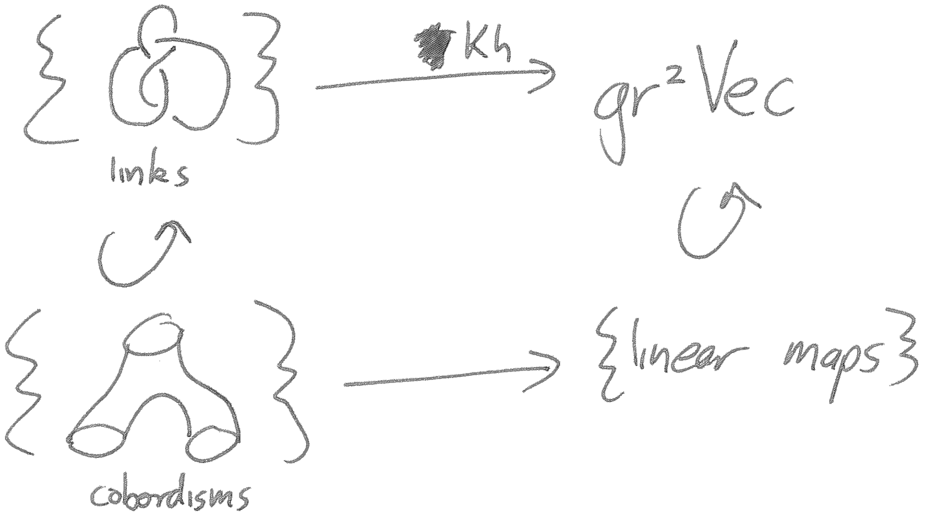
Khovanov homology

- Khovanov homology is a 'categorical' knot invariant



- In particular, it categorifies the Jones polynomial:

$$J(L) = \sum_{i,j} (-1)^i q^j \dim Kh^{i,j}(L)$$

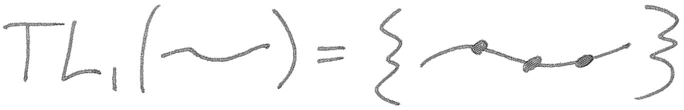
Why does the Jones polynomial exist?

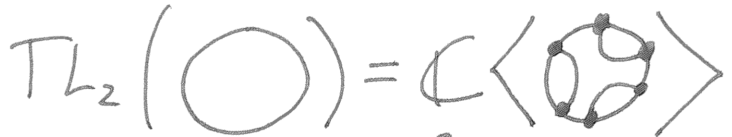
$\Rightarrow$ because of a nice braided tensor category called "Temperley-Lieb"

Let's present it as a 2-category:

$TL_0(\bullet) = \{\bullet\}$

(this layer is 'boring' — forgetting it turns a 2-category into a $\otimes$-category)

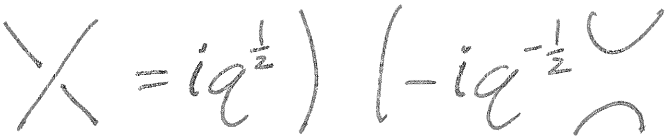


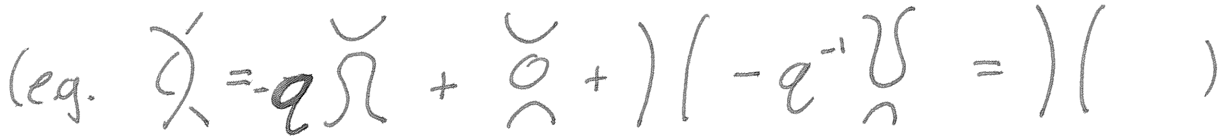


modulo $\bigcirc = q+q^{-1}$

only consider linear combinations with fixed boundary conditions

In fact TL is braided, via





J(L) is just interpreting L as a Temperley-Lieb diagram.

The Jones polynomial also comes from representation theory:

$$\mathrm{Rep}^{\mathrm{uni}}\, U_{-q}\,\mathfrak{sl}_2 \cong TL(q+q^{-1})$$

Why does Khovanov homology exist?

