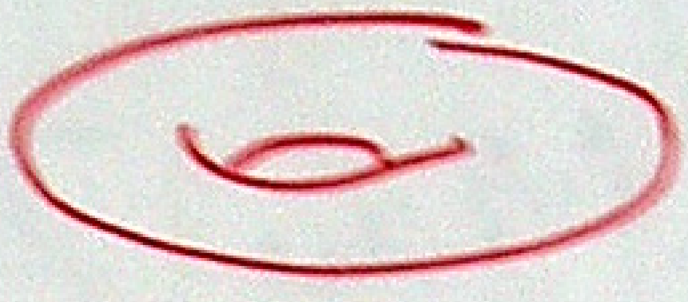
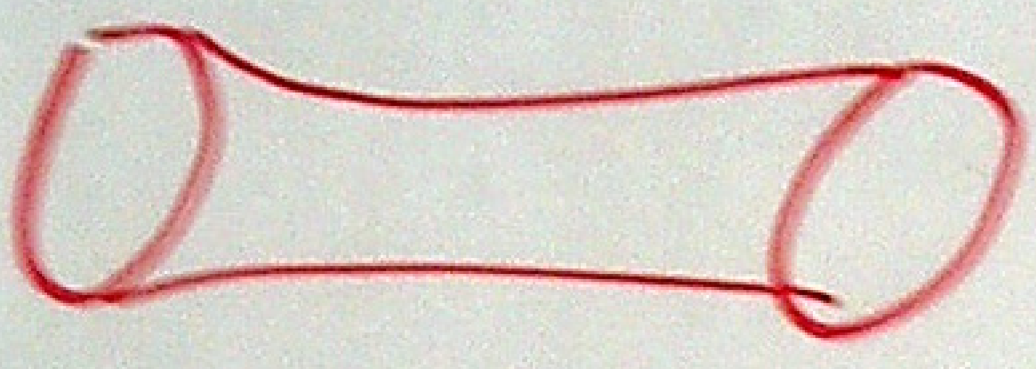
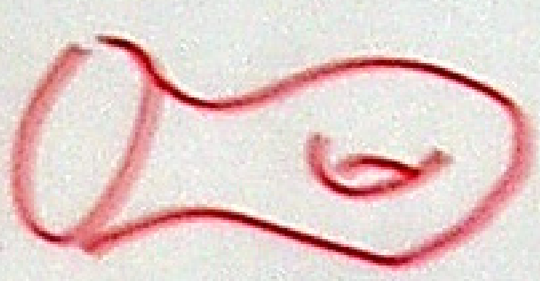
Khovanov homology over $\mathbb{C}$[]

- Recall in Bar-Natan's cobordism model of Khovanov homology, we imposed relations:

 $= 0$  $= 2$

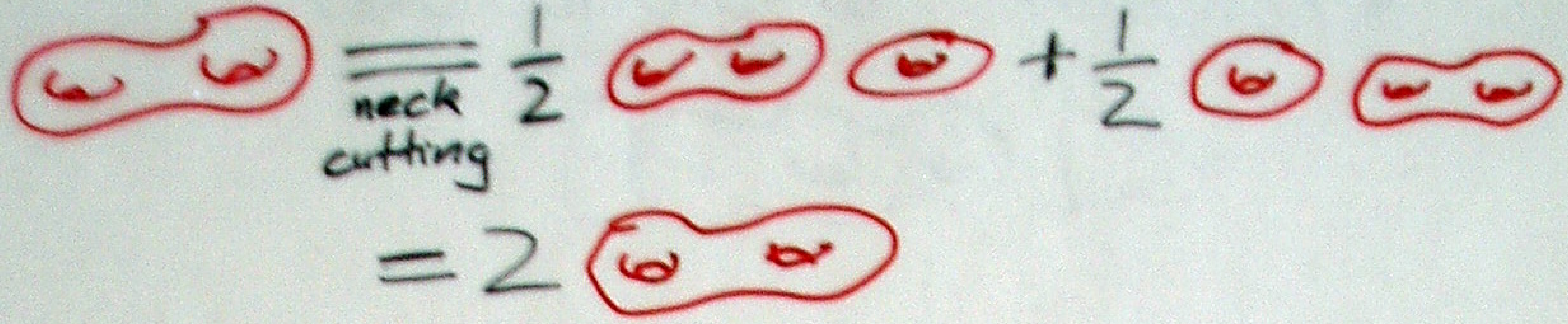
 $= \frac{1}{2}$   $+ \frac{1}{2}$  

and  $= 0$

- The last one is actually <u>unnecessary</u>: it's just there to make Hom spaces finite dimensional over $\mathbb{C}$.

