

The Jones polynomial

Conventions:

$$J: \{ \text{oriented tangles} \} \rightarrow \mathbb{Z}[q, q^{-1}] \left\{ \begin{array}{l} \text{unoriented tangles} \\ \text{without crossings} \\ \text{mod } 0 = q + q^{-1} \end{array} \right.$$

$$\{ \text{oriented links} \} \rightarrow \mathbb{Z}[q, q^{-1}] \cdot \phi$$

$$J \left(\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right) = q^{+1} \left(-q^{+2} \begin{array}{c} \cup \\ \cup \end{array} \right)$$

$$J \left(\begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} \right) = -q^{-2} \begin{array}{c} \cup \\ \cup \end{array} + q^{-1} \left(\begin{array}{c} \cup \\ \cup \end{array} \right)$$

It really is an invariant of tangles:

$$J \left(\begin{array}{c} \uparrow \\ \rho \end{array} \right) = q \left(\begin{array}{c} \cup \\ \cup \end{array} \right) - q^2 \left(\begin{array}{c} \cup \\ \cup \end{array} \right) = (q(q + q^{-1}) - q^2) \left(\begin{array}{c} \cup \\ \cup \end{array} \right) = \left(\begin{array}{c} \cup \\ \cup \end{array} \right) = J \left(\begin{array}{c} \uparrow \\ \uparrow \end{array} \right)$$

$$J \left(\begin{array}{c} \nearrow \searrow \\ \cup \end{array} \right) = \left(\begin{array}{c} \cup \\ \cup \end{array} \right) \left(-(q + q^{-1}) \begin{array}{c} \cup \\ \cup \end{array} \right) + \left(\begin{array}{c} \cup \\ \cup \end{array} \right)$$

$$= \left(\begin{array}{c} \cup \\ \cup \end{array} \right) (= J \left(\begin{array}{c} \uparrow \\ \uparrow \end{array} \right))$$

etc..

You can glue together the invariants of small tangle to find the invariant of a big tangle: "J is a map of planar algebras"

The Jones polynomial of the trefoil.

$$J(\text{trefoil}) =$$

$$\begin{array}{ccccccc}
 & & -q^4 & & q^5 & & \\
 & & \text{trefoil} & & \text{trefoil} & & \\
 & & + & & + & & \\
 q^3 & \text{circle} & + & -q^4 & \text{trefoil} & + & q^5 & \text{trefoil} & + & -q^6 & \text{trefoil} \\
 & & + & & + & & \\
 & & -q^4 & & q^5 & & \\
 & & \text{trefoil} & & \text{trefoil} & & \\
 & & \parallel & & & &
 \end{array}$$

$$+q^3(q+q^{-1})^2 - 3q^4(q+q^{-1}) + 3q^5(q+q^{-1})^2 - q^6(q+q^{-1})^3$$

Thought of like this, the Jones polynomial is always an alternating sum. Perhaps it's the Euler characteristic of some complex?

- We'd want the columns above to be replaced by 'chain groups', whose dimensions (or graded dimensions) give the terms above.
- We'd need differentials, too?

A category for the complex. — "Cob(su₂)".

Objects

Let's begin with what we've got:

unoriented tangles without crossings,
along with a 'factor' of q^k

Examples: $\emptyset, q^2 \emptyset, q^3 \emptyset$

Morphisms

For diagrams D_1 and D_2 , let's declare

$$\text{Hom}_{\text{Cob}(su_2)}(D_1, D_2) =$$

R { cobordisms from D_1 to D_2 rel ∂
modulo relations:

(some ring
with $\frac{1}{2}$)

