Khovanov homology ~~for~~ 4-manifolds (Davis, Jan 31 2012).

Today I'm going to define a vector-space

$$Kh(W^4),$$

an invariant of 4-manifolds.

In fact, the 4-manifold may have boundary, and a link in the boundary:

$$Kh(W^4; L \subset \partial W).$$

When $W^4 = B^4$, this is just the usual Khovanov homology of L

To do this, I'm going to walk you through a recipe for an invariant of n-manifolds, starting with an "n-category". ②

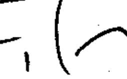
We'll see Khovanov homology can provide a 4-category!

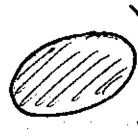
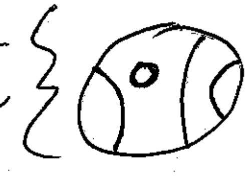
We need one new fact about Khovanov homology but otherwise we're following a standard recipe.

What is a ("disklike") 2-category? ③

- For $k=0,1,2$, a functor from k-balls to Set.

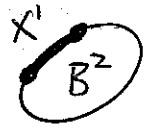
e.g. $F_0(\bullet) = \{\bullet\}$ (the singleton, containing the 0-ball)

F_1() = {  } (finite subsets of the 1-ball)

F_2() = $\mathbb{C}$ {  } (embedded 1-manifolds, transverse to the boundary)

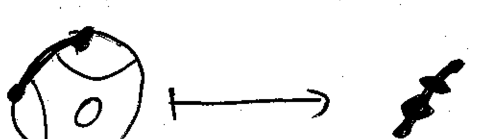
↑ linear combinations, all with the same boundary

- Boundary restrictions: ④

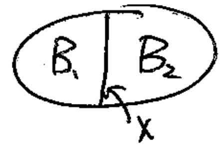
if $X^{k-1} \subset \partial B^k$, e.g. 

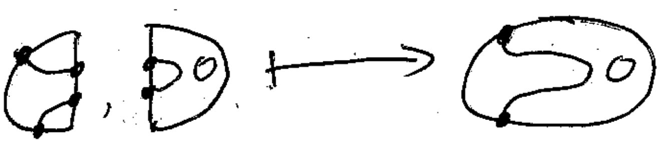
we have functions

$$F_k(B) \xrightarrow{\text{"restriction"}} F_{k-1}(X)$$

e.g. 

- Gluing: whenever we have a ball $B = B_1 \cup_X B_2$

a map $F_k(B_1) \times_{\partial: F(B_i) \to F(X)} F_k(B_2) \to F_k(B)$ 

e.g. 

Usually, we want to impose relations at the top level.

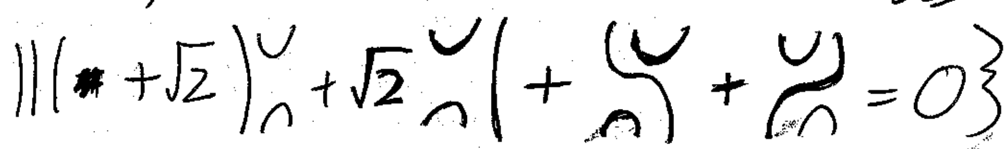
Example $F_2($$) = \mathbb{C}\{\text{embedded 1-manifolds}\} / \sim$

where $\sim$ could be:

$\{$isotopy rel ∂, $O = \delta\}$ (all the vector spaces are finite dimensional)

$\{$as above, and $)(= \,\asymp$, $\delta = 1\}$ (all the vector spaces are 1-dimensional!)