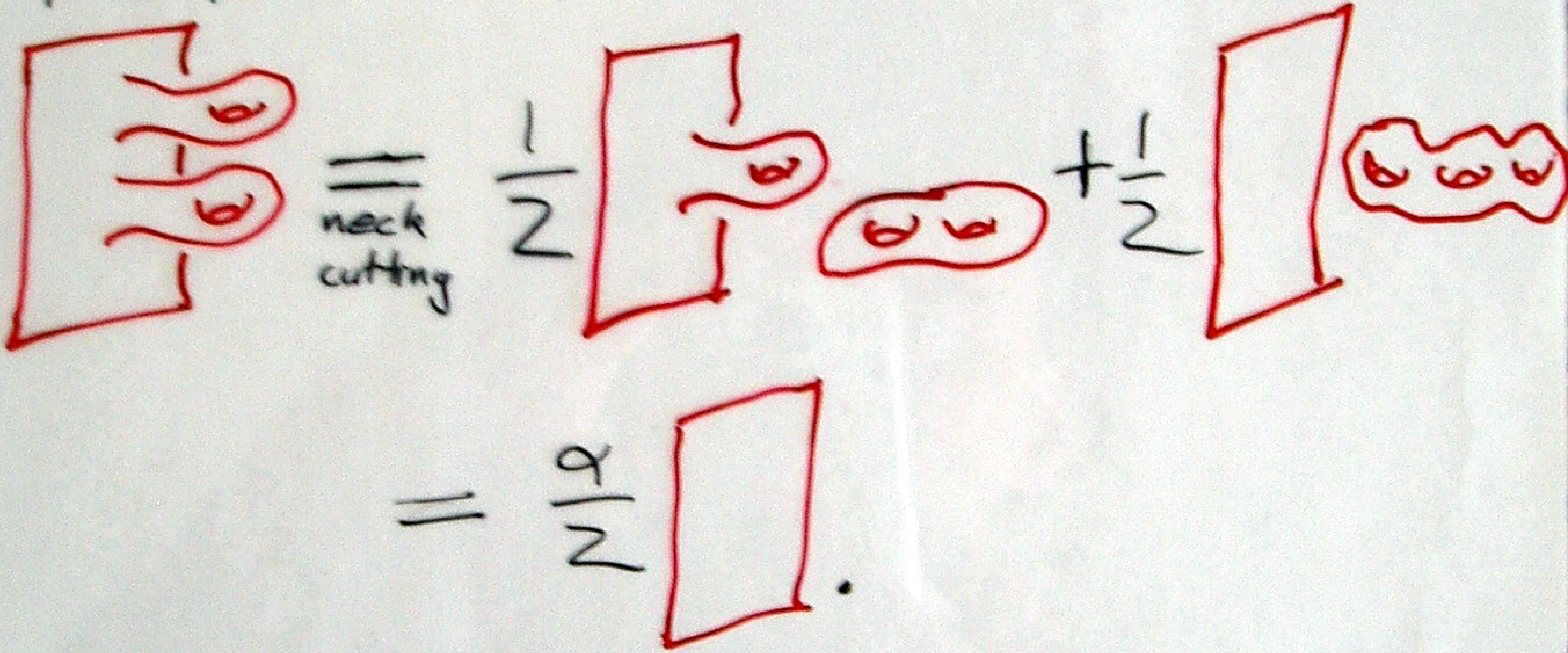
But it's a bad tradeoff — instead, we should write $\alpha =$  and absorb it into the coefficient ring.

- First a little calculation from yesterday:

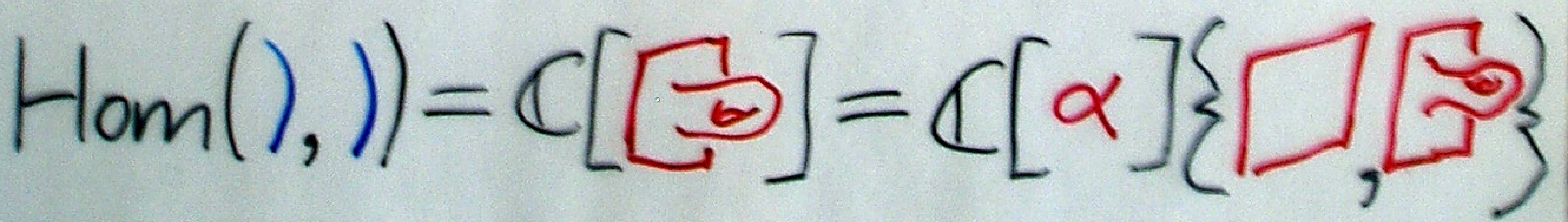


So  = 0.

- Then

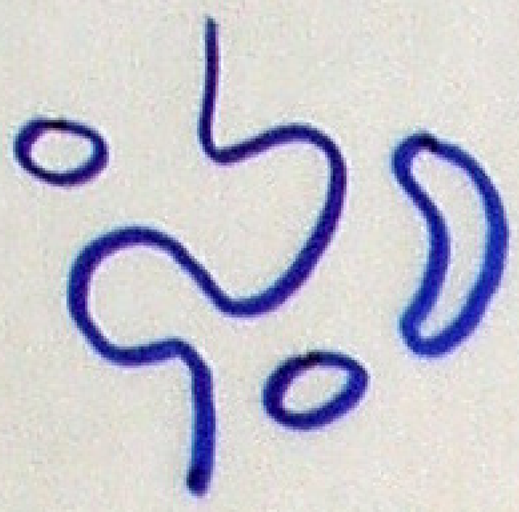


- We can run this both ways, eliminating even numbers of handles for α, or 'reattaching' α to a sheet. Thus we have two descriptions of Hom( ),)):



