Subfactors as quantum symmetries.

- A factor is a von Neumann algebra with trivial centre.

  (Commutative vN algebras are measure spaces; this is the opposite extreme. Every vN algebra is a direct integral of factors.)

- There is a classification:

  type I_n: the $n\times n$ matrix algebras $M_n(\mathbb{C})$

  type I_∞: $B(\mathcal{H})$

  type II_1: There is a trace, with traces of projections taking all values in $[0,1]$

  (no minimal projections, projections are finite)

  type II_∞ ...

  type III ...

We'll be interested in the II_1 world, and in particular in the (unique) hyperfinite II_1 factor R:

$$R = \lim_{n\to\infty} M_2(\mathbb{C})^{\otimes n}$$

(also: $vN(G)$ G countable, amenable icc or group measure space construction for free ergodic actions of countable amenable groups)

A subfactor is merely an inclusion of factors $A \subset B$.

What are subfactors for?

$\Rightarrow$ Subfactors describe 'quantum symmetries'.

First, how do they describe classical symmetries?

Jones (1980) showed that a finite group G has a unique (up to conjugacy) action by outer automorphisms on R

Consider $R \subset R \rtimes G$. (Exercise: the semidirect product is a II_1 factor)

Can we recover G?

Our strategy will be to analyze the bimodules on which these factors act.

① What does $R \rtimes G$ look like as an $R-R$ bimodule?

It is easy to see that it breaks up into $|G|$ simple bimodules.

Each looks just like ${}_R R_R$, but with one of the R actions twisted by the group element.

We can compute the tensor product

$$R^g \underset{R}{\otimes} R^h \underset{R-R}{\cong} R^{gh}$$

② What about $(R \rtimes G) \underset{R}{\otimes} (R \rtimes G)$, as an $R\rtimes G - R \rtimes G$ bimodule?

Essentially, R gets out of the way, and we analyze ${}_G \mathbb{C}\langle G \times G \rangle_G$, and find a simple bimodule for each irrep of G:

$$(R \rtimes G) \underset{R}{\otimes} (R \rtimes G) \cong \bigoplus_{\pi \text{ irrep of } G} \dim\pi \cdot S^{\pi}$$

Given a subgroup $H \subset G$, we can look at

$$\begin{array}{l} A = R \rtimes H \\ \cap \\ B = R \rtimes G. \end{array}$$

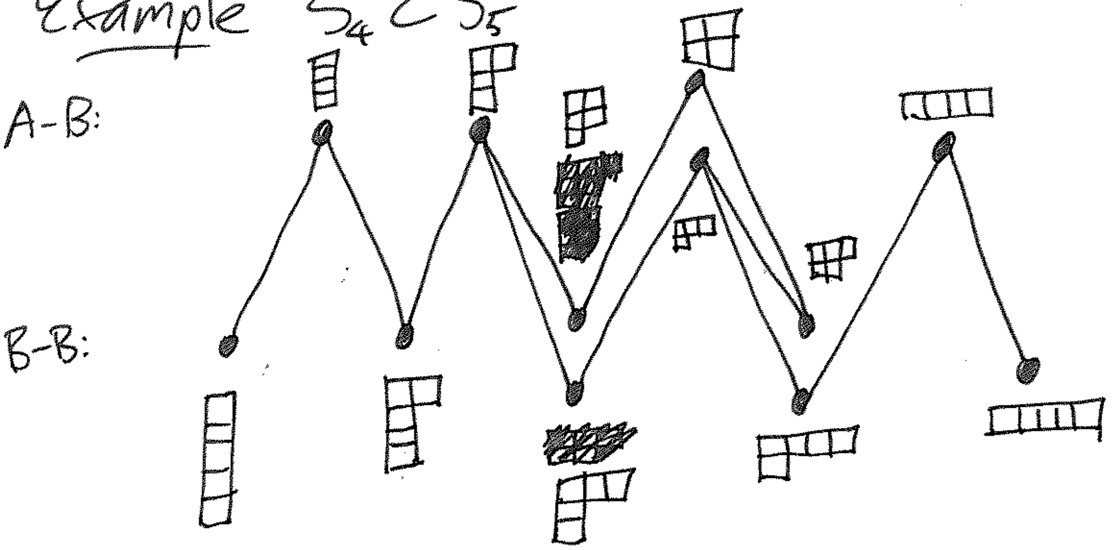
The B-B bimodules give Rep G (with its $\otimes$-structure)

