The Cappell-Shaneson spheres and the s-invariant

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http://tqft.net/counterexample-kiw

Outline

The smooth 4-dimensional Poincaré conjecture

The smooth 4-dimensional Poincaré conjecture is the 'last man standing' in classical geometric topology. It says

Conjecture (SPC4)

A smooth 4-manifold Σ homeomorphic to the 4-sphere, $\Sigma \cong S^4$, is actually diffeomorphic to it, $\Sigma = S^4$.

There's some 'evidence' either way, but I think by now most people think that it's *false*:

Conjecture ($\sim$SPC4)

Somewhere out there, perhaps not far away, there's is a 4-manifold homeomorphic but not diffeomorphic to the 4-sphere.

Outline

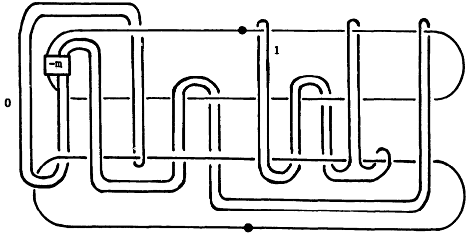
The Cappell-Shaneson spheres

- Consider the 3-torus bundle over S^1 with monodromy $A \in SL(3, \mathbb{Z})$.
- If $\det(I - A) = \pm 1$, surgery on the "zero section" produces a homotopy 4-sphere, denoted W_A.
- Conjugation of A in $GL(3, \mathbb{Z})$ doesn't change W_A. In fact there are finitely many conjugacy classes for each possible trace, and only one when $-4 \leq \mathrm{tr} A \leq 9$.
- We'll consider a family realising every trace:

$$A_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & m+1 \end{pmatrix}$$

Known results

- Kirby-Akbulut conjectured that W_0 was exotic (1985),
- ... but Gompf later showed it was actually standard!
- Gompf also gave a handle presentation for each W_n:



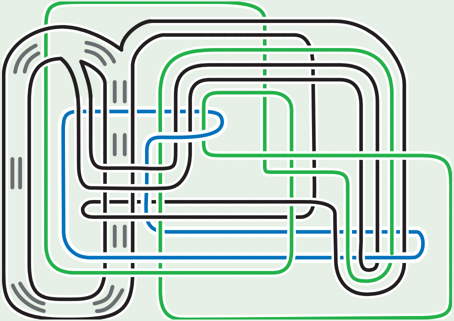
(Unknotted dotted circles indicate 1-handles, knotted circles indicate (framed) attaching curves for 2-handles.)

Localisation

- Sadly, there are no known 4-manifold invariants which can distinguish the Cappell-Shaneson spheres from the standard sphere. (Gauge theory is not good at homotopy spheres.)
- Notice that Gompf's handle presentation has no 3-handles. The 0-, 1- and 2- handles give a homotopy 4-ball, with S^3 boundary.
- The meridians of the 2-handles form a two component link in S^3, which must be slice in the Cappell-Shaneson ball.

Theorem (Freedman-Gompf-Morrison-Walker)

If the two component link L_m



is not slice in B^4, the Cappell-Shaneson ball $\dot{W}_m$ must be exotic.

(Here, the blue component is not 'real'; it represents a 2π twist.)

Outline

What is Khovanov homology?

- Khovanov homology is an invariant of links. It is a doubly-graded vector space, $Kh^{\bullet,\bullet}(L)$.
- The Khovanov polynomial counts the graded dimensions:

$$Kh(L)(q,t) = \sum_{r,j} q^j t^r \dim Kh^{j,r}(L) \in \mathbb{N}[q^{\pm}, t^{\pm}].$$

- The 'euler characteristic' of Khovanov homology is the Jones polynomial:

$$Kh(L)(q,-1) = J(L)(q).$$

The s-invariant gives genus bounds

Other variations of Khovanov homology give more information.

Theorem (Rasmussen)

There is an integer invariant of knots $s(K)$, and

$$|s(K)| \leq g_{slice}(K).$$

Theorem

There is a family of polynomial invariants $f_k(K) \in \mathbb{N}[q^{\pm}, t^{\pm}]$ and

$$Kh(K)(q,t) = q^{s(K)}(q + q^{-1}) + \sum_{k \geq 2} (1 + q^{2k} t) f_k(K)(q,t).$$

A chain of programs (Green/Bar-Natan/Morrison-Shumakovitch) can compute these invariants directly.

Extracting the s-invariant.

