Khovanov homology for 4-manifolds (Davis, Jan 31 2012)

Today I'm going to define a vector-space

\[ Kh(W^4) \]

an invariant of 4-manifolds.

In fact the 4-manifold may have boundary,
and a link in the boundary:

\[ Kh(W^4; L \subset \partial W) \]

When \( W^4 = B^4 \), this is just the usual Khovanov homology of \( L \)

To do this, I'm going to walk you through

a recipe for an invariant of n-manifolds,
starting with an "n-category".

We'll see Khovanov homology can provide a 4-category!

We need one new fact about Khovanov homology
but otherwise we're following a standard recipe.
What is a ("disklike") 2-category?

- For \( k=0,1,2 \), a functor from \( k \)-balls to \( \text{Set} \).
  
  e.g. \( F_0(\bullet) = \{\bullet\} \) (the singleton, containing the 0-ball)
  
  \( F_1(\subset) = \{\{\} \} \) (finite subsets of the 1-ball)
  
  \( F_2(\bullet \bullet) = \{\bigcup \{\} \} \) (embedded 1-manifolds, transverse to the boundary)

- Boundary restrictions:
  
  if \( X^{k-1} \subset \partial B^k \), e.g. \( \bullet \subset B^2 \)
  
  we have functions

  \( F_k(B) \leftarrow \text{"restriction"} \rightarrow F_{k-1}(X) \)

  e.g. \( \bullet \rightarrow \emptyset \)

- Gluing: whenever we have a ball \( B = B_1 \cup B_2 \)

  a map

  \[ F_k(B_1) \times F_k(B_2) \rightarrow F_k(B) \]

  e.g. \( \emptyset, \emptyset \rightarrow \emptyset \)
Usually, we want to impose relations at the top level.

Example \( F_2(\mathbb{N}) = C^3 \text{ embedded 1-manifolds} \)

where \( n \) could be:

- \( \mathbb{E} \) isotopy rel \( d \), \( C = S^3 \) (all the vector spaces are finite dimensional)

- \( \mathbb{E} \) as above, and \( l = \infty \), \( s = 13 \) (all the vector spaces are 1-dimensional!)

- \( \mathbb{E} \) as above, and \( s = \sqrt{2} \),

\[
\| (\| + \sqrt{2}) \| + \sqrt{2} (\| + \|) + \| = 0^3
\]

(vector space with \( 2n \) boundary points, dimension \( 2^{n-1} \))
How do we build invariants of n-manifolds?

**Definition** ($F(W)$, a vector space)

Consider the poset $D(W)$ of "ball decompositions" of $W$.

$F$ gives a functor $D(W) \to \text{Vect}$:

$$F(\bigotimes) = \bigoplus \bigotimes F(\sigma) \otimes F(\sigma') \otimes F(D) \otimes F(D')$$

subject to conditions

$F(\bigotimes \to \bigotimes)$ is the gluing map $F(D) \otimes F(D) \to F(D)$.
\[ F(0) = \epsilon \otimes \mathbb{D} / 0 = \mathbb{Z} \]

\[ \text{Homology with coefficients in a 'quantum finite group'} \]

\[ E(\otimes) = \mathbb{C} \otimes \mathbb{Z} / \mathbb{Z} \]

\[ \mathbb{E}(\otimes) = \mathbb{C} \mathbb{D} / \mathbb{Z} \]

\[ \mathbb{S}(\otimes) = \mathbb{C} \mathbb{D} / 0 = \mathbb{C} \]

\[ \text{Let's examine an example:} \]

\[ E(\otimes) = \mathbb{C} \mathbb{D} / \mathbb{Z} \]
What is the Khovanov 4-category?

\[ \text{Kh}_0(\bullet) = \{, \{, \} \} \]

\[ \text{Kh}_1(\sim) = \{, \{, \} \} \]

\[ \text{Kh}_2(\bigcirc) = \{, \{, \} \} \]

\[ \text{Kh}_3(\circlearrowleft) = \{, \{, \} \} \]

\[ \text{Kh}_4(\text{Bar}) = \text{Kh}(L) \quad \text{a doubly graded vector space} \]
I had a significant difficulty!

We need $Kh(L \subset S^3)$, and in order to be sure...

but Khovanov homology as usually discussed has a combinatorial construction that only gives an invariant of a link in $B^3$.

Before we had 'categorical knot invariants', there was no such distinction —

- from a link in $S^3$, deleting a point gives a link in $B^3$,
  and $L \cong L'$ in $S^3$ iff $L \sim L'$ in $B^3$. 
How do we define the gluing maps?

Given tangles $T_1$, $T_2$, $T_3$, we need maps

$$K_3 (T_1 \! \cup \! T_2) \otimes K_3 (T_2 \! \cup \! T_3) = K_3 (T_1 \! \cup \! T_3)$$

Khovanov homology is "functorial"; every link cobordism gives a linear map.

There is a cobordism:

\[\text{e.g.} \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{example1.png}
\end{array}\]

This gives the desired gluing maps.
For Khovanov homology, we want:
- a particular vector space for each link
- an isomorphism between the vector spaces for each isotopy between links
- if two cobordisms between links are themselves isotopic, we get the same linear map

In $S^3$, cobordisms can be isotopic, but stop being isotopic when we delete a point.

We need to check that the map Khovanov homology gives us is the identity.

(Theorem—there's nothing else in the way of lifting Khovanov homology to $S^3$.)
We need to understand the map $\frac{1}{\bar{g}} \to \frac{1}{\bar{f}}$.

Theorem (mod 2), this map is "diagonal": on each resolution of $\bar{g} \in \bar{\Phi}$, the map is given by the sequence of Reidemeister moves:

$$\frac{1}{\bar{g}} \to \frac{1}{\bar{g}'} \to \frac{1}{\bar{g}''} \to \cdots \to \frac{1}{\bar{g}'''}$$

"Proof" The Reidemeister 3 map $\frac{1}{\bar{X}} \to \frac{1}{\bar{Y}}$

looks like

$$\begin{array}{c}
\frac{1}{\bar{X}} \xrightarrow{\phi} \frac{1}{\bar{Y}} \\
\downarrow \beta \quad \downarrow \gamma
\end{array}$$

and (working mod 2), $\beta$ is just two Reidemeister moves.

Any maps that factor through an "off-diagonal" $\beta$ term contribute zero.

Theorem The sweep around map is the identity.

Putting it all together: mod 2 Khovanov homology gives a disklike 4-category, and the usual recipe gives an invariant of 4-manifolds.