Cutting open the knot

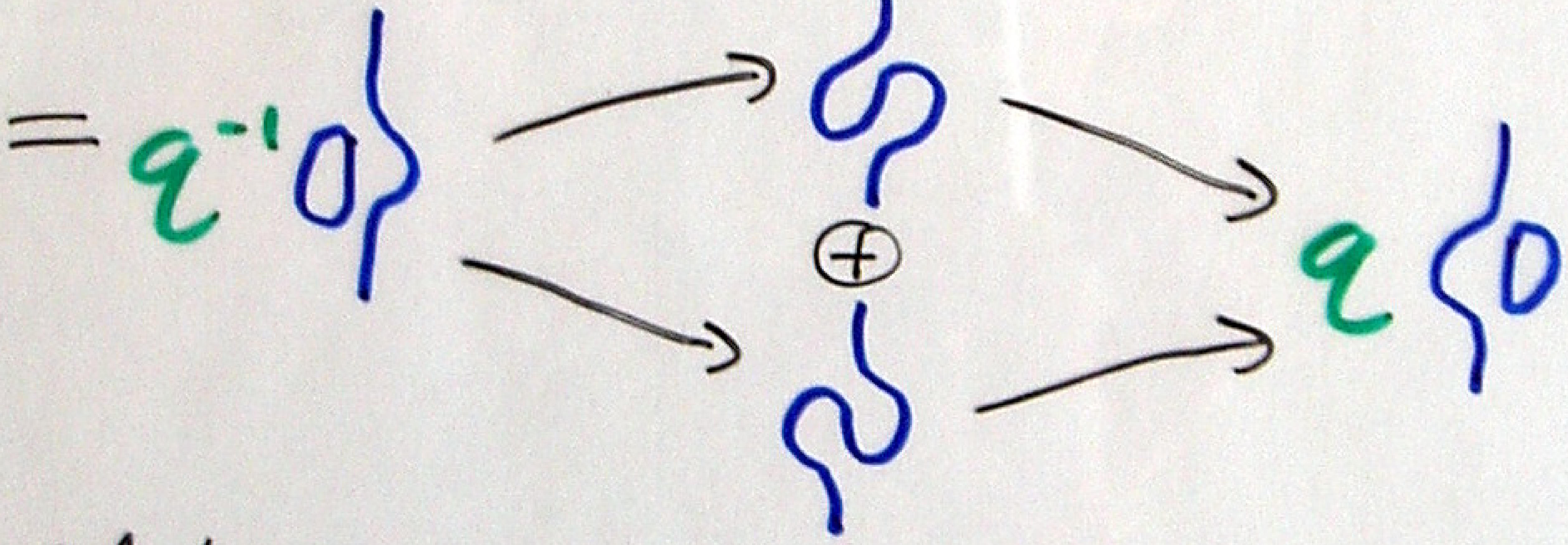
- Before calculating Khovanov homology, cut open the link at a chosen point, producing a "1-1 tangle".
- Each resolution of a 1-1 tangle looks like



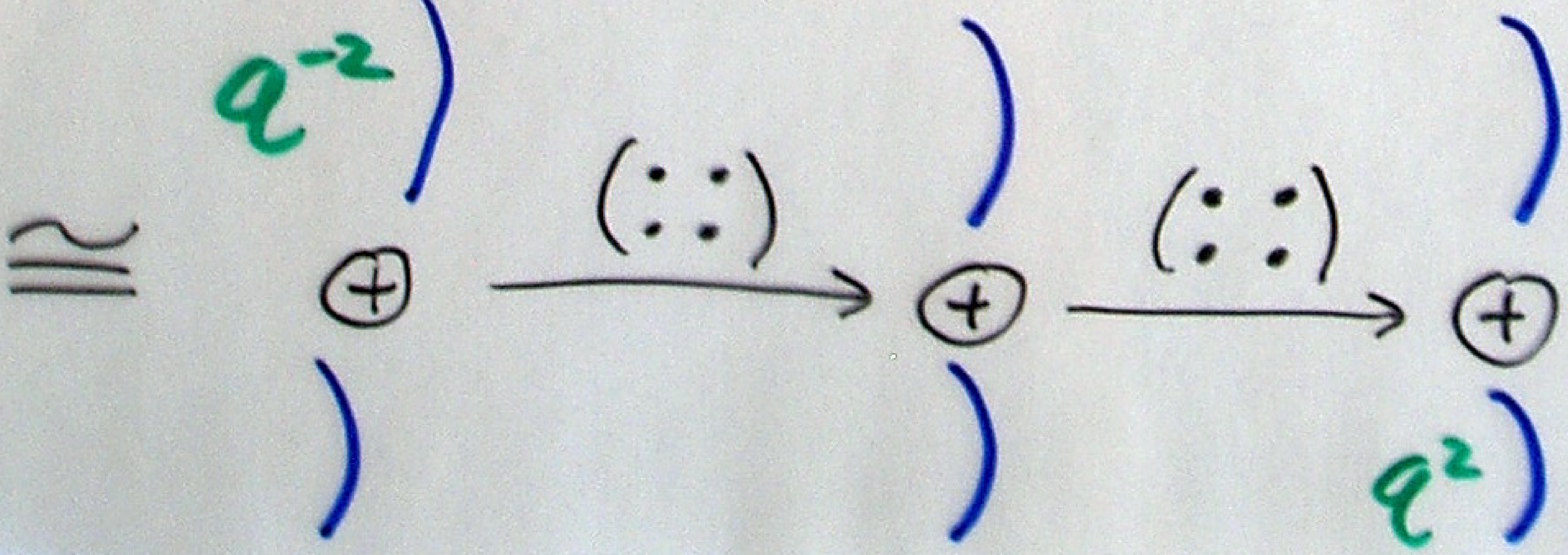
; a single "through strand" with circles on either side.

- We can take the complex for a 1-1 tangle, e.g.