$\Rightarrow$ you can categorify all of $\mathrm{Rep}\, U_q\, \mathfrak{g}$!

(c.f. Rouquier, Webster, Khovanov & Lauda)

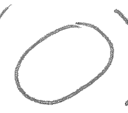
Today, though, we just going to categorify TL directly.

Categorifying Temperley-Lieb (via Bar-Natan)

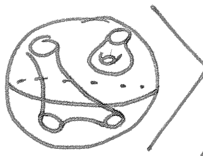
Let's define a 3-category:

$F_0(\bullet) = \{\bullet\}$

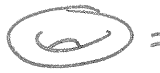
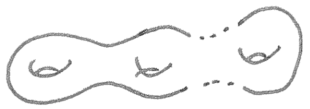
$F_1(\sim) = \{$  $\}$

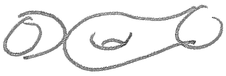
$F_2($  $) = \{$  $\}$

codimension 1 submanifolds

$F_3($  $) = \mathbb{C}\langle$  $\rangle /$ relations

The relations:

 $= 0$,  $= 2$,  $= 0$

 $= \frac{1}{2}$  $+ \frac{1}{2}$ 

We define a grading on F_3 by

$$\deg(\Sigma) = \chi(\Sigma) - \chi(\partial\Sigma)$$

(it's additive with respect to gluing).

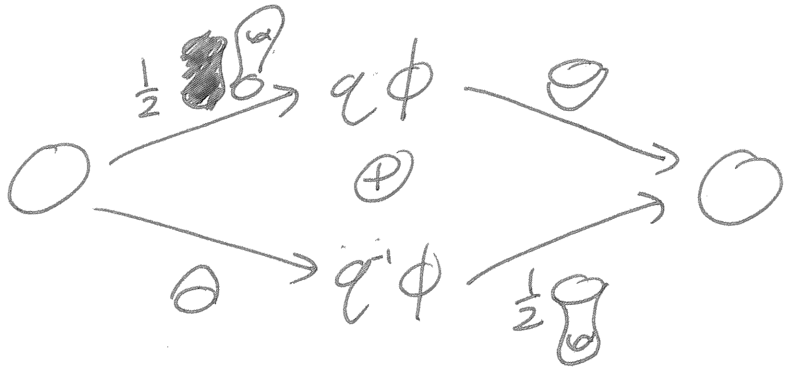
Examples $\deg($  $) = +1$, $\deg($  $) = -1$.

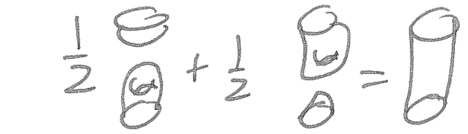
Incredibly, this gadget categorifies Temperley-Lieb!

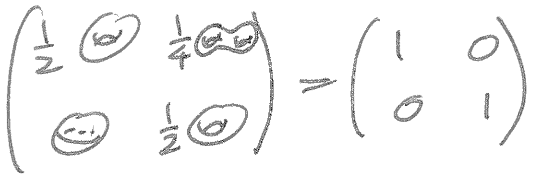
Lemma $\bigcirc = q\phi \oplus q^{-1}\phi$

this means 'the empty diagram shifted down one degree'

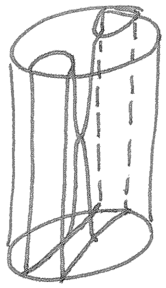
Proof Here's the isomorphism:



the compositions are 

and 

Lemma $q\dim(\mathrm{Hom}(X \to Y)) = X \cup Y$ evaluated in TL

Example  $\in \mathrm{Hom}(\,|\,| \to {}^{\cup}_{\cap})$, with degree $+1$

Proof Use 'neck cutting' so each surface has a single boundary component, and genus either 0 or 1.

Theorem the (split) 'Grothendieck 2-category' of F is exactly **TL**.

Is $\mathcal{F}$ braided?

$\Rightarrow$ No, but $Kom(\mathcal{F})$ is!

