July 16 2012, Maui (1) [redacted]

Connections and Turaev-Viro theory.

The goal for today is to give a nice explanation of some well known facts about connections.

Recall first a graph planar algebra:

Given a graph Γ with positive dimensions d (an eigenvector for the adjacency matrix)

we can define a planar algebra G with

$$G_n = \{\text{loops of length } n \text{ on } \Gamma\}^*$$

and the action of planar tangles by ...

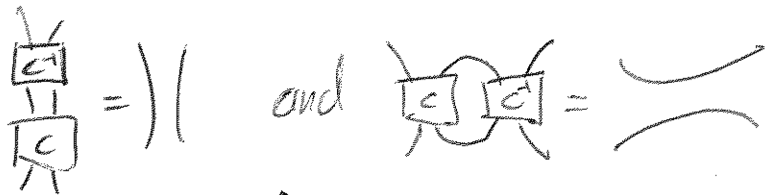
We also have "two-sided" graph planar algebras.

Given a red and blue graph Γ and Γ' on the same set of vertices,

$$\underset{\substack{\uparrow \\ \text{a word in} \\ \text{red \& blue}}}{G_w} = \{w\text{-loops}\}^*$$

and the action as before.

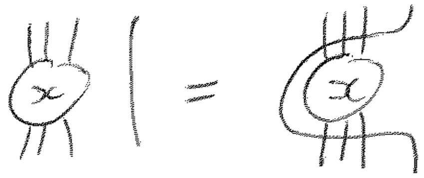
We call an element $c \in G_{rbrb}$ a <u>connection</u> if



(That is, it's reasonable to write c as $\times$)

Note that $\mathcal{G}(\Gamma) \subset \mathcal{G}(\Gamma, \Gamma')$. ②

The "flat subalgebra" $F \subset \mathcal{G}(\Gamma)$ w.r.t. a connection c consists of those elements x with



<u>Theorem</u> Given a planar algebra P, with principal graphs Γ and Γ',

1) there is an embedding $\mathcal{E}: P \hookrightarrow \mathcal{G}(\Gamma)$.
2) there is a connection $c \in$ [redacted] $\mathcal{G}(\Gamma, \Gamma')_{\text{vbvb}}$
3) $F = \mathcal{E}(P)$

What is Turaev-Viro theory?

Given a planar algebra P, we can assign vector spaces to surfaces $A(\Sigma; c) = P\text{-pictures on } \Sigma$

(c: a finite subset of $\partial\Sigma$)

called the "skein modules".

If P is unitary and finite depth, Turaev-Viro provides

1) inner products on $Z(\Sigma; c) = A(\Sigma; c)^*$, the "TQFT Hilbert spaces"
2) elements $Z(M^3) \in Z(\partial M)$ for 3-manifolds

└ these inner products are determined by $\langle a, b\rangle_\Sigma = Z(\Sigma \times I)(a \otimes b)$

Satisfying $Z\Big($  $\Big) = \sum_{e_i} \dfrac{Z\big(\text{ }$  $\text{ }\big)}{\langle e_i, e_i \rangle}$

(e_i: a basis)

(This formula actually uniquely fixes Z in terms of $Z(B^3) \in Z(S^2)$ ~~and the inner product on $Z(S^2)$.)~~ as long as that gives a positive definite inner product on $A(D^2)$)

Actually, we get slightly more:

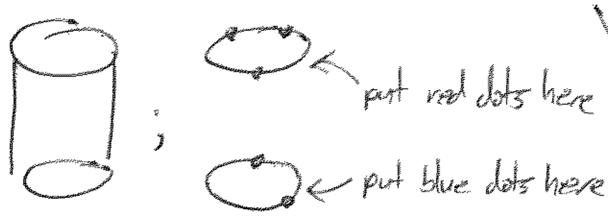
- if $\partial M = \Sigma_{in} \cup \Sigma_{out} \cup \Sigma_{\cancel{\text{decorated}}\ \text{vertical}}$

