Khovanov homology for 4-manifolds and the blob complex

"Khovanov homology" is a gadget associating

- to each link $L \subset B^3$ a chain complex $Kh_\bullet(L)$, and
- to each cobordism $\Sigma : L_0 \to L_1$

  a chain map $Kh(\Sigma) : Kh_\bullet(L_0) \to Kh_\bullet(L_1)$.

such that

- isotopic cobordisms give homotopic maps, and
- there is an exact triangle

$$\cdots \longrightarrow Kh(\asymp) \longrightarrow Kh(\times) \longrightarrow Kh()() \to \cdots$$

(For small links it's often computable directly from the exact triangle.)

Moreover there is an extension to tangles:

$Kh(T)$ is a complex in a category $\mathcal{B}_n$

(where T has n boundary points, and

$\mathcal{B}_0 = \mathrm{Vec}$ recovers the invariant of links)

and when $T = T_1 \cup T_2$ we can compute

$Kh(T)$ from $Kh(T_1)$ and $Kh(T_2)$.

Khovanov homology is intrinsically 4-dimensional.

Can we use it to understand 4-manifolds?

We'll assemble all this algebraic structure into a <u>disklike 4-category</u> from which we immediately obtain TQFT invariants, associating a vector space $Kh(W^4; L \subset \partial W)$ to each 4-manifold (possibly with a link in the boundary).

It generalizes Khovanov homology:

$$Kh(B^4; L \subset S^3) = Kh(L).$$

Today:

- Khovanov homology
- disklike n-categories and n-dimensional TQFT
- Khovanov homology as a disklike 4-category
- why we need more — the blob complex

A <u>disklike n-category</u> consists of

- functors $\mathcal{C}_k : \{k\text{-balls}\} \longrightarrow \mathrm{Set}$ for $0 \le k \le n$
  $\circlearrowleft$
  $\{\text{homeomorphisms}\}$
  "$\mathcal{C}_k(X)$ is "the set of k-morphisms of shape X"

[illegible]

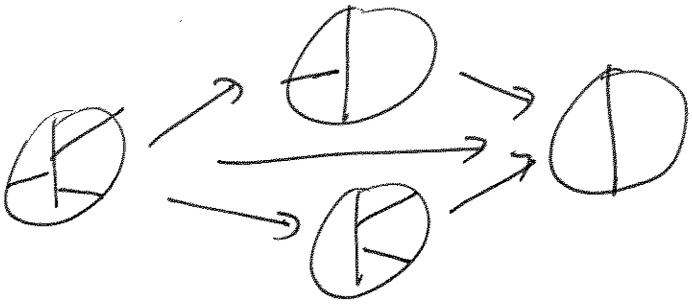
- restriction maps $\mathcal{C}_k(X) \to \mathcal{C}_{k-1}(Y)$ when $Y \subset \partial X$.
- gluing maps $\mathcal{C}_k(X_1) \underset{\mathcal{C}_{k-1}(Y)}{\times} \mathcal{C}_k(X_2) \longrightarrow \mathcal{C}_k(X_1 \underset{Y}{\cup} X_2)$

such that

- at level n, isotopic homeomorphisms act identically
- the gluing maps are strictly associative.

From a <u>disklike n-category</u> we immediately obtain invariants of n-manifolds.

Given M an n-manifold, consider $\mathbb{D}(M)$, the poset of ball decompositions