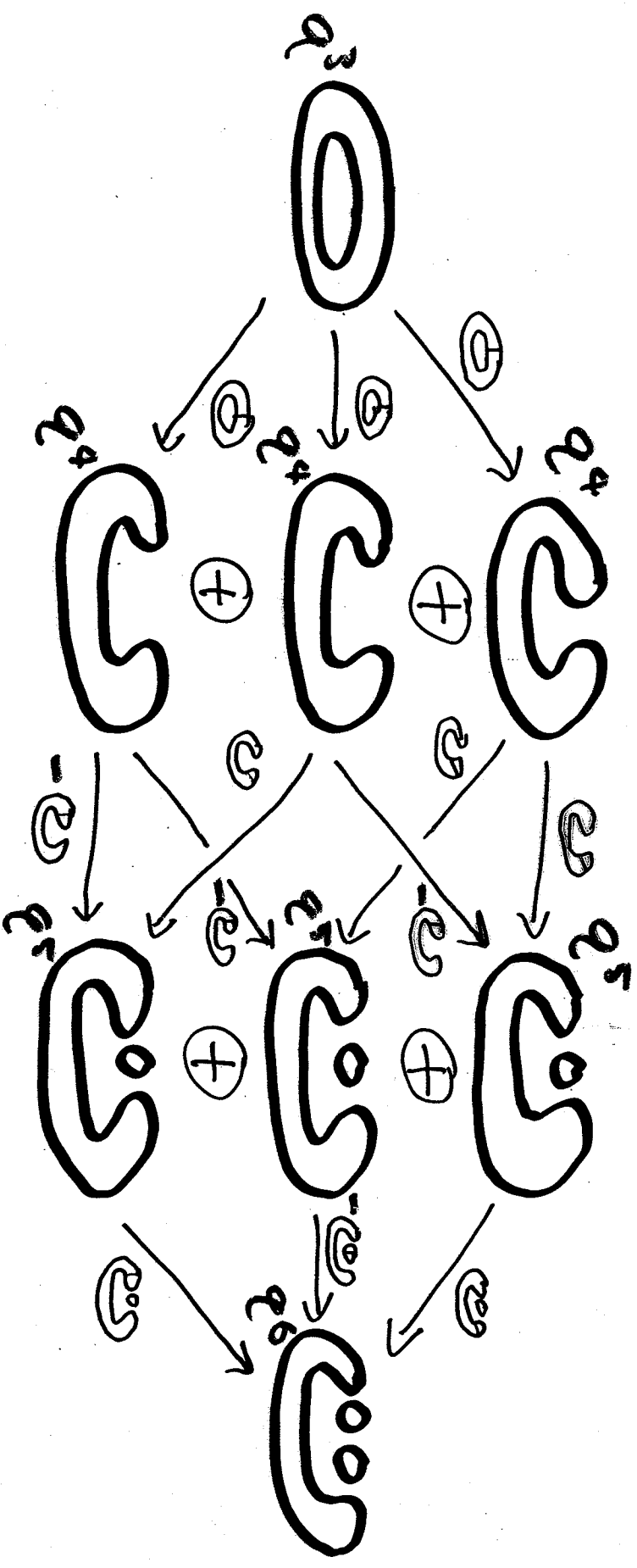
$$\text{circle with two dots} = 0, \quad \text{circle with one dot} = 2$$

$$\text{cylinder} = \frac{1}{2} \text{circle with one dot} \text{ circle} + \frac{1}{2} \text{circle} \text{ circle with one dot}$$

We'll come back to these relations
in a moment!

The category $\text{Cob}(su_2)$ can't be quite
right, however: we want to have
'chain groups' which are 'sums' of diagrams

$$[\text{Diagram}] =$$



$$[\cdot J]^0 \xrightarrow{d^0} [\cdot J]^1 \xrightarrow{d^1} [\cdot J]^2 \xrightarrow{d^2} [\cdot J]^3$$

Let's try $\text{Mat}(\text{Cob}(su_2))$ instead.

- For any category \mathcal{C} whose morphism spaces are additive, we can define $\text{Mat}(\mathcal{C})$.

Objects 'formal direct sums' of objects in \mathcal{C} .

Morphisms matrices of morphisms in \mathcal{C} .

(each matrix entry is a morphism from a direct summand of the source to a direct summand of the target)

In this category, there's a natural complex to write down for a knot!

"Chain groups" as in the calculation of the Jones polynomial.