$\{$as above, and $\delta = \sqrt{2}$,


$\}$

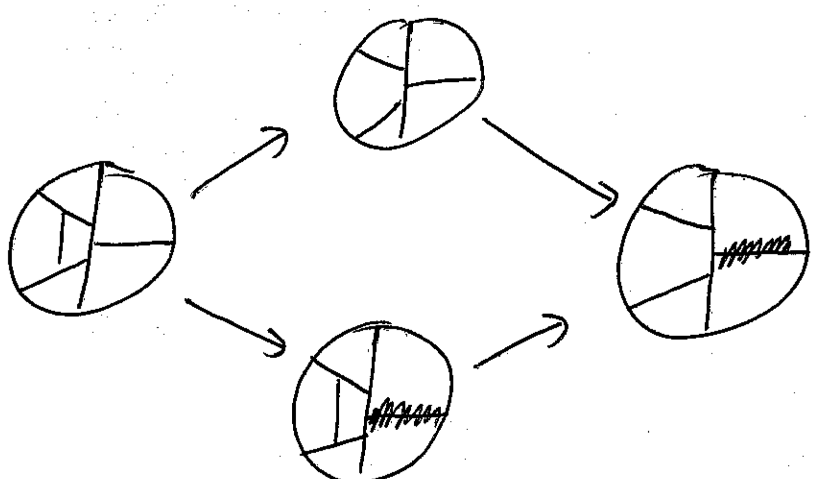
(vector space with $2n$ boundary points dimension 2^{n-1})

1
1
2
4
8
16
32

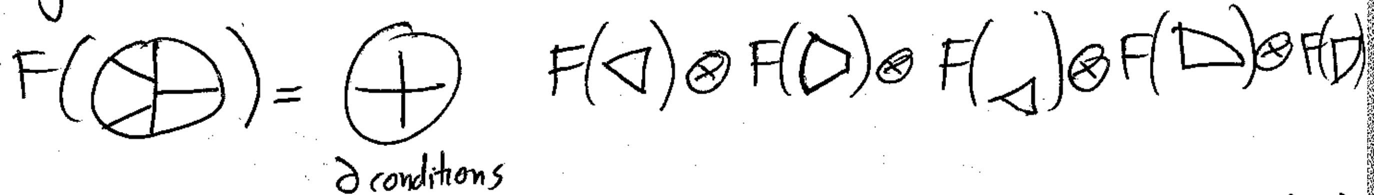
How do we build invariants of n-manifolds?

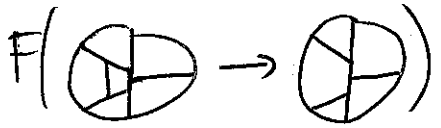
__Definition__ ($\underrightarrow{F}(W^n)$, a vector space)

Consider the poset $D(W)$ of "ball decompositions" of W.



F gives a functor $D(W) \to \mathrm{Vect}$:





is the gluing map

$$\underrightarrow{F}(W) = \lim_{D(W)} F.$$

Let's examine an example:

$F_2(\bigcirc) = \mathbb{C}\{\ldots\}/_{\bigcirc = 1,\ )(= \asymp}$

$\underrightarrow{F_2}\Big($  $\Big) = \mathbb{C}\Big\{$  $\Big\}/_{\bigcirc = 1,\ )(= \asymp}$

embedded 1-manifolds

$= \mathbb{C}\langle H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})\rangle$ (4-dimensional for a torus)

with $F_2(\bigcirc) = \mathbb{C}\{\ldots\}/_{\bigcirc = \sqrt{2}}$, $||| + \frac{1}{\sqrt{2}} \asymp | + \frac{1}{\sqrt{2}} | \asymp + \ldots + \ldots$,

$\underrightarrow{F_2}\Big($  $\Big)$ is 9-dimensional.

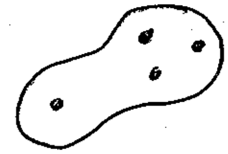
"homology with coefficients in a 'quantum finite group'"

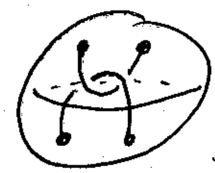
These are examples of Turaev-Viro theories, and indeed this formalism captures "all of" Turaev-Viro, Reshetikhin-Turaev, and Crane-Yetter TQFT.

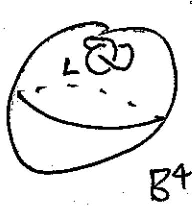
What is the Khovanov 4-category?

