we can extract $s(K)$ directly from the Khovanov polynomial, by analysing possible decompositions into the polynomials f_k.
- When this works, it is much faster than calculating the actual decomposition.
- It is now possible to compute $s(K)$ for knots K with 50 or more crossings; previously 10-15 was the limit.

A plausible theorem dooms this approach

- Recently, connections have been found between Khovanov homology and knot Floer homology.
- Experience suggests gauge theoretic approaches can't detect smooth structure near a point.
- Thus the following plausible result would kill this approach:

Conjecture

If a knot K is slice in any homotopy 4-ball, then $s(K) = 0$.

- On the other hand, Khovanov homology relies on picking coordinates, and using projections of links and cobordisms. This is both an obstacle to 'geometric' interpretations, and some cause for hope that it is sensitive to smooth structure.

The future: global obstructions

We can also define 4-manifold invariants.

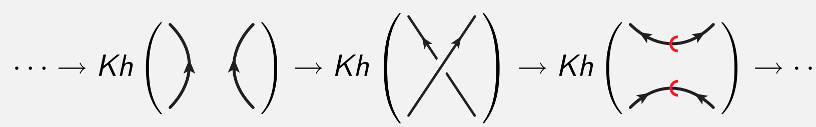
- Khovanov homology has the structure of a 4-*category with duals* (modulo a conjecture about the S^3 movie move).
- Standard topological quantum field theory constructions give the skein module invariant

$$Z(W^4, L \subset \partial W) \in \mathcal{V}\mathrm{ect}^{\bullet,\bullet}.$$

- Perhaps this can distinguish the Cappell-Shaneson spheres directly?

Exact triangles and blob homology

- Unfortunately computing the skein module invariant $Z(W)$ for Khovanov homology is very hard.
- The main tool for computing $Kh(L)$ is the exact triangle

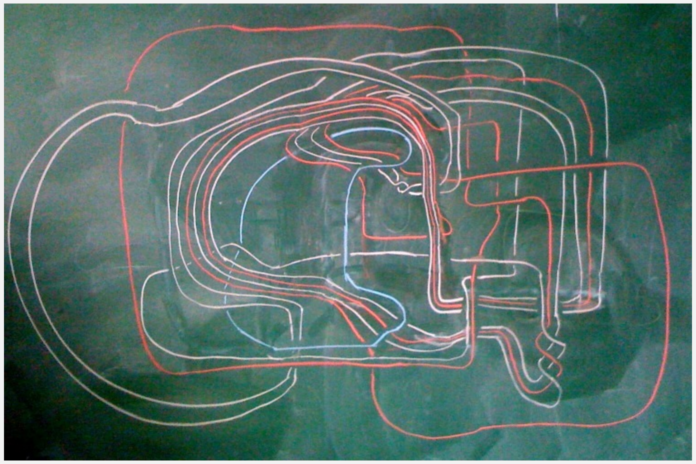


which *fails* for the skein module (essentially because taking quotients is not an exact functor).

- With Kevin Walker, I'm working on 'blob homology', a simultaneous generalisation of TQFT skein modules and Hochschild homology, which may be more computable for Khovanov homology.

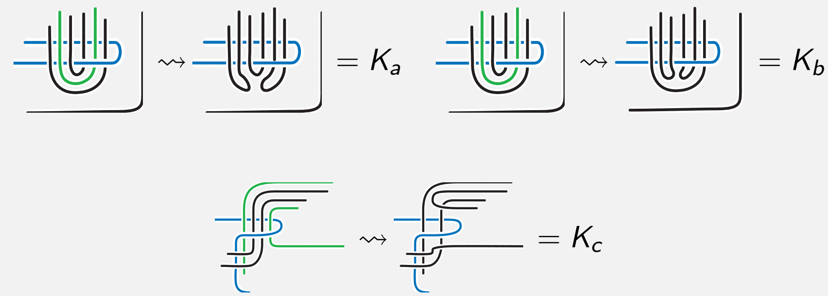
L_1 is huge

Unfortunately the two component link L_m is huge; even L_1 has ~ 222 crossings; even worse, its *girth* is ~ 24.