$Kom(\mathcal{F})_2$ = chain complexes in $\mathcal{F}$

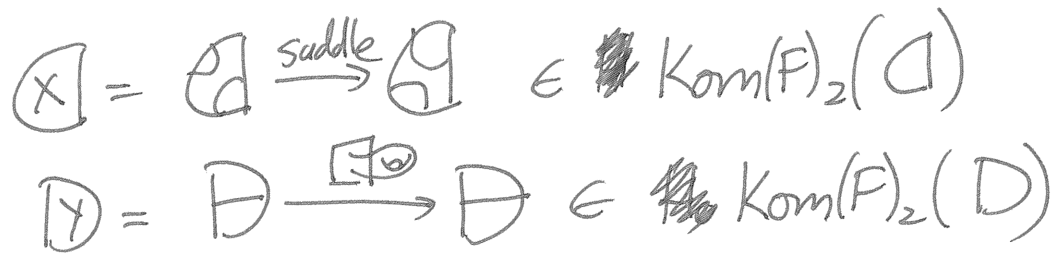
$$\cdots \longrightarrow \underset{\substack{\cap \\ \mathcal{F}_2}}{X} \xrightarrow[\substack{\cap \\ \mathcal{F}_3}]{\partial} Y \longrightarrow \cdots$$

(let's not worry too much about $Kom(\mathcal{F})_3$ yet; roughly it's chain maps up to homotopy)

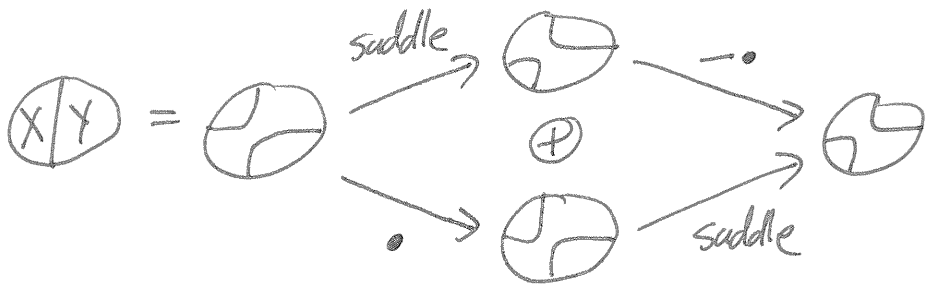
How do we glue chain complexes together?

$\Rightarrow$ take the tensor products of the complexes, combining objects & morphisms according to the gluing rules for $\mathcal{F}$.