The [redacted] n-category $\mathcal{C}$ gives a functor $\mathbb{D}(M) \to \mathcal{S}et$ and the TQFT skein module is

$$A(M;\mathcal{C}) = \operatorname*{colim}_{\mathcal{D}(M)} \mathcal{C}$$

Examples

1. $\mathcal{C}_{k<n}(X^k) = \mathrm{Maps}(X \to T)$, $\mathcal{C}_n(X^n) = [X \to T]_{\mathrm{rel}\,\partial}$

   and then $A(M;\mathcal{C}) = [M \to T]_{\mathrm{rel}\,\partial}$

2. $\mathcal{C}$ a fusion category,

   $A(\Sigma^2;\mathcal{C})$ is the Turaev-Viro vector space for a surface

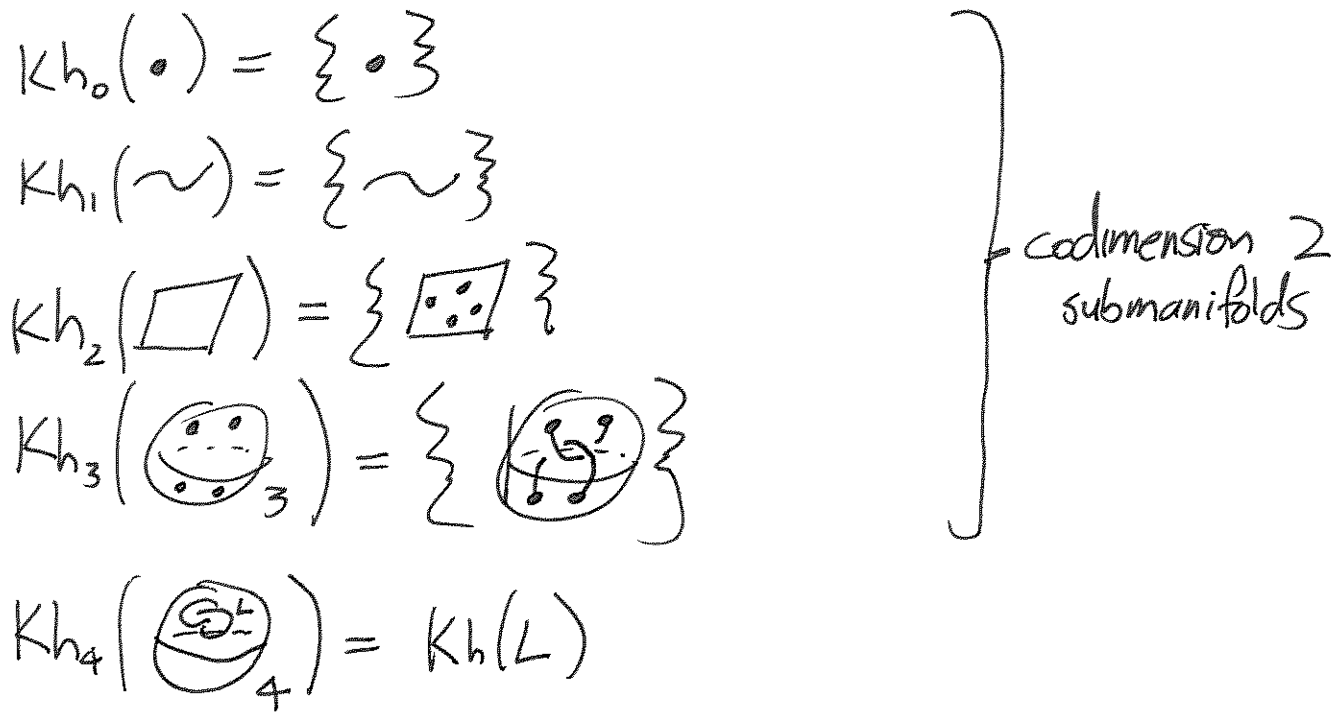
3. $\mathcal{C}$ a modular tensor category,

   $A(M^3;\mathcal{C})$ is the Reshetikhin-Turaev vector space for ∂M.

4. Let's try to make Khovanov homology a 4-category!

Khovanov homology as a 4-category

We specify the morphisms on k-balls
(with boundaries decorated by k-1 morphisms):



What are the gluing maps?
Given $T_1 \# T_2$ and $T_2 \# T_3$ we need a map

$$Kh(T_1 \# T_2) \otimes Kh(T_2 \# T_3) \longrightarrow Kh(T_1 \# T_3)$$

and there is an obvious cobordism implementing this.