Conjecture

Only f_2 is nonzero, and the s-invariant is determined by the Khovanov polynomial, via

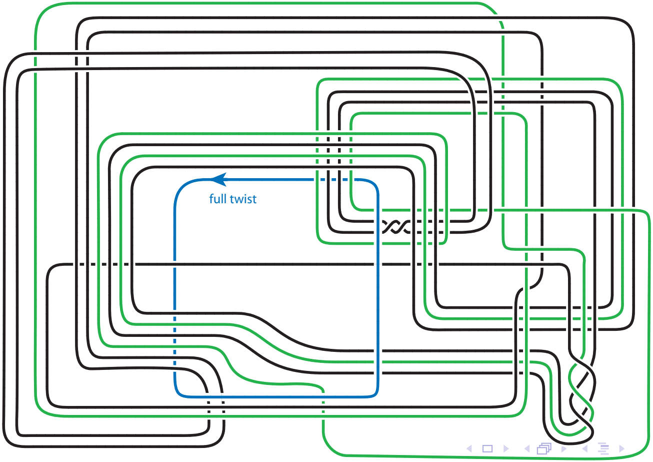
$$q^{s(K)}(q + q^{-1}) = Kh(K)(q, -q^{-4}).$$

- Even without this conjecture, often we can extract $s(K)$ directly from the Khovanov polynomial, by analysing possible decompositions into the polynomials f_k.
- When this works, it is much faster than calculating the actual decomposition.
- It is now possible to compute $s(K)$ for knots K with 50 or more crossings; previously 10-15 was the limit.

Outline

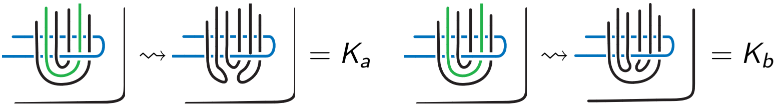
L_1 is huge

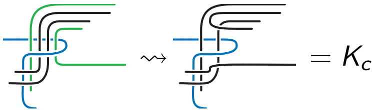
Unfortunately the two component link L_m is huge; even L_1 has ~ 222 crossings; even worse, its *girth* is ~ 24.



Band moves

- Let's take a risk, and look for *band* connect sums that become simpler. If the resulting knot is not slice, the original link can't be either.
- We'll consider the following three bands on L_1, and call the resulting knots K_a, K_b and K_c:





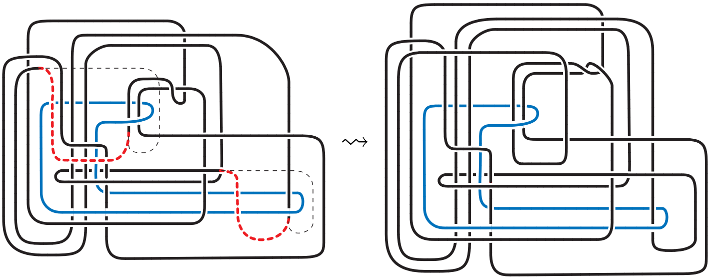
Corollary

If any of $s(K_a)$, $s(K_b)$ or $s(K_c)$ is non-zero, then the smooth 4-dimensional Poincaré conjecture is false.

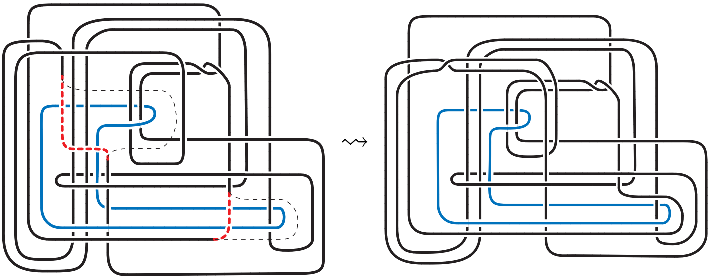
Are these s-invariants computable? In principle "yes":

- We have a combinatorial implementation of the decomposition of Khovanov homology, which gives the s-invariant directly.
- We have a much faster program that just calculates $Kh(K_\bullet)(q,t)$, and it may be possible to extract the s-invariant from this.

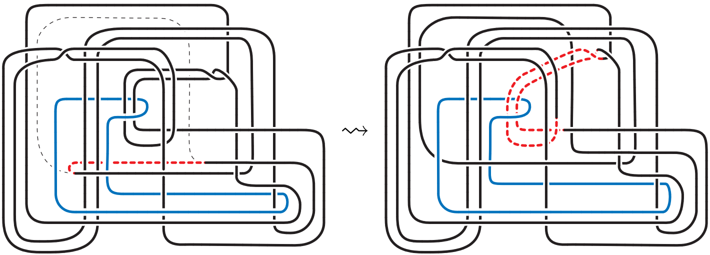
Simplifying K_b, I



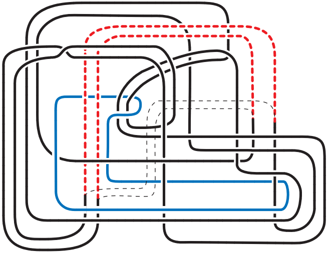
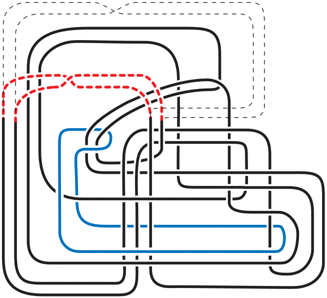
Simplifying K_b, II