<u>Example</u>

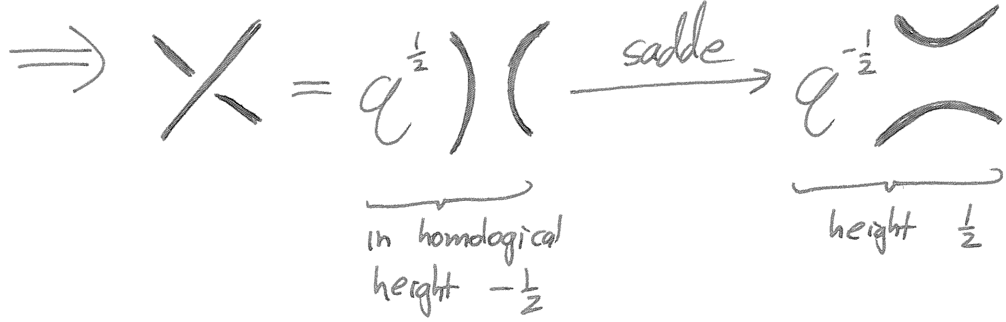


then

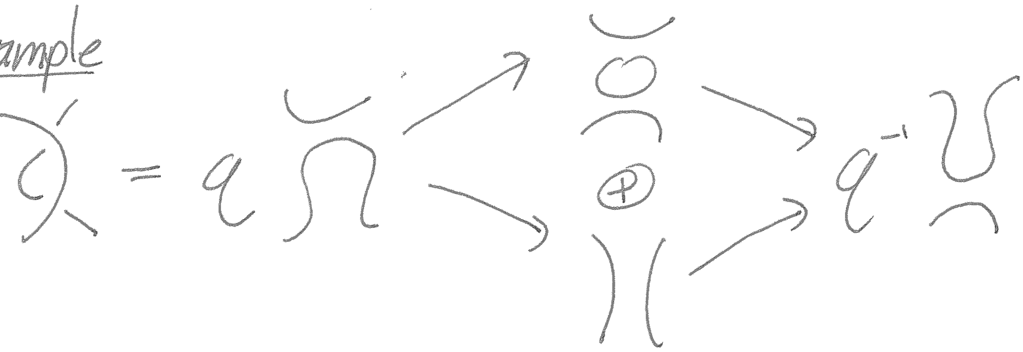


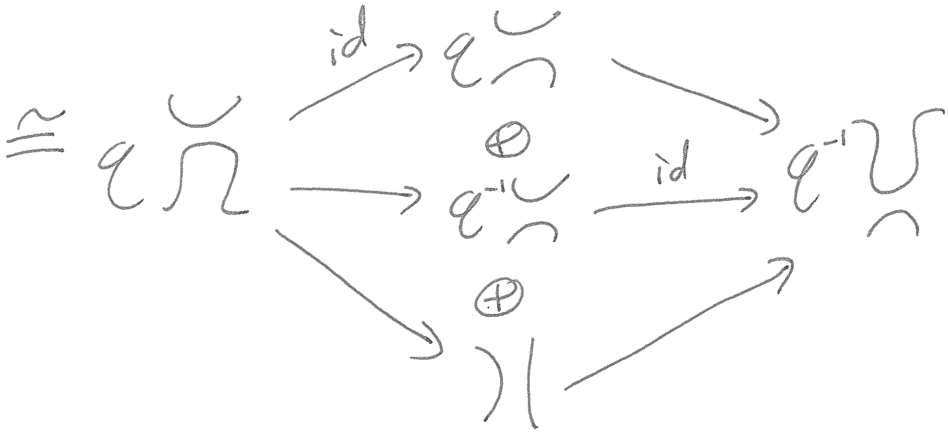
<u>Theorem</u> the triangulated Grothendieck 2-category of $Kom(\mathcal{F})$ is still TL.

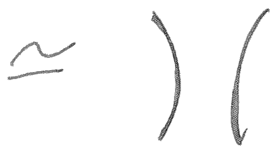
What is the 'braiding'?



<u>Example</u>







What is Khovanov homology?

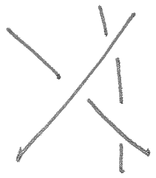
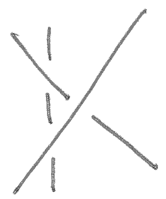
$$Kh(L) = \mathrm{Hom}_F(\phi, L)$$

doubly graded, by homological grading and euler characteristic.

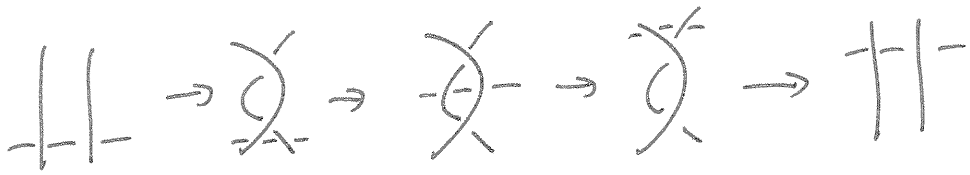
When we say this is a 'braiding' we mean:
there are chain homotopy equivalences

R1:  $\simeq q^{3/2} t^{-\frac{1}{2}}$ 

R2:  $\simeq$ 

R3:  $\simeq$ 

satifying the 'movie move' coherence conditions (up to homotopy)

<u>Example</u> 

$= \mathrm{id}$

(in today's version, this only works mod 2).

These coherence conditions imply that a link cobordism (up to isotopy) gives a well-defined chain map (up to homotopy).

In my next talk, I'll reinterpret the braiding, the Reidemeister moves and the coherence conditions in terms of a 4-category.