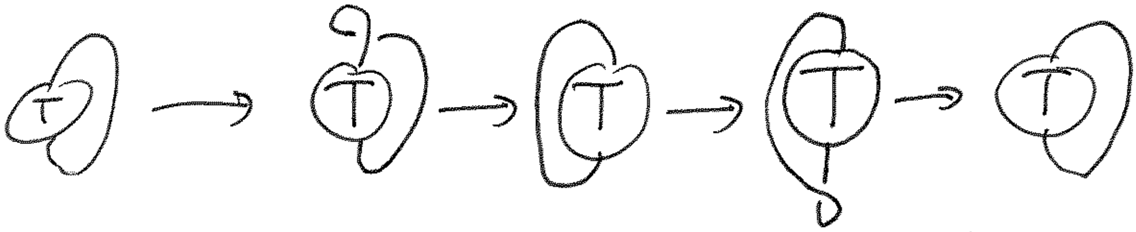
There is a significant technical detail here:

~~define Kh(...)~~

Khovanov homology is defined ~~by via link diagrams~~
via chain complexes for link diagrams
and chain equivalences for link isotopies.

As we need an action of $\mathrm{Diff}(B^4)$ on $Kh(L)$,
we require slightly more than is usually meant by
"functoriality of Khovanov homology".

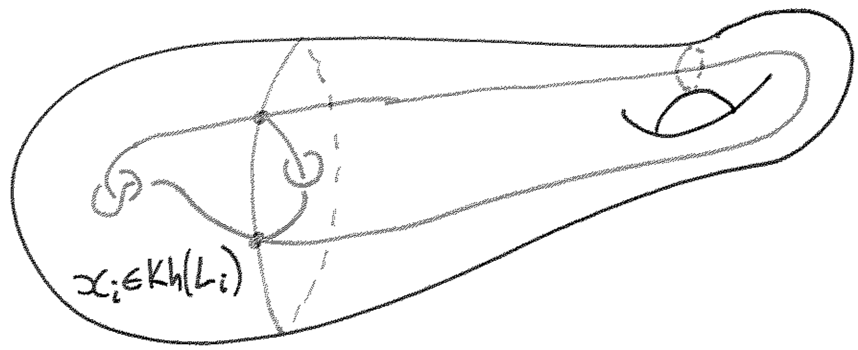
In particular, the following isotopy in B^3



must induce the identity on Khovanov homology, because
it is isotopic to the identity in S^3.

We can only prove this with $\mathbb{Z}/2\mathbb{Z}$ coefficients
at this point.

Thus $Kh(W^4; L)$ is spanned by diagrams $\mathcal{B}, \mathcal{T}, x$

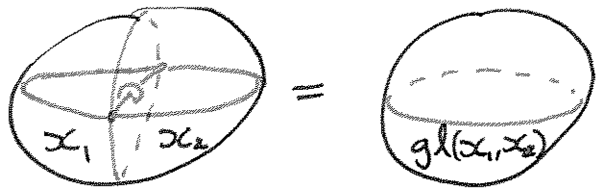


$\mathcal{B}$ a decomposition of W into balls

$\mathcal{T}$ a tangle on each interface between balls

x an element of the Khovanov homology of the link on each boundary sphere.

The relations are given by identifying subdivisions;



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Computations

There is a gluing formula, as for all TQFT invariants

$$Kh(W_1 \cup_M W_2) = Kh(W_1) \otimes_{Kh(M)} Kh(W_2)$$

but this requires understanding the categories

$$Kh(M)(\sim) = Kh(M \times \sim).$$

This is feasible for $M = (B^3, \cdot\,\cdot)$, but seems too hard otherwise.

What about the exact triangle that made Kh computable in the first place?

Given a link $L_{\times}$ in ∂W and the resolutions $L_{)(}$ and $L_{\asymp}$ of one crossing, there is an exact triangle of functors $D(W) \to \mathrm{Vect}$.

$$\cdots \to (Kh, L_{\asymp}) \to (Kh, L_{\times}) \to (Kh, L_{\asymp}) \to \cdots$$

but this does not descend to the 4-manifold invariants simply because colimits are not exact.