$Kh_{cut}($  $) = Kh($  $)$



and "deloop" it, so every object is a direct sum of strands:



Gaussian elimination over $\mathbb{C}[t]$

- We saw yesterday how to simplify the complex for a tangle by "Gaussian elimination", successively stripping off invertible matrix entries, turning them into contractible direct summands.
- Today, this won't get us as far:  is not invertible. Nevertheless, we can strip off the lowest power of .

If $k \leq l, m \in \mathbb{N}$, and φ is an isomorphism, there's an isomorphism of complexes:

$$\begin{tikzcd}
A \arrow[r, "\begin{pmatrix}\beta\\ \gamma\end{pmatrix}"] \arrow[d] & B \oplus C \arrow[r, "\begin{pmatrix} t^k\varphi & t^l\lambda \\ t^m\mu & \nu \end{pmatrix}"] \arrow[d] \arrow[dr, phantom, "\cong"] & D \oplus E \arrow[r, "(\delta \;\; \epsilon)"] \arrow[d] & F \arrow[d] \\
A \arrow[r, "\begin{pmatrix}0\\ \gamma\end{pmatrix}"] & B \oplus C \arrow[r, "\begin{pmatrix} t^k\varphi & 0 \\ 0 & \nu - t^{l+m-k}\mu\varphi^{-1}\lambda \end{pmatrix}"] & D \oplus E \arrow[r, "(0 \;\; \epsilon)"] & F
\end{tikzcd}$$

- Since our matrix entries are homogeneous elements of $\mathbb{C}[$$]$, they're all either 0 or $^k\varphi$ for some $k \in \mathbb{N}$, $\varphi \in \mathbb{C}^\times$.

- The Gaussian elimination lemma decomposes our complex into direct sums of the following short complexes:

$$E = \,)$$

$$C_n = q^{-2n}\,) \xrightarrow{\quad{}^{n}\quad} \underline{)}$$

each with grading a homological height shifts attached. Thus at the level of complexes,

$$Kh^{cut}_{\circ\circ\circ}(L) \cong \sum_{x,r} a_{x,r}\, q^x t^r E + \sum_{x,r,n\geq 1} b_{x,r,n}\, q^x t^r C_n$$

- Taking homology (implicitly replacing $)$ with $\mathrm{Hom}(\,),\,)\,) = \mathbb{C}[t]$)

$$H^\bullet(\,)\,) = \mathbb{C}[t] \qquad (\text{in height } \bullet = 0)$$

$$H^\bullet\left(q^{-2n}\,) \xrightarrow{\quad{}^{n}\quad} \,)\right) = \mathbb{C}[t]/_{t^n=0}$$

There's one generator in homology for each indecomposable complex, and it's either "free" or "t-torsion".

The fancy version of all this would say

- "$\mathbb{C}[t]$ has homological dimension 1"
- "Kom($\mathbb{C}[t]$-modules) is Krull-Schmidt," i.e. has unique decomposition into indecomposables

Observe E and C_n are not simple.

(exercise, write down all the chain maps between them. which ones are homotopically trivial?)

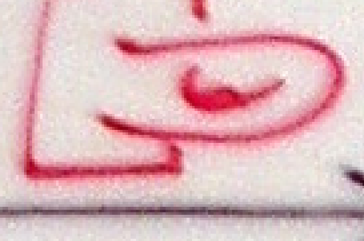
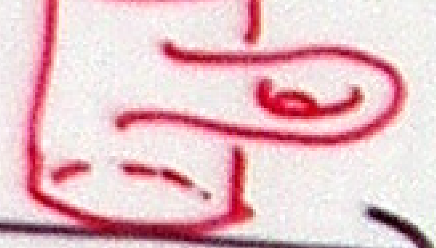
Be careful with "torsion" here: everything is over $\mathbb{C}$, so this is orthogonal to the also-interesting notion of $\mathbb{Z}$-torsion in Khovanov homology. Nothing I'm saying today really works over $\mathbb{Z}$.

Recovering the usual invariant:

- We need to close the cut open invariant, and set $\alpha = 0$.

$$E = \,) \underset{\text{close}}{\rightsquigarrow} \bigcirc \overset{\text{delooping}}{\cong} q\emptyset \oplus q^{-1}\emptyset$$

$$\underset{\text{take Hom}}{\rightsquigarrow} q\mathbb{C} \oplus q^{-1}\mathbb{C}$$

$C_1 = q^{-2}\,)$  $\longrightarrow \,) \underset{\text{close}}{\rightsquigarrow} q^{-2}\bigcirc$  $\longrightarrow \bigcirc$

$$\underset{\text{delooping}}{\cong} \begin{matrix} q^{-3}\emptyset \\ \oplus \\ q^{-1}\emptyset \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}} \begin{matrix} q^{-1}\emptyset \\ \oplus \\ q\emptyset \end{matrix}$$