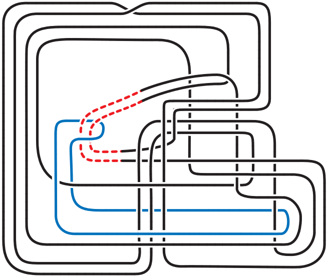
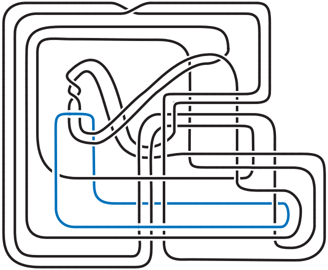
Simplifying K_b, III



Simplifying K_b, IV

$\rightsquigarrow$  $\rightsquigarrow$ 

Simplifying K_b, V

$\rightsquigarrow$  $\rightsquigarrow$  .

The knots K_a, K_b and K_c

- A little work by hand shows K_a is ribbon, and hence slice.
- The Alexander polynomials are all 1; by a theorem of Freedman this means they're all *topologically* slice.
- But how big are they?

	apparent crossings	apparent girth
K_a	67	14
K_b	78	14
K_c	86	16

- This is still scarily large, but perhaps plausible! The biggest computation of the Khovanov polynomial so far is in Bar-Natan's "I've computed $Kh(T(8,7))$ and I'm happy"; that has girth 14 but only 48 crossings. Computations seem to scale at least exponentially in the number of crossings, and *really badly* in the girth.

Improving JavaKh

We started with Jeremy Green's program `JavaKh`, and made many improvements:

New interface
: Progress reports, saving to disk.

Memory optimisations
: Caching, 'bit flipping', paging to disk.

Minimising girth
: Better algorithms to find small girth presentations.

A better algorithm
: Cancelling blocks of isomorphisms, not just one at a time.

At the end, we had something that can compute $Kh(K_b)$; it takes almost a week on a fast machine with 32gb of RAM!