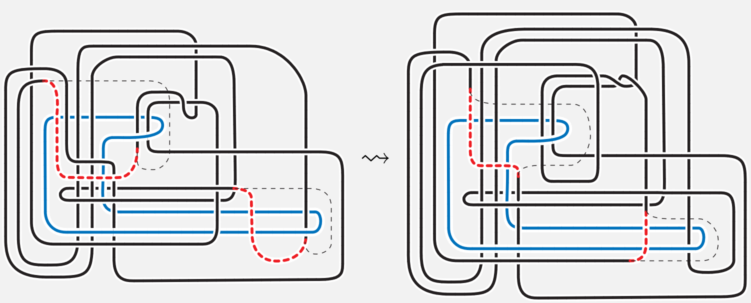


Band moves

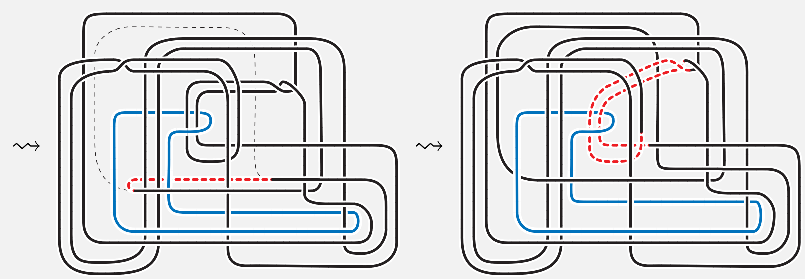
- Let's take a risk, and look for *band* connect sums that become simpler. If the resulting knot is not slice, the original link can't be either.
- We'll consider the following three bands on L_1, and call the resulting knots K_a, K_b and K_c:



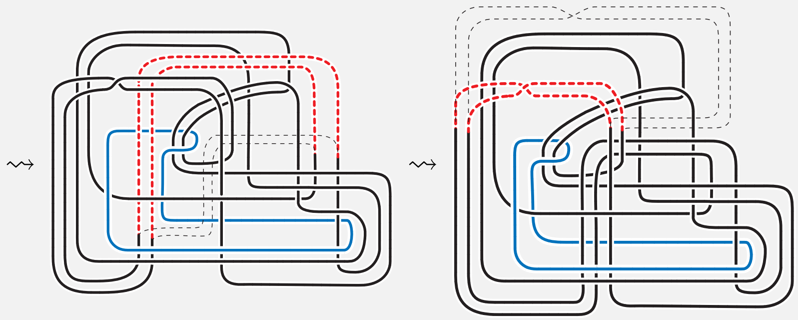
Simplifying K_b, I



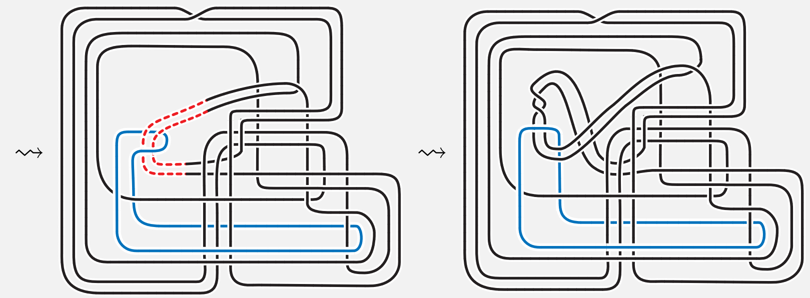
Simplifying K_b, II



Simplifying K_b, III



Simplifying K_b, IV



The knots K_a, K_b and K_c

- A little work by hand shows K_a is ribbon, and hence slice.
- The Alexander polynomials are all 1; by a theorem of Freedman this means they're all *topologically* slice.
- But how big are they?

	apparent crossings	apparent girth
K_a	67	14
K_b	78	14
K_c	86	16

- This is still scarily large, but perhaps plausible! The biggest computation of the Khovanov polynomial so far is in Bar-Natan's "I've computed $Kh(T(8,7))$ and I'm happy"; that has girth 14 but only 48 crossings. Computations seem to scale at least exponentially in the number of crossings, and *really badly* in the girth.