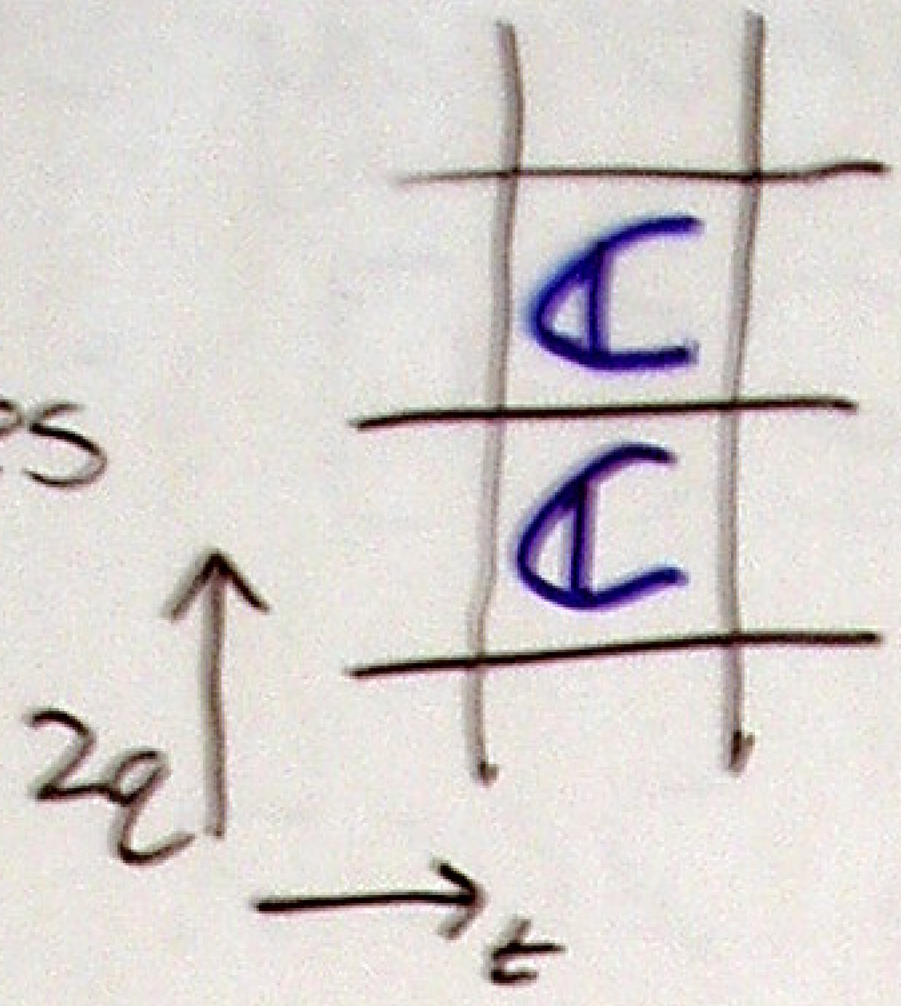
$$\underset{\text{htpy}}{\simeq} q^{-3}\emptyset \xrightarrow{\alpha=0} q\emptyset$$

$$\underset{\text{take Hom}}{\rightsquigarrow} q^{-3}t^{-1}\mathbb{C} \oplus q\mathbb{C}$$

(without setting $\alpha = 0$, we would have just got $q\mathbb{C}[\alpha]/\alpha = 0$.)