Differentials "pairs of pants" merging or splitting circles

(locally, saddles changing the resolution of a crossing)

(we'll need to sprinkle some minus signs, in order to get $d^2=0$)

Homology, homotopy, and all that

- Obviously, the complex itself can't be a link invariant.
- Complexes are rarely invariants anyway; we expect to have to take homology.
 - That's not an option here. The category isn't abelian (cobordisms don't have kernels or images).
- Instead, we'll settle for the homotopy type of the complex.
 - This isn't as bad as it sounds; it turns out every link has a unique homotopy representative with no differentials....
- Actually, we can do something simpler — take the simple homotopy type.
 - What is simple homotopy?
Adding or deleting contractible direct summands:

$$(A \xrightarrow{\text{d an iso}} B) \overset{\sim}{\underset{\text{simple homotopy}}{}} \bullet$$

The Khovanov/Bar-Natan invariant for tangles.

For single crossings, we make the definitions

$$\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \\ + \end{array} \right] = q \left(\begin{array}{c} \square \\ \longrightarrow \\ q^2 \end{array} \right) \begin{array}{c} \cup \\ \cap \end{array}$$

$$\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \\ - \end{array} \right] = \underbrace{q^{-2} \begin{array}{c} \cup \\ \cap \end{array}}_{[\cdot]^{-1}} \underbrace{\begin{array}{c} \square \\ \longrightarrow \\ q^{-1} \end{array}}_{[\cdot]^{0}} \underbrace{\begin{array}{c} \cup \\ \cap \end{array}}_{[\cdot]^{1}}$$

For a larger tangle (or link), we take the tensor product of the complexes for each crossing, gluing together diagrams and cobordisms in the same way the crossings were glued together to form the tangle.

Example

$$\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \\ - \end{array} \right] \left[\begin{array}{c} \searrow \nearrow \\ \swarrow \nwarrow \\ + \end{array} \right] = q^{-1} \left(\begin{array}{c} \cup \\ \cap \end{array} \right) \oplus \left(\begin{array}{c} \cap \\ \cup \end{array} \right) \left(\begin{array}{c} \cup \\ \cap \end{array} \right)$$

$\underbrace{\quad}_{[\cdot]^{-1}} \quad \underbrace{\quad}_{[\cdot]^{0}} \quad \underbrace{\quad}_{[\cdot]^{1}}$

(We say: "[\cdot]" is a map of planar algebras")

The "delooping" isomorphism

- In $\text{Mat}(\text{Cob}(su_2))$ there's an interesting isomorphism between \mathbb{O} and $e\phi \oplus e^{-1}\phi$ given by the matrices

$$\mathbb{O} \xrightarrow{\begin{pmatrix} \frac{1}{2} \text{C} \\ \text{A} \end{pmatrix}} \begin{matrix} e\phi \\ \oplus \\ e^{-1}\phi \end{matrix} \xrightarrow{\begin{pmatrix} \text{B} & \frac{1}{2} \text{C} \end{pmatrix}} \mathbb{O}$$

(notice everything here is in grading zero—)
 $\chi(\text{C}) + 1 = 0 = \chi(\text{A}) - 1$

- Why are these matrices inverses?

$$\begin{pmatrix} \text{B} & \frac{1}{2} \text{C} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \text{C} \\ \text{A} \end{pmatrix} = \frac{1}{2} \text{C} + \frac{1}{2} \text{A} \stackrel{\text{N.C.}}{=} \text{C}$$

$$\begin{pmatrix} \frac{1}{2} \text{C} \\ \text{A} \end{pmatrix} \begin{pmatrix} \text{B} & \frac{1}{2} \text{C} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \text{C} & \frac{1}{4} \text{C} \text{C} \\ \text{A} & \frac{1}{2} \text{C} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Here we've used $\text{C} = 2$, $\text{A} = 0$, and derived

$$\begin{aligned} \text{C} &\stackrel{\text{N.C.}}{=} \frac{1}{2} \text{C} \text{C} + \frac{1}{2} \text{C} \text{C} \\ &= 2 \text{C} \text{C} = 0 \end{aligned}$$