Improving `JavaKh`

We started with Jeremy Green's program `JavaKh`, and made many improvements:

New interface Progress reports, saving to disk.

Memory optimisations Caching, 'bit flipping', paging to disk.

Minimising girth Better algorithms to find small girth presentations.

A better algorithm Cancelling blocks of isomorphisms, not just one at a time.

At the end, we had something that can compute $Kh(K_b)$; it takes almost a week on a fast machine with 32gb of RAM!

Results for $Kh(K_b)$

$Kh(K_b)(q,t) =$

$$
\begin{aligned}
&q^{-45}t^{-32} + q^{-41}t^{-31} + q^{-39}t^{-29} + q^{-35}t^{-28} + q^{-37}t^{-27} + q^{-37}t^{-26} + q^{-33}t^{-26} + \\
&q^{-35}t^{-25} + q^{-33}t^{-25} + q^{-35}t^{-24} + 2q^{-31}t^{-24} + q^{-33}t^{-23} + 2q^{-31}t^{-23} + q^{-27}t^{-23} + \\
&q^{-33}t^{-22} + 2q^{-29}t^{-22} + q^{-27}t^{-22} + q^{-31}t^{-21} + 3q^{-29}t^{-21} + q^{-25}t^{-21} + q^{-31}t^{-20} + \\
&3q^{-27}t^{-20} + 2q^{-25}t^{-20} + 4q^{-27}t^{-19} + 2q^{-23}t^{-19} + q^{-27}t^{-18} + 2q^{-25}t^{-18} + 4q^{-23}t^{-18} + \\
&4q^{-25}t^{-17} + q^{-23}t^{-17} + 3q^{-21}t^{-17} + q^{-19}t^{-17} + 4q^{-25}t^{-16} + 2q^{-23}t^{-16} + 6q^{-21}t^{-16} + \\
&q^{-17}t^{-16} + 4q^{-23}t^{-15} + 5q^{-21}t^{-15} + 3q^{-19}t^{-15} + 2q^{-17}t^{-15} + q^{-23}t^{-14} + q^{-21}t^{-14} + \\
&8q^{-19}t^{-14} + q^{-17}t^{-14} + q^{-15}t^{-14} + 3q^{-21}t^{-13} + 6q^{-19}t^{-13} + 3q^{-17}t^{-13} + 4q^{-15}t^{-13} + \\
&q^{-21}t^{-12} + 2q^{-19}t^{-12} + 9q^{-17}t^{-12} + 5q^{-15}t^{-12} + 2q^{-13}t^{-12} + 7q^{-17}t^{-11} + 4q^{-15}t^{-11} + \\
&7q^{-13}t^{-11} + 3q^{-17}t^{-10} + 7q^{-15}t^{-10} + 7q^{-13}t^{-10} + 2q^{-11}t^{-10} + q^{-9}t^{-10} + 8q^{-15}t^{-9} + \\
&6q^{-13}t^{-9} + 9q^{-11}t^{-9} + q^{-9}t^{-9} + 3q^{-15}t^{-8} + 5q^{-13}t^{-8} + 13q^{-11}t^{-8} + 4q^{-9}t^{-8} + \\
&2q^{-7}t^{-8} + 5q^{-13}t^{-7} + 8q^{-11}t^{-7} + 9q^{-9}t^{-7} + 5q^{-7}t^{-7} + q^{-5}t^{-7} + 5q^{-11}t^{-6} + 13q^{-9}t^{-6} + \\
&6q^{-7}t^{-6} + 4q^{-5}t^{-6} + q^{-11}t^{-5} + 8q^{-9}t^{-5} + 11q^{-7}t^{-5} + 8q^{-5}t^{-5} + q^{-3}t^{-5} + 2q^{-9}t^{-4} + \\
&12q^{-7}t^{-4} + 10q^{-5}t^{-4} + 6q^{-3}t^{-4} + 7q^{-7}t^{-3} + 9q^{-5}t^{-3} + 12q^{-3}t^{-3} + 2q^{-1}t^{-3} + \\
&9q^{-5}t^{-2} + 12q^{-3}t^{-2} + 8q^{-1}t^{-2} + q^{1}t^{-2} + 3q^{-5}t^{-1} + 7q^{-3}t^{-1} + 15q^{-1}t^{-1} + 5q^{1}t^{-1} + \\