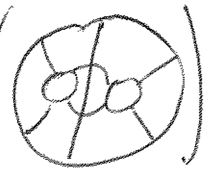
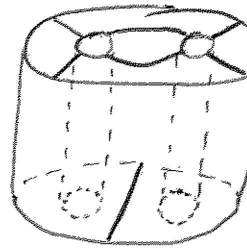
  and $a \in A(\Sigma_{\cancel{\text{decorated}}\ \text{vertical}})$

  we get $Z(M) : Z(\Sigma_{in}) \longrightarrow Z(\Sigma_{out})$

- we can relax the condition that **P** be finite depth, but then we only get $Z(M)$ for **M** buildable using 0- and 1-handles (with no 2-handles!) (or stronger, relative to Σ_{in})

Turaev-Viro theory immediately gives a two-sided planar algebra

$T_\omega = Z\Bigg($  $\Bigg)$

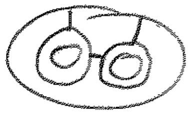
$T\Bigg($  $\Bigg) = Z\Bigg($  $\Bigg)$

thought of as a map from the inner cylinders to the outer cylinders, with top and bottom surfaces decorated.

<u>Lemma 1</u> This planar algebra only depends on some combinatorial data from **P**: in fact

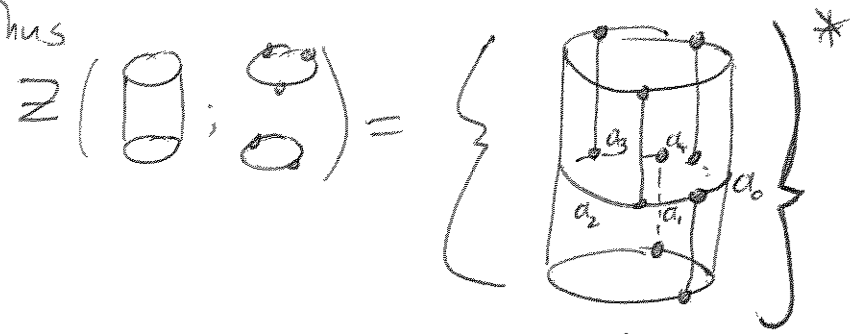
$$T \simeq \mathcal{G}(\Gamma(\mathbf{P})) \quad \text{(ignoring the connection)}$$

<u>Sketch</u> the spine lemma:

for any 'spine' for (Σ, c), e.g. 

there is a basis for $A(\Sigma, c)$ given by labelling edges by simple idempotents, and vertices by intertwiners.

Thus



this is exactly an ω-loop

Verifying that the actions of tangles agree is a calculation using the gluing formula. * (see ⑥)

Next, observe that the 3-ball gives adjoint maps

$$B^3 : T_\omega \rightleftarrows P_{red(\omega)} \otimes P_{blue(\omega)}$$

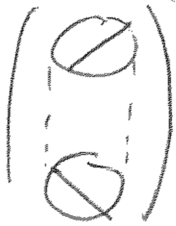
$T_\omega = Z($$)$ $\qquad$ $P_{red(\omega)} \otimes P_{blue(\omega)} = Z($$)$

The graph planar algebra embedding map is

$$\mathcal{E}(S) = B^3(S, \phi) \in T.$$

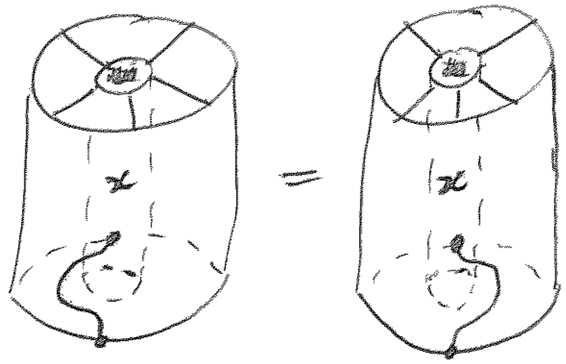
This is a map ~~of~~ planar algebras — since $\mathcal{E}(\phi)(\phi)$ is just $B^3(\phi \in S^2) \neq 0$, this map is injective as soon as P is nondegenerate.

Turaev-Viro theory immediately supplies the connection

$\left(c \in T_{rbrb}\right) = Z\Bigg($  $\Bigg)$

Our remaining task is to understand why $F = \mathcal{E}(P)$.