$Kh_0(\cdot) = \{\cdot\}$

$Kh_1(\sim) = \{\sim\}$

$Kh_2($  $) = \{$  $\}$

$Kh_3($  $) = \{$  $\}$

$Kh_4($  $) = Kh(L)$ ⇝ a doubly graded vector space

I hid a significant difficulty!

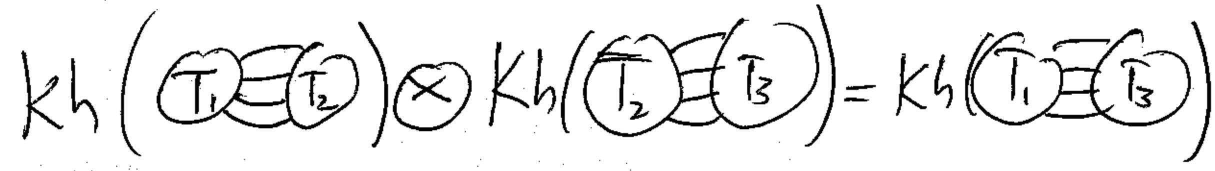
We need $Kh(L \subset S^3)$~~, and in order to be sure~~
~~[illegible]~~
but Khovanov homology as usually discussed has a combinatorial construction that only gives an invariant of a link in B^3.

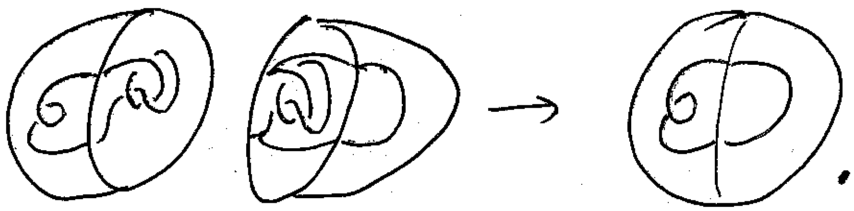
Before we had 'categorical knot invariants', there was no such distinction —

- from a link in S^3, deleting a point gives a link in B^3, and $L \underset{\text{isotopy}}{\sim} L'$ in S^3 iff $L \sim L'$ in B^3.

How do we define the gluing maps?

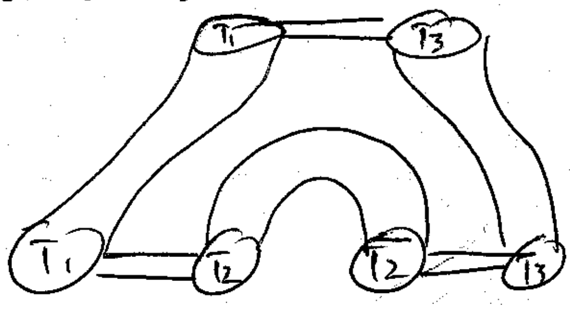
Given tangles $T_1, T_2, T_3,$ we need maps



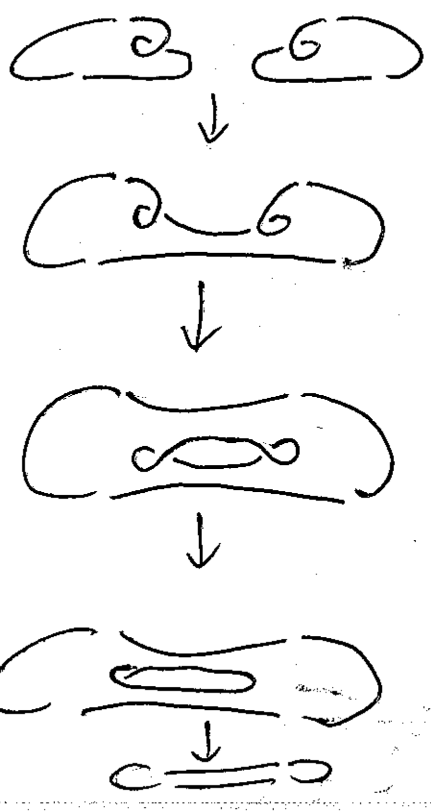


Khovanov homology is "functorial"; every link cobordism gives a linear map.