The A-B bimodules give Rep H

We can draw the "principal graph" with these vertices, and an edge between ${}_A X_B$ and ${}_B Y_B$ for each copy of Y inside $B \otimes_A X$.

For the subgroup case, we get the induction restriction graph:

<u>Example</u> $S_4 \subset S_5$



You can think of these subfactors as encoding a transitive group action $G \curvearrowright X$ (with $H = \mathrm{stab}\, x$).

Planar algebras (paragroups, λ-lattices)

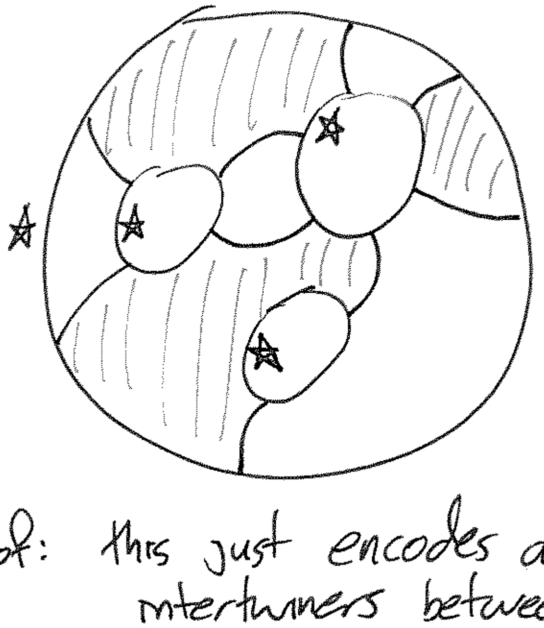
Let's formalize this process of 'extracting the symmetries'.

Given a subfactor $A \subset B$, define vector spaces

$$P_{n,+} = \mathrm{Hom}_{A-A}\big(A \longrightarrow \underbrace{B \otimes_A \cdots \otimes_A B}_{n \text{ tensor factors}}\big)$$

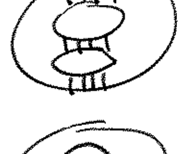
and $$P_{n,-} = \mathrm{Hom}_{B-B}\big(B \longrightarrow \underbrace{B \otimes_A \cdots \otimes_A B}_{n+1 \text{ tensor factors}}\big)$$

Theorem (Jones) These vector spaces form a planar algebra, i.e. there is an action of shaded planar tangles:



$$: P_{2,+} \otimes P_{3,-} \otimes P_{1,+} \to P_{3,+}$$

Proof: this just encodes all the natural operations on intertwiners between bimodules.

e.g. $P_{2n,+} \cong \mathrm{End}_{A-A}(B^{\otimes n})$ and  is multiplication.

 is trace

...

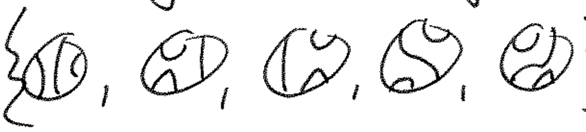
Theorem (Popa)

From a spherical, unitary shaded planar algebra, we can reconstruct a subfactor.

The planar algebra is a complete invariant of amenable subfactors of R.