&q^{3}t^{-1} + 3q^{-3}t^{0} + 14q^{-1}t^{0} + 10q^{1}t^{0} + 6q^{3}t^{0} + q^{-3}t^{1} + 5q^{-1}t^{1} + 11q^{1}t^{1} + 10q^{3}t^{1} + 2q^{5}t^{1} + \\
&q^{-1}t^{2} + 8q^{1}t^{2} + 10q^{3}t^{2} + 8q^{5}t^{2} + 2q^{1}t^{3} + 7q^{3}t^{3} + 10q^{5}t^{3} + 5q^{7}t^{3} + 4q^{3}t^{4} + 7q^{5}t^{4} + 6q^{7}t^{4} + \\
&3q^{9}t^{4} + q^{3}t^{5} + 5q^{9}t^{5} + 2q^{5}t^{6} + 5q^{7}t^{6} + 7q^{9}t^{6} + 4q^{11}t^{6} + 4q^{5}t^{5} + 8q^{7}t^{5} + q^{7}t^{7} + 5q^{9}t^{7} + \\
&4q^{11}t^{7} + 3q^{13}t^{7} + 2q^{9}t^{8} + 4q^{11}t^{8} + 3q^{13}t^{8} + 3q^{11}t^{9} + 4q^{13}t^{9} + 3q^{15}t^{9} + q^{11}t^{10} + q^{13}t^{10} + \\
&3q^{15}t^{10} + 2q^{17}t^{10} + q^{13}t^{11} + 2q^{15}t^{11} + q^{17}t^{11} + q^{13}t^{12} + 2q^{17}t^{12} + q^{19}t^{12} + 2q^{17}t^{13} + \\
&q^{21}t^{13} + q^{17}t^{14} + q^{19}t^{14} + q^{21}t^{14} + q^{19}t^{15} + q^{21}t^{15} + q^{23}t^{15} + q^{23}t^{16} + q^{23}t^{17} + q^{27}t^{18}
\end{aligned}
$$

Extracting $s(K_b)$

- There are thousands of possible decompositions of $Kh(K_b)$ of the form
$$Kh(K_b)(q,t) = q^{s(K_b)}(q+q^{-1}) + \sum_{k\geq 2} f_k(K_b)(q,t)(1+q^{2k}t).$$
- Exactly one has $f_k = 0$ for $k > 2$, in agreement with our conjecture, and this is presumably the actual decomposition.
- Nevertheless, every decomposition gives $s = 0$, so for this knot we find no obstruction.

What next?

Obviously this is disappointing. On the other hand, we've only turned over the first stone.

- Computations for K_c are running right now!
- It looks like L_{-1} might be simpler than L_1, but we've only just started searching for nice bands.
- With present technology (algorithm, implementation, hardware), there are probably several more accessible cases. (But only several.)

Conclusions

- Certain 'local' slice problems for links imply that SPC4 is false.
- Khovanov homology may provide obstructions. Even with recent advances, the calculations are hard, so we use bands to turn the links into smaller knots.
- The first s-invariant we could calculate didn't produce an obstruction. Other bands, and other Cappell-Shaneson spheres, are running as we speak!
- We can define 'global' 4-manifold invariants using Khovanov homology, and using 'blob homology' these may be computable.