Reidemeister 1 invariance.

$$[\uparrow \rho] = \rho \circ \xrightarrow{d = \text{loop}} \rho^2$$

||| delooping

$$\begin{array}{ccc}) & \xrightarrow{\frac{1}{2} \square \oplus} & \rho^2 \\ \oplus & & \\ \rho^2) & \xrightarrow{\square} & \rho^2 \end{array}$$

via a "change of basis":

$$\begin{array}{ccc}) & \xrightarrow{\begin{pmatrix} \square & 0 \\ -\frac{1}{2} \square \oplus & \square \end{pmatrix}} & \oplus \\ \oplus & & \\ \rho^2) & \xrightarrow{\begin{pmatrix} \square & 0 \\ \frac{1}{2} \square \oplus & \square \end{pmatrix}} & \oplus \\ \rho^2) & & \end{array}$$

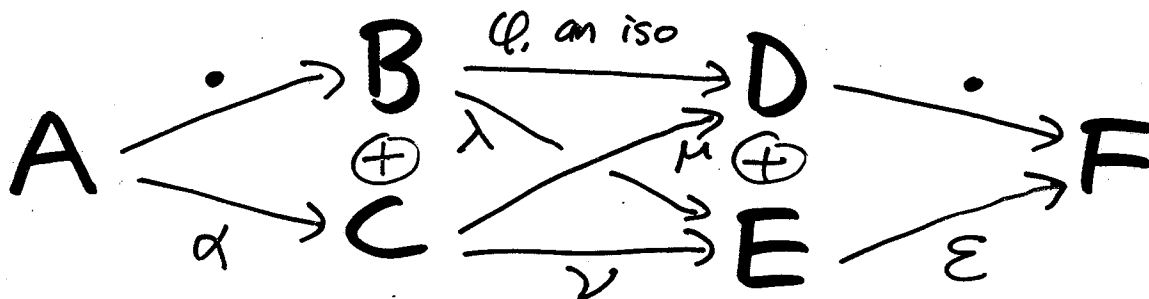
$$\begin{array}{ccc}) & \xrightarrow{0} & \rho^2 \\ \oplus & & \\ \rho^2) & \xrightarrow{\square} & \rho^2 \end{array}$$

||| simple homotopy

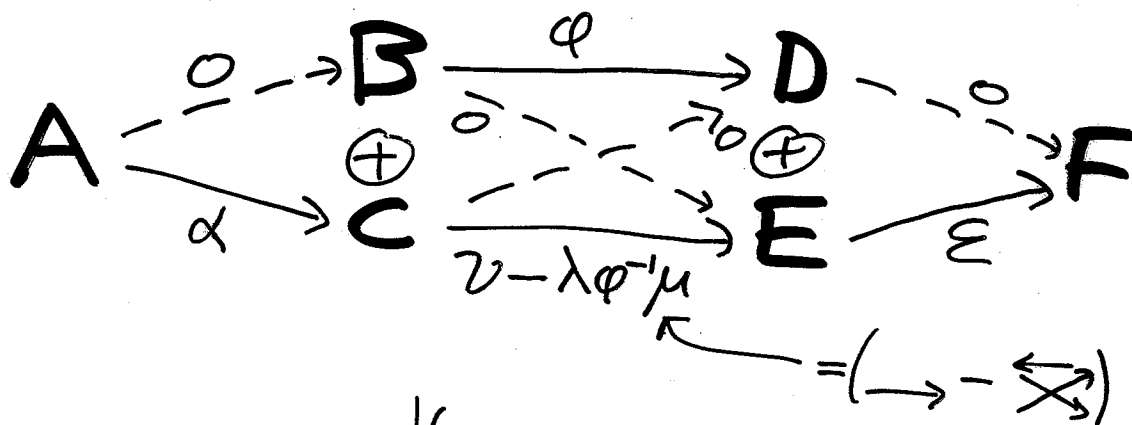
$$) = [\uparrow]$$

Gaussian elimination for complexes.