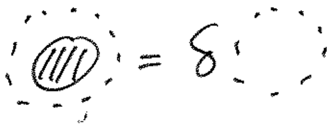
The Temperley-Lieb planar algebra

$TL_{n,+} = \{$crossingless matchings on $2n$ points$\}$

e.g. $TL_{3,+} = \{$  $\}$

sits inside every plana algebra.

The index $[B:A]$ is (equivalently)

1. if  in $P_{0,+}$, δ^2
2. $\|\Gamma\|^2$ (the square of the largest eigenvalue of the principal graph's adjacency matrix)
3. the Murray-von Neumann dimension of B as an A-module.

Clearly $[R \rtimes G : R \rtimes H] = [G:H]$.

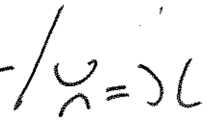
<u>Theorem</u> (Jones)

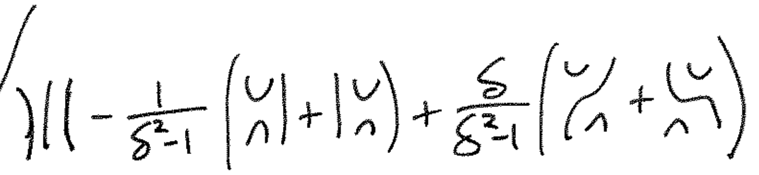
$$[B:A] \in \{4\cos^2 \pi/n\}_{n\geq 3} \cup [4, \infty)$$

<u>Sketch proof</u> Otherwise, the TL subalgebra can't be unitary.

There's a beautiful ADE classification (Ocneanu, Izumi, Kawahigashi) for $[B:A] < 4$.

For the A series, there's nothing but TL, and in fact it's a quotient.

e.g. $A_1 = TL/$  (every vector space in the planar algebra is 1-d)

$A_2 = TL/$ 

⋮

For D and E, we have to add new generators.

There's an "affine ADE" classification at index 4. (Popa)

We're been trying to understand what else is out there.

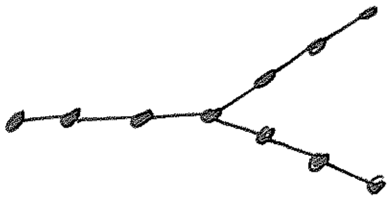
* Examples from finite group data, all at integer index, not super exciting
* Examples coming from quantum groups at roots of unity (well understood, includes A series below index 4)
* "quantum subgroups" of the above (e.g. DE cases below 4)
* There are also <u>sporadic</u> examples, which have only been discovered through classification projects.

-Bigelow-Haagerup-Asaeda

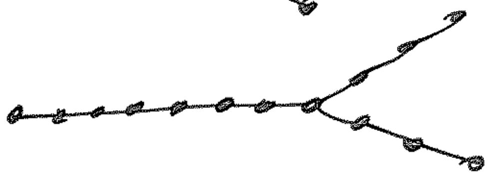
<u>Theorem</u> (M-Snyder-Peters-Penneys-Tener-Jones-Izumi-Calegari-Ostrik)

There are 10 subfactors (of R) with index between 4 and 5, besides TL (at every index), coming in 5 pairs:

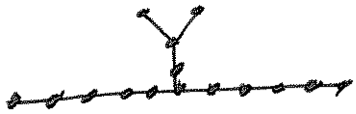
"Haagerup"



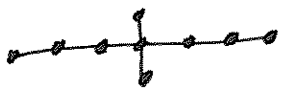
"Extended Haagerup"



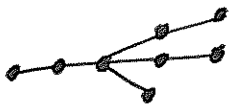
"Asaeda-Haagerup"



"3311"



"2221"



How do we prove such a theorem?

1) Graph combinatorics and geometric number theory, to constrain the principal graphs

2) Representation theory and "skein theory" (=2d topology) to constrain the possible planar algebras

3) "Applied algebraic geometry" to produce candidate planar algebras as subalgebras of 'standard' ones

4) More skein theory to prove the candidate is what you'd hoped.