The flatness condition is now:



so certainly everything in $\mathcal{E}(P)$ is flat.

Some motivation:

$Z(\;)$ is the annular category.

It's a bimodule over itself, by stacking cylinders.

We can decompose it into irreps, and what we're actually proving here is that F and $\mathcal{E}(P)$ are the trivial component of $Z(\;)$ under the action from below.

Now, given a flat $x \in Z(\;)$, define $\alpha \in Z(D^2; c)$ as follows -

for $a \in A(D^2; c)$, write $a = a' \circ \phi$ with $a' \in A(\;)$, and

let $\alpha(a) = x(a')$.

This is well-defined by flatness!

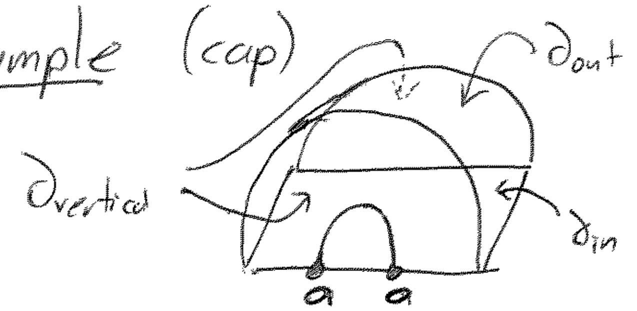
Finally:

Sometimes we have a graph planar algebra $\mathcal{G}(\Gamma, \Gamma')$ and a connection c, but the flat subalgebra doesn't have principal graphs Γ, Γ'. (E.g. complex 4-4 Hadamard matrices give a connection on  and different parameters give each of the affine D_n's).

What can we say there?

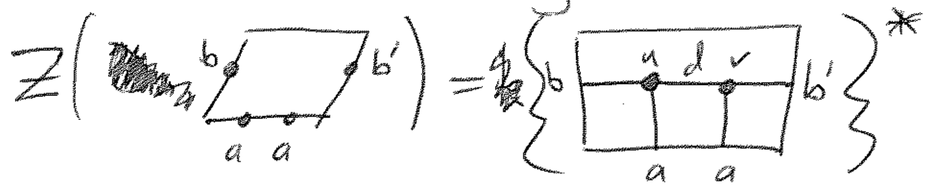
(*) How do we see that the actions of tangles agree?

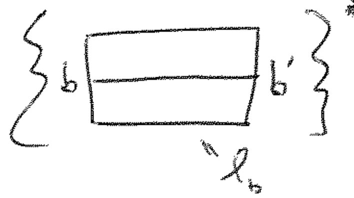
Example (cap)



What is the basis for $Z(\partial_{in})$?

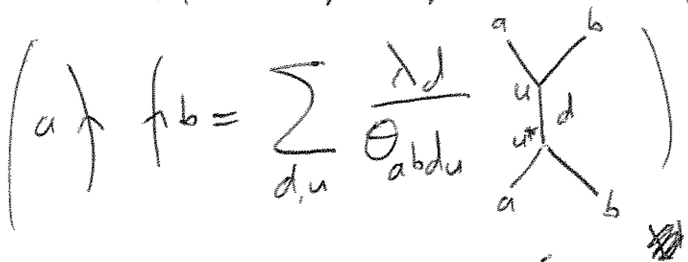
we need to consider arbitrary labels for the undecorated boundary:



$Z(\partial_{out}; b, b') =$  (b must equal b')

$$Z(M_{cap})(X_{duv})(l_b) = X_{duv}(l_b \bullet l_a)$$

(since M_{cap}, undecorated, is just a cylinder)



$$= \delta_{vu^*} \frac{\lambda_d}{\theta_{abdu}}$$

and $Z(M_{cap})(X_{duv}) = \delta_{bb'} \delta_{vu^*} \frac{\lambda_d}{\theta_{abdu}} X_b$

These quantities $\frac{\lambda_d}{\theta_{abdu}}$ give all the structure coefficients in the GPA.
Choosing different conventions for normalising θ's gives e.g. spherical and lopsided coefficients.