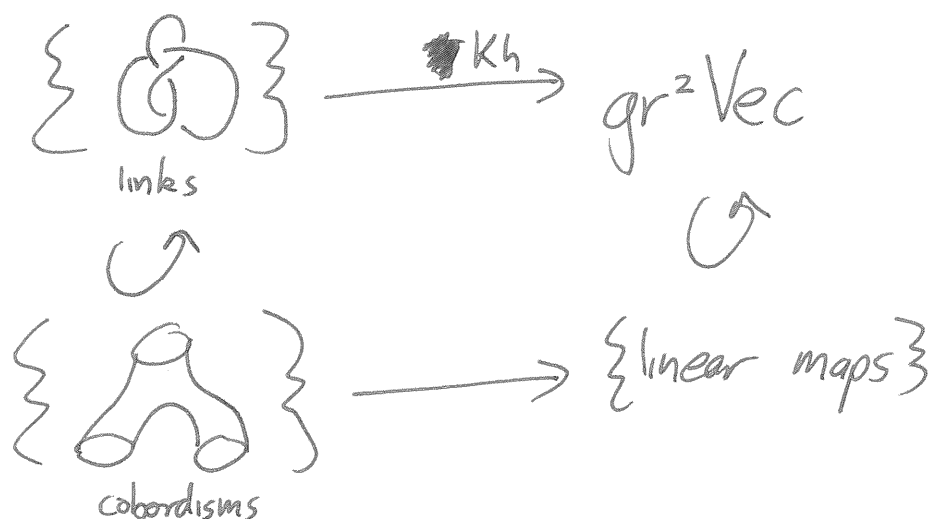


Khovanov homology

- Khovanov homology is a 'categorical' knot invariant



- In particular, it categorifies the Jones polynomial:

$$J(L) = \sum_{i,j} (-1)^i q^j \dim Kh^{i,j}(L)$$

Why does the Jones polynomial exist?

$\Rightarrow$ because of a nice braided tensor category called "Temperley-Lieb"

Let's present it as a 2-category:

$$\pi_{L_0}(\bullet) = \{\bullet\}$$

(this layer is 'boring' - forgetting it turns a 2-category into a $\mathcal{C}$ -category)

$$TL_1(\sim) = \{ \text{diagram with 3 dots on a line} \}$$

$$TL_2(\bigcirc) = \mathbb{C} \langle \text{diagram} \rangle \text{ modulo } 0 = q + q^{-1}$$

↑ only consider linear combinations with fixed boundary conditions

In fact TL is branded, via

$$\times = iq^{\frac{1}{2}}) (-iq^{-\frac{1}{2}})$$

$$(eq. \quad \check{\mathcal{D}} = -q \check{\mathcal{H}} + \check{\mathcal{O}} +) (- q^{-1} \check{\mathcal{U}} =) (\quad)$$

$J(L)$ is just interpreting L as a Temperley-Lieb diagram.

The Jones polynomial also comes from representation theory:

$$\text{Rep}^{\text{uni}} U_q \mathfrak{sl}_2 \cong \text{TL}(q+q^{-1})$$

Why does Khovanov homology exist?

$\Rightarrow$ you can categorify all of $\text{Rep} U_q \mathfrak{g}$!

(c.f. Rouquier, Webster, Khovanov & Lauda)

Today, though, we just going to categorify TL directly.

Categorifying Temperley-Lieb (via Bar-Natan)

Let's define a 3-category:

$$F_0(\bullet) = \{\bullet\}$$

$$F_1(\sim) = \{ \text{arc} \}$$

$$F_2(\bigcirc) = \{ \text{cap/cup} \}$$

$$F_3(\text{disk}) = \mathbb{C} \langle \text{diagrams} \rangle / \text{relations}$$

} codimension 1
submanifolds

The relations:

$$\text{cap} = 0, \quad \text{cup} = 2, \quad \text{cup} \circ \text{cap} = 0$$

$$\text{cup} \circ \text{cup} = \frac{1}{2} \text{cup} \circ \text{cup} \circ \text{cup} + \frac{1}{2} \text{cup} \circ \text{cup} \circ \text{cup}$$

We define a grading on F_3 by

$$\deg(\Sigma) = \chi(\Sigma) - \chi(\partial\Sigma)$$

(it's additive with respect to gluing).

Examples $\deg(\bigcirc) = +1, \quad \deg(\text{cap}) = -1.$

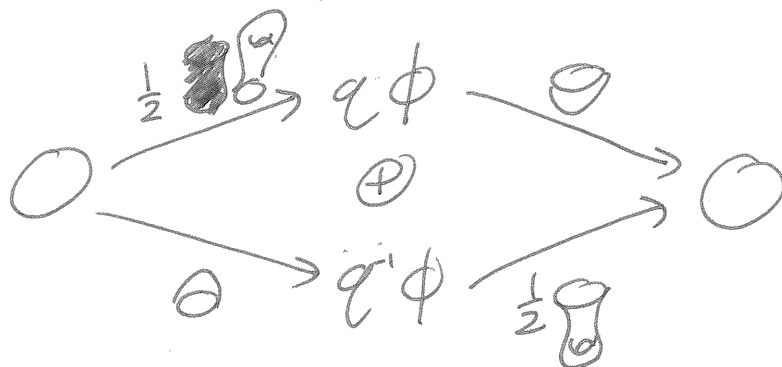
Incredibly, this gadget categorifies Temperley-Lieb!