There is a cobordism:



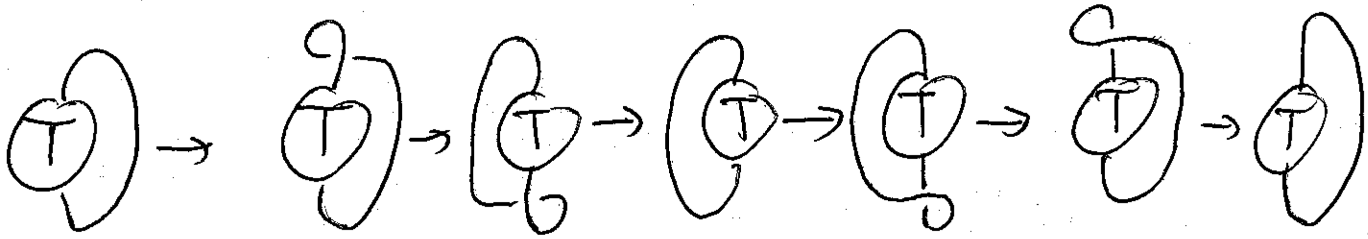
e.g.



This gives the desired gluing maps.

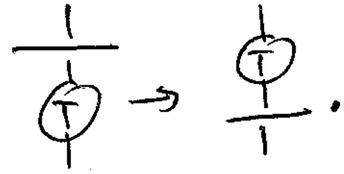
- For Khovanov homology, we want
  - a particular vector space for each link
  - a isomorphism between the vector spaces for each isotopy between links
  - if two cobordisms between links are themselves isotopic, we get the same linear map

In S^3, cobordisms can be isotopic, but stop being isotopic when we delete a point.

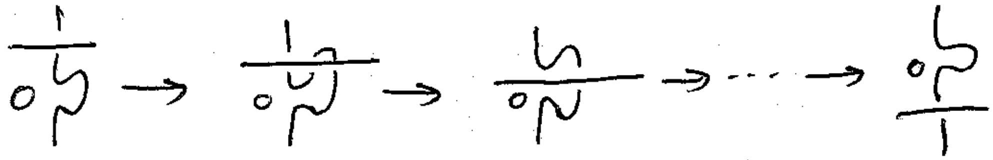


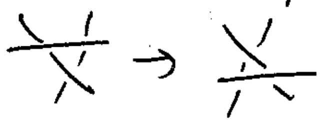
We need to check that the map Khovanov homology gives us is the identity!

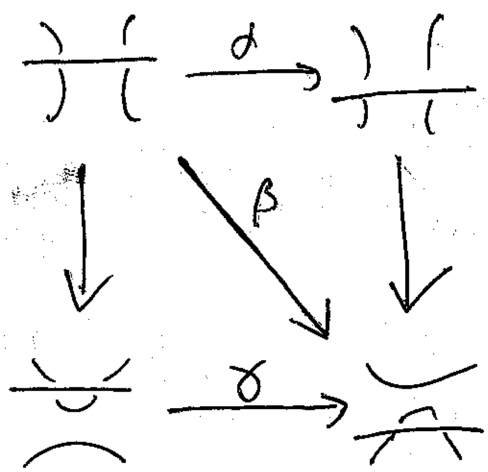
(Theorem—there's nothing else in the way of lifting Khovanov homology to S^3.)

We need to understand the map .

<u>Theorem</u> mod 2, this map is 'diagonal': on each resolution , the map is given by the sequence of Reidemeister moves:



"Proof" The Reidemeister 3 map  looks like



and (working mod 2), γ is just two Reidemeister moves.

Any maps that factor through an "off-diagonal" β term contribute zero.

<u>Theorem</u> The sweep around map is the identity.

Putting it all together: mod 2 Khovanov homology gives a disklike 4-category, and the usual recipe gives an invariant of 4-manifolds.