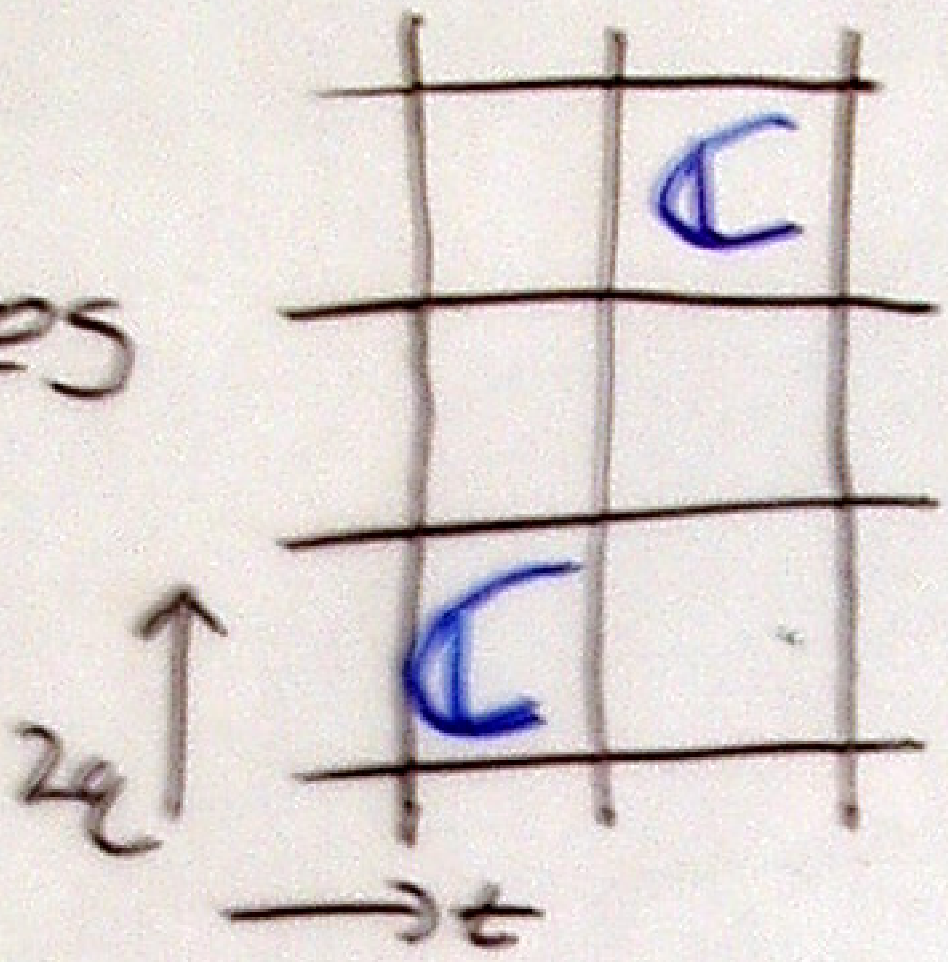
- Thus in classical Khovanov homology

E contributes



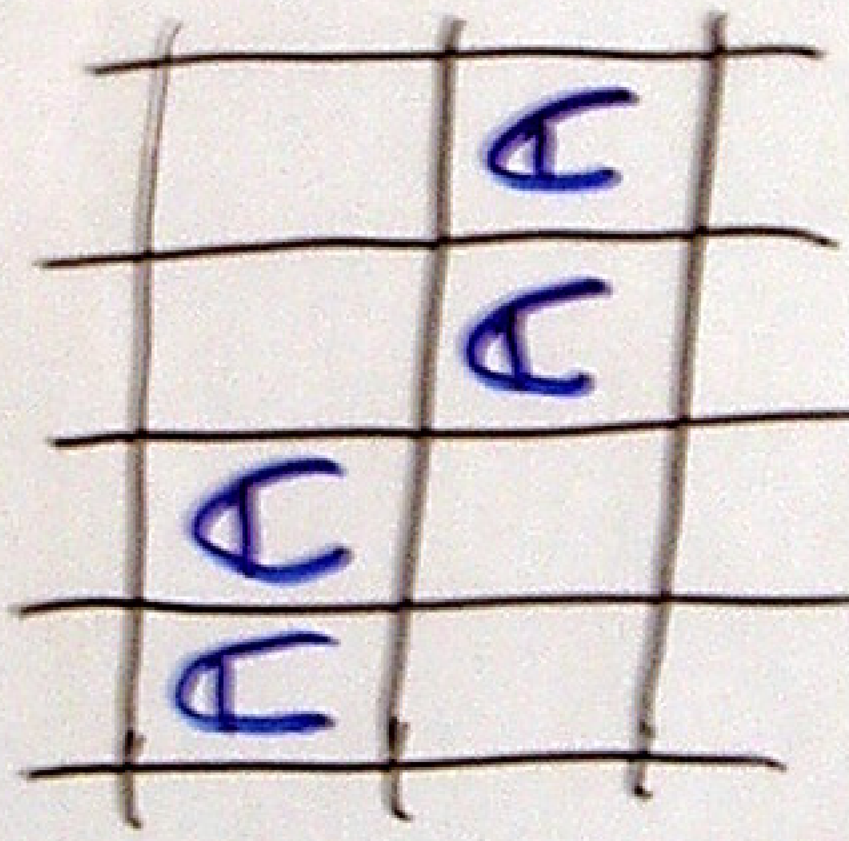
often called an "exceptional pair"

C_1 contributes



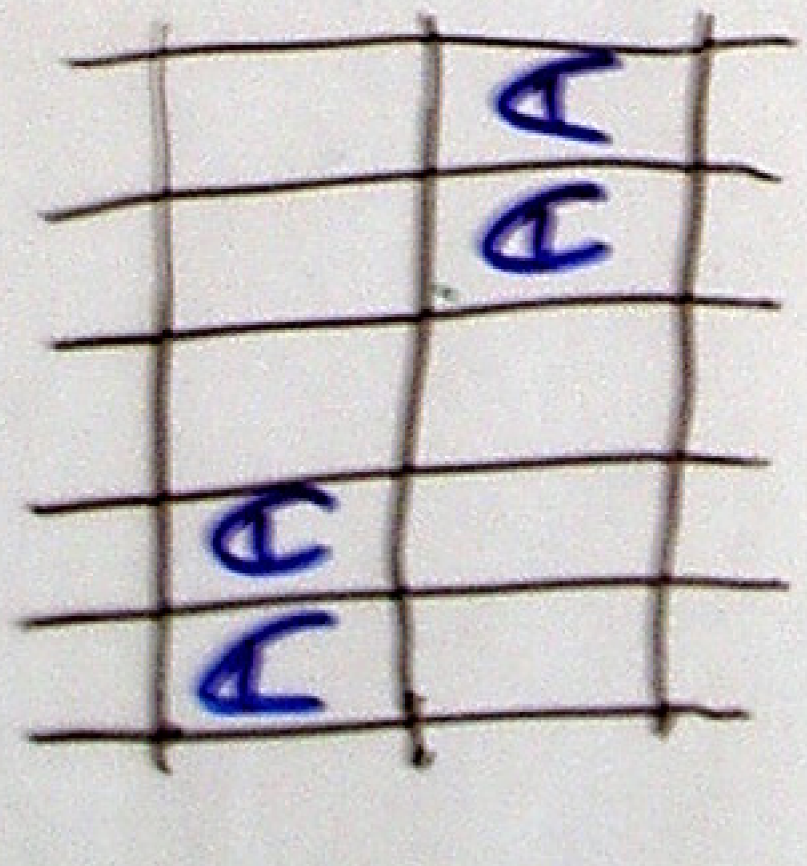
called a "knight's move"

C_2 contributes



, indistinguishable from a pair of knight's moves.

C_3 contributes



, and so on.

(However, as far as I know $C_{\geq 3}$ hasn't yet been seen in the wild.)

Some "phenomenology"

- In the homology of a knot, there's a single copy of E, in homological height 0. It's q-grading is the "Rasmussen s-invariant".