We propose replacing colimits with homotopy colimits.

This prescription realizes any extended TQFT invariant as the 0-th homology of a chain complex, the "bbb complex".

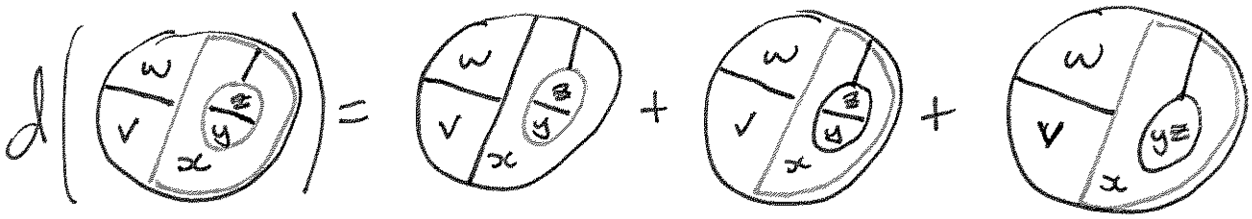
A k-chain of the blob complex consists of

- k balls ("blobs") in W, pairwise nested or disjoint
- a compatible decomposition of W into balls
- a labelling of the balls by n-morphisms from $\mathcal{C}$.

The differential is a sum over

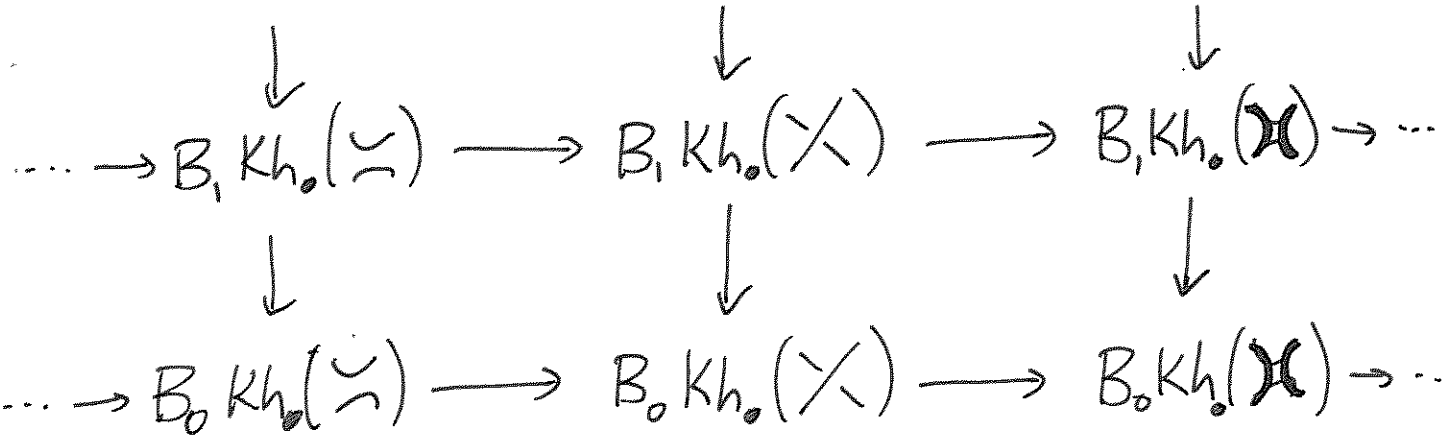
a) ways to forget a blob, and

b) ways to forget an innermost blob, gluing up its contents.



It is easy to see H_0 recovers the original colimit.

Now, exact sequences of boundary conditions give a double complex



Taking homology horizontally gives zero, taking homology vertically gives the blob homologies, so we have a spectral sequence starting at the blob homologies and converging to zero.