4-manifold invariants from Khovanov homology.

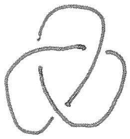
- Today I'll define $\underrightarrow{Kh}(W^4)$, a vector-space valued invariant of a smooth 4-manifold

- There's also a relative version $\underrightarrow{Kh}(W^4; L)$ for L a link in ∂W.

- It's a natural extension ~~of~~ Khovanov homology
$$\underrightarrow{Kh}(B^4; L) \cong Kh(L).$$

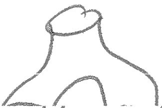
What is Khovanov homology?

Khovanov homology is a 'categorical' knot invariant

$\Big\{$  $\Big\}$ $\longrightarrow \mathrm{gr}^2\mathrm{Vec}$

links

$\circlearrowright \qquad\qquad \circlearrowright$

$\Big\{$  $\Big\}$ $\longrightarrow$ {linear maps}

categorifying the Jones polynomial

$$J(L) = \sum_{i,j} (-1)^i q^j \dim Kh^{i,j}(L).$$

It is defined combinatorially:

$$\begin{array}{ccccc}
\{\text{link diagrams}\} & \xrightarrow{Kh} & \left\{\begin{array}{c}\text{chain complexes in}\\ \text{graded vector spaces}\end{array}\right\} & \xrightarrow{H_*} & \left\{\begin{array}{c}\text{doubly graded}\\ \text{vector spaces}\end{array}\right\} \\
\circlearrowright & & \circlearrowright & & \circlearrowright \\
\left\{\begin{array}{c}\text{Reidemeister and}\\ \text{Morse moves}\end{array}\right\} & \xrightarrow{Kh} & \{\text{chain maps}\} & \xrightarrow{H_*} & \{\text{linear maps}\} \\
\circlearrowright & & \circlearrowright & & \\
\{\text{movie moves}\} & \xrightarrow{Kh} & \{\text{chain homotopies}\} & &
\end{array}$$

There's a very general construction called

__topological quantum field theory__

{n-categories with duals} $\longrightarrow$ {vector space valued invariants of n-manifolds}

(sometimes also numerical invariants of (n+1)-manifolds)

We could build a 4-category from Khovanov homology, then follow this recipe.

Today I'll explain a special case, using __lasagna algebras__

(~3 slides, which you can find in several of my previous talks!)

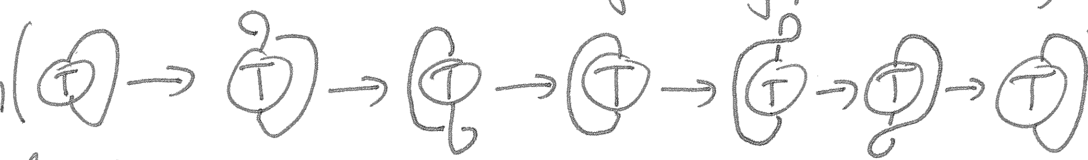
Khovanov homology natively lives in B^3;
we need to define $Kh(L \subset S)$. ← a 3-sphere

<u>Definition</u> $Kh(L \subset S)$ is the flat sections of

$$\{Kh(L \subset S \setminus \{x\})\}$$
$$\downarrow$$
$$S \setminus L$$

with parallel transport along $\gamma \subset S \setminus L$ given by the cobordism $L \times I \subset S \times I \setminus \text{graph}(\gamma)$.

<u>Theorem</u> this bundle is flat (i.e, there are nonzero flat sections, and evaluating at any point is an iso)

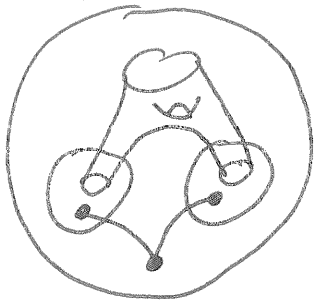
iff $Kh\Big($  $\Big)$

is the identity

<u>Theorem</u> for Khovanov homology mod 2, this is the case.

⇒ we don't know if this is true over the integers! :(

We define the action of the lasagna diagrams by picking arcs avoiding Σ:



gives a cobordism

$$\sqcup L_i \to L_0$$

in $B_0 \setminus \cup B_i \setminus \gamma \cong B^4$

so Khovanov homology provides a linear map.

(The above theorems guarantee independence of γ.)

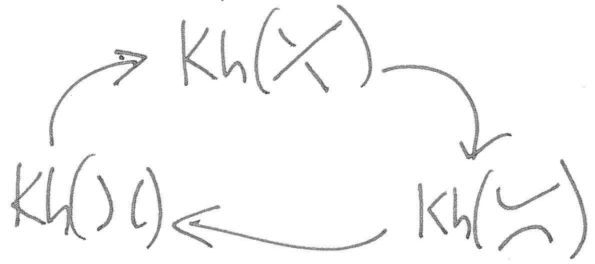
Thus Khovanov homology provides a lasagna algebra and hence a 4-manifold invariant.

Computations

- We can also define $\underrightarrow{Kh}(M^3)$, a category for each 3-manifold!
- If $M \subset \partial W^4$, $\underrightarrow{Kh}(W)$ becomes a module over $\underrightarrow{Kh}(M)$
- If $M \sqcup M^{op} \subset \partial W^4$, $\underrightarrow{Kh}(W)$ becomes a bimodule over $\underrightarrow{Kh}(M)$,

and:
$$\underrightarrow{Kh}(W \underset{M}{\cup} \circlearrowleft) \cong \underrightarrow{Kh}(W) \underset{\underrightarrow{Kh}(M)}{\otimes} \circlearrowleft$$

Sadly, the main tool for computing Kh on links is missing: the exact triangle



(Essentially, our construction is not an exact functor w.r.t. boundary conditions.)

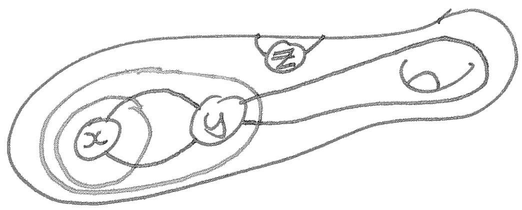
Suprisingly, we can realize the vector spaces $\underrightarrow{Kh}(W, L)$ as the 0-th homology of an (up-to-homotopy) invariant chain complex, on which the exact triangle survives!

(cf. "the blob complex"; arXiv:1009.5025)

$Kh_0 = \{$labelled lasagna diagrams$\}$

$Kh_k = \{$labelled lasagna diagrams, with k 'blobs', pairwise nested or disjoint, enclosing some of the labelled balls$\}$

<u>Example</u> (supressing dimensions)



The differential $\partial : Kh_k \to Kh_{k-1}$ is a sum over ways to

- forget a blob
- forget an 'innermost' blob, replacing its contents with the lasagna algebra evaluation.

Easily, $H_0 Kh_\bullet(W, L) = \underrightarrow{Kh}(W, L)$

The double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & Kh_1()() \arrow[r] \arrow[d] & Kh_1(\times) \arrow[r] \arrow[d] & Kh_1(\asymp) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Kh_0()() \arrow[r] & Kh_0(\times) \arrow[r] & Kh_0(\asymp) \arrow[r] & \cdots
\end{tikzcd}$$

gives a spectral sequence.

Computing homology by 'horizontal first, then vertical' gives zero, since the rows are exact.

Computing homology by 'vertical first*, then horizontal' shows the first page of the spectral sequence contains the blob complex invariants of $(W,)()$, $(W, \times)$ and $(W, \asymp)$.

Concrete calculations ...?