- In alternating knots, only C_1 shows up, and they're all "on diagonal":

$$Kh(K^{\text{alternating}}) = q^s E + \sum_r b_r q^{s+2r} t^r C_1$$

- C_2 first appears in 8_{19}, and is eventually common in non-alternating knots, especially closures of positive braids and torus knots.

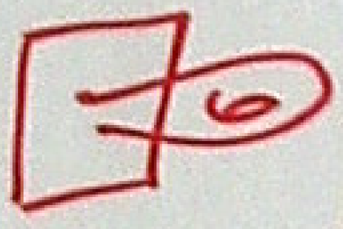
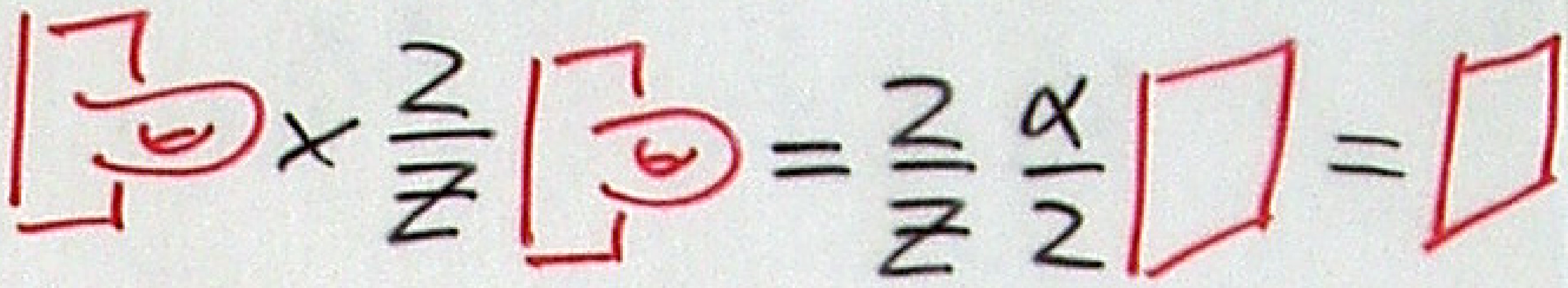
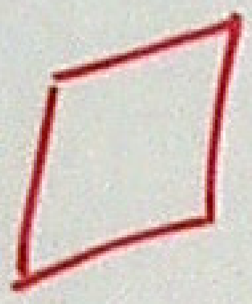
- I haven't seen a C_3 yet, but there's no reason not to expect them?

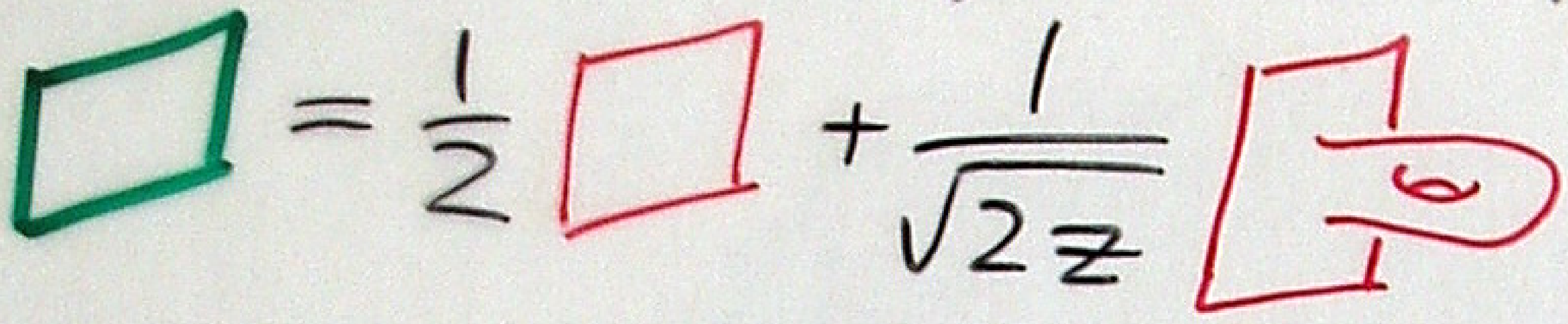
Lee homology

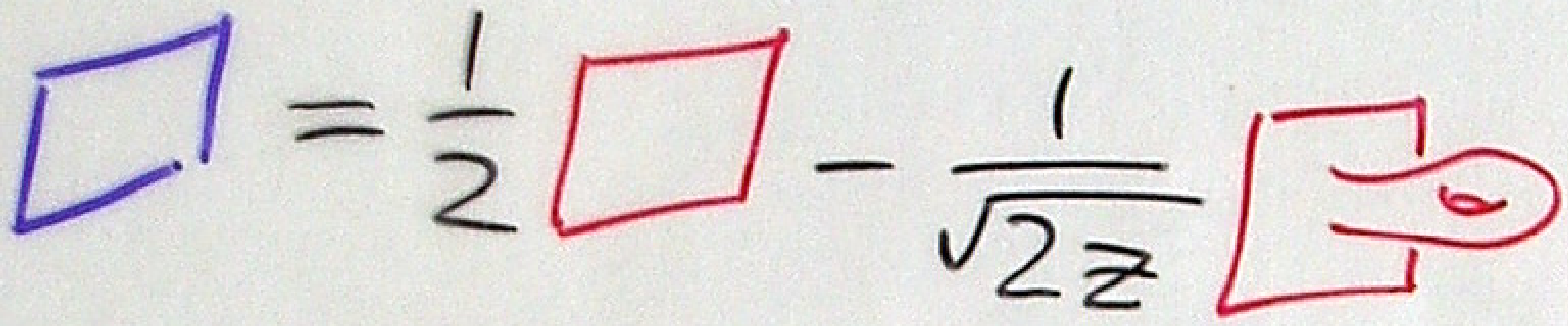
- We can recover Lee homology by setting

 $= Z \in \mathbb{C}^{\times}$.