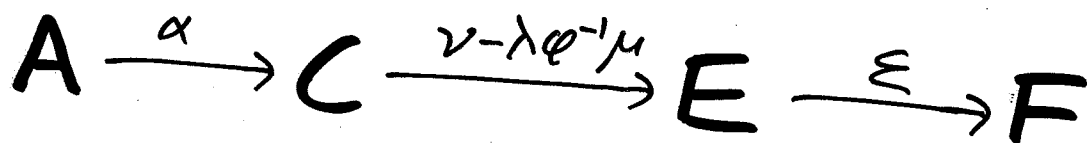
- Anytime you see an isomorphism as a matrix entry in a differential, there's a contractible direct summand to strip off!



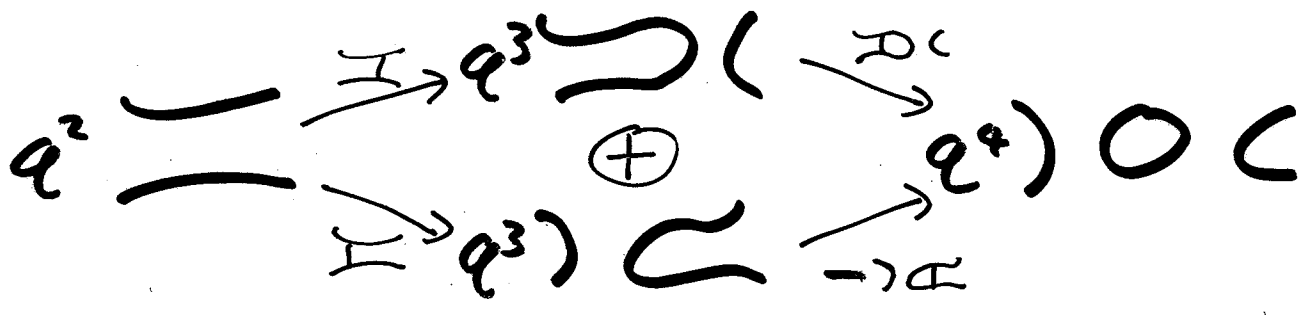
||| via a change of basis



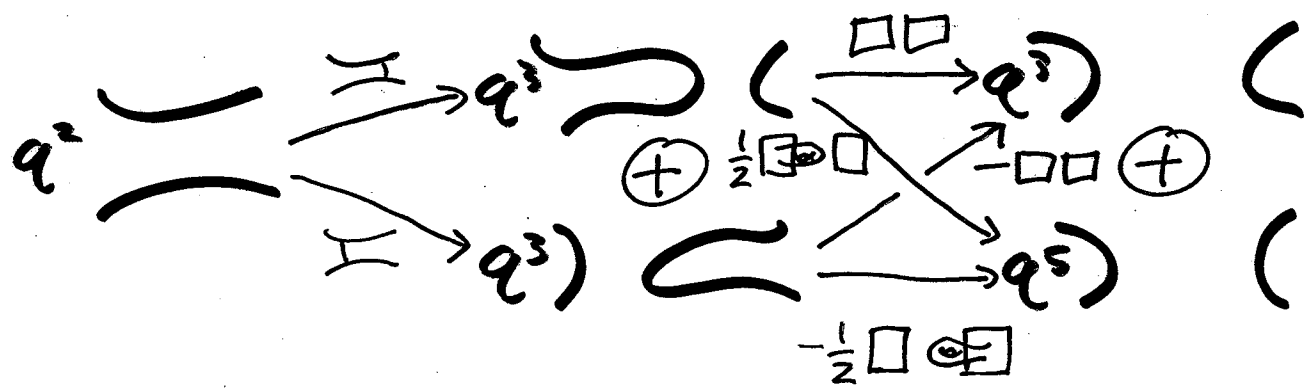
|S simple homotopy



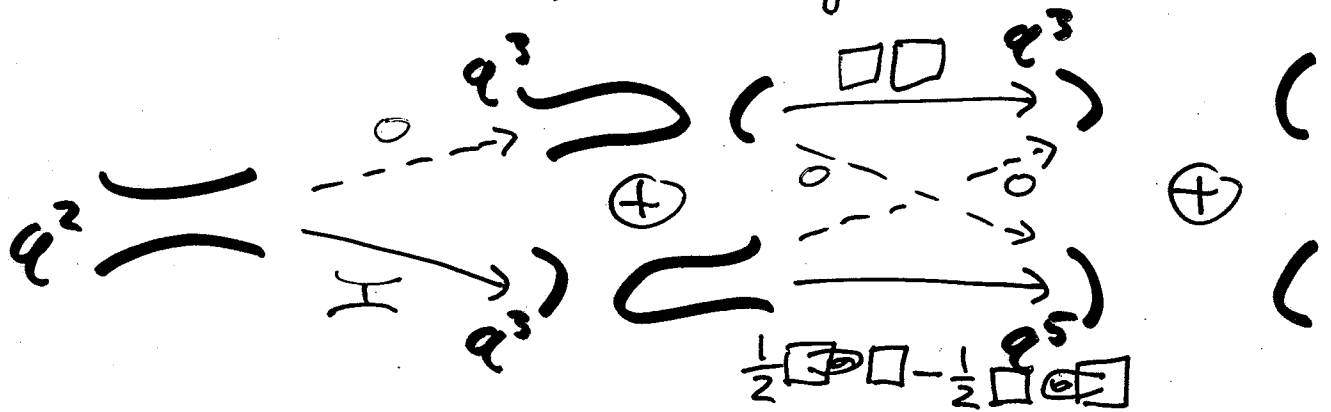
$$[\mathbb{Z}\pi] =$$



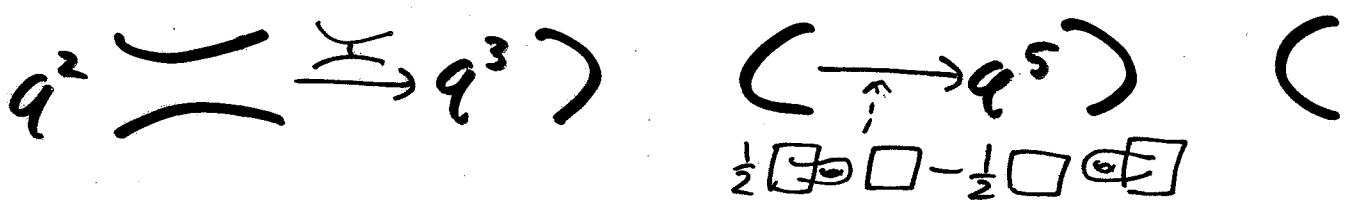
||S (delooping)



||S (change of basis)



||S (simple homotopy)



The Hopf link:



$$q^2 \bigcirc \xrightarrow{\bigcirc} q^3 \bigcirc \xrightarrow{\circ} q^5 \bigcirc$$

||S delooping

$$\begin{array}{ccc} q^2 \bigcirc & \xrightarrow{\frac{1}{2} \text{cylinder}} & q^3 \bigcirc \xrightarrow{\circ} q^5 \bigcirc \\ \oplus & & \\ q^3 \bigcirc & \xrightarrow{\text{cylinder}} & \end{array}$$

||S simple homotopy

$$q^2 \bigcirc \xrightarrow{\circ} \cdot \xrightarrow{\circ} q^5 \bigcirc$$

||S delooping again

$$\begin{array}{ccc} | & & q^9 \\ \oplus & \xrightarrow{\circ} \cdot \xrightarrow{\circ} & \oplus \\ q^2 & & q^6 \end{array}$$

$$\text{Kh}(\text{Hopf link}) = t^0(1+q^2) + t^2(q^4+q^6)$$

$$J(\text{Hopf link}) = 1+q^2+q^4+q^6 \quad (\text{set } t=-1)$$