Results for $Kh(K_b)$

$$
\begin{aligned}
Kh(K_b)(q,t) = \\
& q^{-45}t^{-32} + q^{-41}t^{-31} + q^{-39}t^{-29} + q^{-35}t^{-28} + q^{-37}t^{-27} + q^{-37}t^{-26} + q^{-33}t^{-26} + \\
& q^{-35}t^{-25} + q^{-33}t^{-25} + q^{-35}t^{-24} + 2q^{-31}t^{-24} + q^{-33}t^{-23} + 2q^{-31}t^{-23} + q^{-27}t^{-23} + \\
& q^{-33}t^{-22} + 2q^{-29}t^{-22} + q^{-27}t^{-22} + q^{-31}t^{-21} + 3q^{-29}t^{-21} + q^{-25}t^{-21} + q^{-31}t^{-20} + \\
& 3q^{-27}t^{-20} + 2q^{-25}t^{-20} + 4q^{-27}t^{-19} + 2q^{-23}t^{-19} + q^{-27}t^{-18} + 2q^{-25}t^{-18} + 4q^{-23}t^{-18} + \\
& 4q^{-25}t^{-17} + q^{-23}t^{-17} + 3q^{-21}t^{-17} + q^{-19}t^{-17} + 4q^{-25}t^{-16} + 2q^{-23}t^{-16} + 6q^{-21}t^{-16} + \\
& q^{-17}t^{-16} + 4q^{-23}t^{-15} + 5q^{-21}t^{-15} + 3q^{-19}t^{-15} + 2q^{-17}t^{-15} + q^{-23}t^{-14} + q^{-21}t^{-14} + \\
& 8q^{-19}t^{-14} + q^{-17}t^{-14} + q^{-15}t^{-14} + 3q^{-21}t^{-13} + 6q^{-19}t^{-13} + 3q^{-17}t^{-13} + 4q^{-15}t^{-13} + \\
& q^{-21}t^{-12} + 2q^{-19}t^{-12} + 9q^{-17}t^{-12} + 5q^{-15}t^{-12} + 2q^{-13}t^{-12} + 7q^{-17}t^{-11} + 4q^{-15}t^{-11} + \\
& 7q^{-13}t^{-11} + 3q^{-17}t^{-10} + 7q^{-15}t^{-10} + 7q^{-13}t^{-10} + 2q^{-11}t^{-10} + q^{-9}t^{-10} + 8q^{-15}t^{-9} + \\
& 6q^{-13}t^{-9} + 9q^{-11}t^{-9} + q^{-9}t^{-9} + 3q^{-15}t^{-8} + 5q^{-13}t^{-8} + 13q^{-11}t^{-8} + 4q^{-9}t^{-8} + \\
& 2q^{-7}t^{-8} + 5q^{-13}t^{-7} + 8q^{-11}t^{-7} + 9q^{-9}t^{-7} + 5q^{-7}t^{-7} + q^{-5}t^{-7} + 5q^{-11}t^{-6} + 13q^{-9}t^{-6} + \\
& 6q^{-7}t^{-6} + 4q^{-5}t^{-6} + q^{-11}t^{-5} + 8q^{-9}t^{-5} + 11q^{-7}t^{-5} + 8q^{-5}t^{-5} + q^{-3}t^{-5} + 2q^{-9}t^{-4} + \\
& 12q^{-7}t^{-4} + 10q^{-5}t^{-4} + 6q^{-3}t^{-4} + 7q^{-7}t^{-3} + 9q^{-5}t^{-3} + 12q^{-3}t^{-3} + 2q^{-1}t^{-3} + \\
& 9q^{-5}t^{-2} + 12q^{-3}t^{-2} + 8q^{-1}t^{-2} + q^{1}t^{-2} + 3q^{-5}t^{-1} + 7q^{-3}t^{-1} + 15q^{-1}t^{-1} + 5q^{1}t^{-1} + \\
& q^{3}t^{-1} + 3q^{-3}t^{0} + 14q^{-1}t^{0} + 10q^{1}t^{0} + 6q^{3}t^{0} + q^{-3}t^{1} + 5q^{-1}t^{1} + 11q^{1}t^{1} + 10q^{3}t^{1} + 2q^{5}t^{1} + \\
& q^{-1}t^{2} + 8q^{1}t^{2} + 10q^{3}t^{2} + 8q^{5}t^{2} + 2q^{1}t^{3} + 7q^{3}t^{3} + 10q^{5}t^{3} + 5q^{7}t^{3} + 4q^{3}t^{4} + 7q^{5}t^{4} + 6q^{7}t^{4} + \\
& 3q^{9}t^{4} + q^{3}t^{5} + 5q^{9}t^{5} + 2q^{5}t^{6} + 5q^{7}t^{6} + 7q^{9}t^{6} + 4q^{11}t^{6} + 4q^{5}t^{5} + 8q^{7}t^{5} + q^{7}t^{7} + 5q^{9}t^{7} + \\
& 4q^{11}t^{7} + 3q^{13}t^{7} + 2q^{9}t^{8} + 4q^{11}t^{8} + 3q^{13}t^{8} + 3q^{11}t^{9} + 4q^{13}t^{9} + 3q^{15}t^{9} + q^{11}t^{10} + q^{13}t^{10} + \\
& 3q^{15}t^{10} + 2q^{17}t^{10} + q^{13}t^{11} + 2q^{15}t^{11} + q^{17}t^{11} + q^{13}t^{12} + 2q^{17}t^{12} + q^{19}t^{12} + 2q^{17}t^{13} + \\
& q^{21}t^{13} + q^{17}t^{14} + q^{19}t^{14} + q^{21}t^{14} + q^{19}t^{15} + q^{21}t^{15} + q^{23}t^{15} + q^{23}t^{16} + q^{23}t^{17} + q^{27}t^{18}
\end{aligned}
$$

Extracting $s(K_b)$

- There are thousands of possible decompositions of $Kh(K_b)$ of the form

$$Kh(K_b)(q,t) = q^{s(K_b)}(q+q^{-1}) + \sum_{k\geq 2} f_k(K_b)(q,t)(1+q^{2k}t).$$

- Every decomposition gives $s = 0$, so for this knot we find no obstruction.

What next?

Obviously this is disappointing. On the other hand, we've only turned over the first stone.

- Computations for K_c are running right now!
- It looks like L_{-1} might be simpler than L_1, but we've only just started searching for nice bands.
- With present technology (algorithm, implementation, hardware), there are probably several more accessible cases. (But only several.)

Conclusions

- Certain 'local' slice problems for links imply that SPC4 is false.
- Khovanov homology may provide obstructions. Even with recent advances, the calculations are hard, so we use bands to turn the links into smaller knots.
- The first s-invariant we could calculate didn't produce an obstruction. Other bands are running as we speak, and we're about to try other Cappell-Shaneson spheres.