First, notice

-  is now invertible: 
-  splits into orthogonal idempotents:





- What happens?

  E survives

  each C_n becomes contractible.

- People used to say "Lee homology is boring."

  ($2^{\#\text{components}-1}$ copies of **E**)

  "but boring in an interesting way."

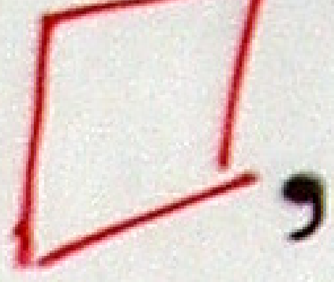
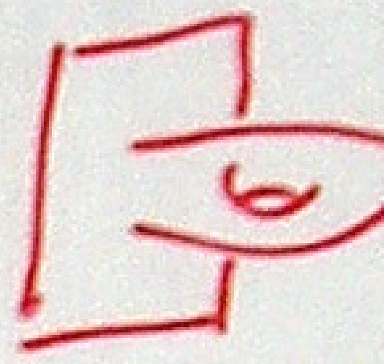
  There's a spectral sequence

  $$E_{*} = CKh, \quad E_2 = Kh, \quad \dots \quad E_{\infty} = Kh_{Lee}$$

- This is easy to understand using Kh!

- After delooping the complex for a H tangle, each matrix entry appearing in a differential is an element of

$$\mathrm{Hom}(\;), \;) \;) = \mathbb{C}[\alpha]\{ \quad , \quad \}$$

We can write the differential as

$$d = d_0 + \alpha d_4 + \alpha^2 d_8 + \cdots$$

- Associated to this is a spectral sequence

$$(CKh, d_0) \rightsquigarrow (H^\bullet(C, d_0), d_4^*) \rightsquigarrow (H^\bullet(H^\bullet(C, d_0), d_4^*), d_8^*) \rightsquigarrow$$

The second page is normal Khovanov homology; at $\alpha = 0$, $d_0 = d$.

The ∞-page is $H^\bullet(CKh, d_0 + d_4 + d_8 + \cdots)$, which is Lee homology; at $\alpha = 1$, $d = d_0 + d_4 + d_8 + \cdots$.

Don't be scared, each step of this spectral sequence really is a complex!

First, write

$$d^2 = d_0^2 + \alpha(d_4 d_0 + d_0 d_4) + \alpha^2(d_8 d_0 + d_4 d_4 + d_0 d_8) + \cdots$$
$$= 0.$$

Next, observe $d_4 : \ker d_0 \circlearrowleft$.

$$d_0 x = 0 \Rightarrow d_0 d_4 x = -d_4 d_0 x = 0.$$

On $\ker d_0 / \operatorname{im} d_0$, $d_4^2 = 0$:

$$d_4^2 x = -d_8 d_0 x - d_0 d_8 x = -d_0 d_8 x \in \operatorname{im} d_0.$$

and so on for higher levels....

Genus bounds from the s-invariant.

- Write $E(L)$ for the "E part" of $Kh(L)$
- A cobordism $\Sigma : L_1 \longrightarrow L_2$ gives a map of grading $\chi(\Sigma)$; $Kh(\Sigma) : Kh(L_1) \to Kh(L_2)$

If Σ is connected, this map is nonzero when restricted to $E(L_1) \to E(L_2)$

(there's some work to do there!)

- For $L_1 = O$ and $L_2 = K$, a knot,

$$Kh(\Sigma) : E \xrightarrow{\ k\ } q^{s(K)} E$$

Calculating gradings, $\chi(\Sigma) = s(K) - 2k$,

so $\chi(\Sigma) \leqslant s(K)$

The same argument with the mirror image says $\chi(\Sigma) \leqslant -s(K)$, so

$$\chi(\Sigma) \leqslant -|s(K)|.$$

- For a cobordism $\Sigma' : \phi \longrightarrow K$ we have

$$\chi(\Sigma') \leqslant -|s(K)| + 1$$

Genus bounds for links, too!

- First swith to the 'framing grading' $t \rightsquigarrow \tau^2 q^{-3}$
- To obtain a genus bound for surfaces inducing an orientation differing in writhe from the original by Δ, look at all copies of E in height $\tau^{-\Delta}$:

$$\boxed{\begin{array}{l} \chi(\Sigma) \leq -\min\limits_{\tau^{-\Delta} q^r E}(r) + 1 \\ \text{and} \\ \chi(\Sigma) \leq \max\limits_{\tau^{-\Delta} q^r E}(r) + 1 \end{array}}$$

- Unfortunately this can be very weak. For example, if $\{r\} = \{-1, 1\}$ we only learn $\chi(\Sigma) \leq 2$.

An Example.

Kh ( L4a1) $= q^3 E + \tau^{-6} C_1 + q^{-1}\tau^{8} E$

- Thus $\chi \leqslant -3+1=-2$ for surfaces inducing the given orientation, which is sharp:

- The other orientation differs in writhe by -8, so we look at the $q^{-1}\tau^{8}E$ term, obtaining $\chi \leqslant 0$, also sharp: