Khovanov homology is a gadget associating

- to each link \( L \subset \mathbb{B}^3 \) a chain complex \( \text{Kh}_0(L) \), and
- to each cobordism \( \Sigma : L_0 \to L_1 \)
  a chain map \( \text{Kh}(\Sigma) : \text{Kh}_0(L_0) \to \text{Kh}_0(L_1) \),
such that

- isotopic cobordisms give homotopic maps, and
- there is an exact triangle

\[
\cdots \to \text{Kh}(\Sigma) \to \text{Kh}(\Xi) \to \text{Kh}(\iota) \to \cdots
\]

(For small links it's often computable directly from the exact triangle.)
Moreover there is an extension to tangles:

$\text{Kh}(T)$ is a complex in a category $B_n$

(where $T$ has $n$ boundary points, and

$B_0 = \text{Vec}$ recovers the invariant of links)

and when $T = T_1 \cup T_2$ we can compute

$\text{Kh}(T)$ from $\text{Kh}(T_1)$ and $\text{Kh}(T_2)$. 
Khoranov homology is intrinsically 4-dimensional.

Can we use it to understand 4-manifolds?

We'll assemble all this algebraic structure into a dislikle 4-category from which we immediately obtain TQFT invariants, associating a vector space $\text{Kh}(W^4; \Delta S^3)$ to each 4-manifold (possibly with a link in the boundary).

It generalizes Khoranov homology:

$$\text{Kh}(B^4; L \subset S^3) = \text{Kh}(L).$$
Today:

- Khovanov homology
- Disklike n-categories and n-dimensional TQFT
- Khovanov homology as a disklike 4-category
- Why we need more — the blob complex
A disklike \( n \)-category consists of

- functors \( E_k : \mathcal{E}k \rightarrow \text{Set} \) for \( 0 \leq k \leq n \)
- \( \mathcal{E} \) homeomorphisms

\( E_k(X) \) is "the set of \( k \)-morphisms of shape \( X \)"

- restriction maps \( E_k(X) \rightarrow E_k(Y) \) when \( Y \subseteq X \)
- gluing maps

\[
\begin{align*}
E_k(X_1) \times E_k(X_2) & \rightarrow E_k(X_1 \cup X_2) \\
E_{k-1}(Y) & 
\end{align*}
\]

such that

- at level \( n \), isotopic homeomorphisms act identically
- the gluing maps are strictly associative.
From a dislikable $n$-category we immediately obtain invariants of $n$-manifolds.

Given $M$ an $n$-manifold, consider $\mathcal{D}(M)$, the poset of ball decompositions.

The $n$-category $\mathcal{E}$ gives a functor $\mathcal{D}(M) \to \text{Set}$ and the TQFT sheaf module is

$$A(M; \mathcal{E}) = \operatorname{colim}_{\mathcal{D}(M)} \mathcal{E}$$
Examples

1. $\mathcal{C}_{\mathrm{ren}}(X) = \text{Maps}(X \to T)$, $\mathcal{C}_n(X) = [X \to T]_{\text{rel}}$

and then $\mathcal{A}(M; \mathcal{C}) = [M \to T]_{\text{rel}}$

2. $\mathcal{C}$ a fusion category

$\mathcal{A}(\Sigma^2; \mathcal{C})$ is the Turaev-Viro vector space for a surface

3. $\mathcal{C}$ a modular tensor category,

$\mathcal{A}(M^3; \mathcal{C})$ is the Reshetikhin-Turaev vector space for $\partial M$.

4. Let's try to make Khovanov homology a 4-category!
Khovanov homology as a 4-category

We specify the morphisms on $k$-balls (with boundaries decorated by $k+1$ morphisms):

$$\begin{align*}
Kh_0(\bullet) &= \xi \circ \Xi \\
Kh_1(\sim) &= \xi \sim \Xi \\
Kh_2(\Box) &= \xi \Box \Xi \\
Kh_3(\bullet\bullet\bullet) &= \xi \bullet\bullet\bullet \Xi \\
Kh_4(\bigcirc\bigcirc) &= Kh(\mathbb{L})
\end{align*}$$

What are the gluing maps?

Given $T_1 \# T_2$ and $T_2 \# T_3$ we need a map

$$Kh(T_1 \# T_2) \otimes Kh(T_2 \# T_3) \to Kh(T_1 \# T_3)$$

and there is an obvious cobordism implementing this.
Khovanov homology is defined by
via chain complexes for link diagrams
and chain equivalences for link isotopies.
As we need an action of Diff$(\mathbb{R}^3)$ on $Kh(L)$,
we require slightly more than is usually meant by
"functionality of Khovanov homology".
In particular, the following isotopy in $\mathbb{R}^3$
\[
\begin{align*}
\begin{array}{c}
\includegraphics{image1} \\
\end{array}
\end{align*}
\]
must induce the identity on Khovanov homology, because
it is isotopic to the identity in $S^3$.
We can only prove this with $\mathbb{Z}/2\mathbb{Z}$ coefficients
at this point.
Thus \( Kh(W^4, L) \) is spanned by diagrams \( \beta, \gamma, x \) (a)

\[
\beta \text{ a decomposition of } W \text{ into balls} \\
\gamma \text{ a tangle on each interface between balls} \\
x \text{ an element of the Khovanov homology of the link on each boundary sphere.}
\]

The relations are given by identifying subdivisions:

\[
\begin{align*}
& x_1 \cdot x_2 = \gamma(x_1, x_2) \\
& x_1 \cdot x_2 = \gamma(x_1, x_2).
\end{align*}
\]
There is a gluing formula, as for all TQFT invariants

\[ \text{Kh}(W_1 \cup W_2) = \text{Kh}(W_1) \otimes \text{Kh}(W_2) \]

but this requires understanding the categories

\[ \text{Kh}(M)(\sim) = \text{Kh}(M \times \sim) \].

This is feasible for \( M = (B^3, \cdot) \), but seems too hard otherwise.

What about the exact triangle that made \( \text{Kh} \) computable in the first place?
Given a link $L_x$ in $\mathcal{W}$ and the resolutions $L_x^c$ and $L_x^\gamma$ of one crossing, there is an exact triangle of functors $D(W) \to \text{Vect}$.

$$\cdots \to (Kh, L_x^c) \to (Kh, L_x^\gamma) \to (Kh, L_x^\gamma) \to \cdots$$

but this does not descend to the 4-manifold invariants simply because colimits are not exact.

We propose replacing colimits with homotopy colimits.

This prescription realizes any extended TQFT invariant as the 0-th homology of a chain complex, the "bobo complex".
A $k$-chain of the blob complex consists of
- $k$ balls ("blobs") in $W$, pairwise nested or disjoint
- a compatible decomposition of $W$ into balls
- a labelling of the balls by $n$-morphisms from $C$.

The differential is a sum over
a) ways to forget a blob, and
b) ways to forget an innermost blob, gluing up its contents.

$$d \left( \begin{array}{c}
\circ \vspace{1cm} \\
\end{array} \right) = \left( \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \right) + \left( \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \right) + \left( \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \right)$$

It is easy to see $H_0$ recovers the original colimit.
Now, exact sequences of boundary conditions give a double complex

$$\vdots \rightarrow B_1 K_{h_0}(\sqcup) \rightarrow B_1 K_{h_0}(\wedge) \rightarrow B_1 K_{h_0}(\amalg) \rightarrow \cdots$$

$$\vdots \rightarrow B_0 K_{h_0}(\sqcup) \rightarrow B_0 K_{h_0}(\wedge) \rightarrow B_0 K_{h_0}(\amalg) \rightarrow \cdots$$

Taking homology horizontally gives zero,

taking homology vertically gives the blob homologies,

so we have a spectral sequence starting at the blob homologies and converging to zero.