Lemma

$$0 = q\phi \oplus q^{-1}\phi$$

→ this means 'the empty diagram shifted down one degree'

Proof Here's the Isomorphism:

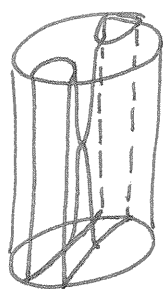


the compositions are $\frac{1}{2} \begin{array}{c} \text{cup} \\ \text{cap} \end{array} + \frac{1}{2} \begin{array}{c} \text{cap} \\ \text{cup} \end{array} = \text{cylinder}$

and $\begin{pmatrix} \frac{1}{2} \text{cup} & \frac{1}{4} \text{cup} \\ \text{cap} & \frac{1}{2} \text{cup} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Lemma $q\dim(\text{Hom}(X \rightarrow Y)) = X \cup Y$ evaluated in TL

Example



$\in \text{Hom}(1 \rightarrow \cup)$, with degree +1

Proof Use 'neck cutting' so each surface has a single boundary component, and genus either 0 or 1.

Theorem the $\vee$ 'Grothendieck 2-category' of F is exactly TL.
(split)

Is $\mathcal{F}$ braided?

$\Rightarrow$ No, but $\text{Kom}(\mathcal{F})$ is!

$\text{Kom}(\mathcal{F})_2 = \text{chain complexes in } \mathcal{F}$

$$\cdots \longrightarrow X \xrightarrow{\partial} Y \longrightarrow \cdots$$

$$\uparrow \quad \quad \uparrow$$

$$\mathcal{F}_2 \quad \quad \mathcal{F}_3$$

(let's not worry too much about $\text{Kom}(\mathcal{F})_3$ yet; roughly it's chain maps up to homotopy)

How do we glue chain complexes together?

$\Rightarrow$ take the tensor products of the complexes, combining objects & morphisms according to the gluing rules for $\mathcal{F}$.

Example $\boxed{X} = \text{circle} \xrightarrow{\text{saddle}} \text{circle} \in \text{Kom}(\mathcal{F})_2(\text{circle})$

$\boxed{Y} = \text{circle} \xrightarrow{\text{circle} \oplus \text{circle}} \text{circle} \in \text{Kom}(\mathcal{F})_2(\text{circle})$

then

$$\boxed{X/Y} = \text{circle} \begin{matrix} \xrightarrow{\text{saddle}} \text{circle} \\ \xrightarrow{\bullet} \text{circle} \end{matrix} \begin{matrix} \xrightarrow{\text{circle} \oplus \text{circle}} \text{circle} \\ \xrightarrow{\text{saddle}} \text{circle} \end{matrix}$$

Theorem the triangulated Grothendieck 2-category of $\text{Kom}(\mathcal{F})$ is still TL.

What is the 'braiding'?

$$\Rightarrow \underbrace{\times = q^{\frac{1}{2}}}_{\text{in homological height } -\frac{1}{2}} \xrightarrow{\text{saddle}} \underbrace{q^{-\frac{1}{2}}}_{\text{height } \frac{1}{2}}$$

Example

$$\text{Diagram} = q \text{Diagram} \rightarrow \begin{matrix} \text{Diagram} \\ \oplus \\ \text{Diagram} \end{matrix} \rightarrow q^{-1} \text{Diagram}$$

$$\cong q \text{Diagram} \xrightarrow{\text{id}} q \text{Diagram} \rightarrow q^{-1} \text{Diagram}$$

$$\xrightarrow{\text{id}} q^{-1} \text{Diagram} \xrightarrow{\text{id}} q^{-1} \text{Diagram}$$

$$\cong \text{Diagram}$$

What is Khovanov homology?

$$Kh(L) = \text{Hom}_F(\phi, L)$$

↑
doubly graded, by homological grading
and Euler characteristic.

When we say this is a 'braiding' we mean:

there are chain homotopy equivalences

$$R1: \rho \simeq q^{3/2} t^{-1/2}$$

$$R2: \text{crossing} \simeq \text{parallel}$$

$$R3: \text{triple crossing} \simeq \text{triple crossing}$$

satisfying the 'movie move' coherence conditions (up to homotopy)

Example $||| \rightarrow \text{crossing} \rightarrow \text{crossing} \rightarrow \text{crossing} \rightarrow |||$
 $= \text{id}$

(in today's version, this only works mod 2).

These coherence conditions imply that a link cobordism (up to isotopy) gives a well-defined chain map (up to homotopy).

In my next talk, I'll reinterpret the braiding, the Reidemeister moves and the coherence conditions in terms of a 4-category.