Fusion categories as quantum symmetries

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A fusion category is a 'noncommutative' finite group.

Definition

A fusion category is a semisimple $\otimes$-category with duals, with finitely many simple objects.

If we insist that the $\otimes$-structure is symmetric, we recover the representation theory of some finite group.

Finite groups give examples, quantum groups at roots of unity give examples, but there are also other, 'exotic' examples.

Why study fusion categories?

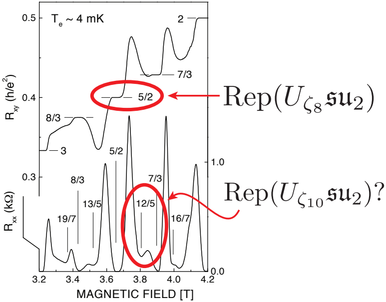
In many physical systems, the fundamental particles are described by representations of the symmetry group of the system.

In '2d topological phases of matter', the fundamental excitations of the system correspond to the simple objects of some fusion category.

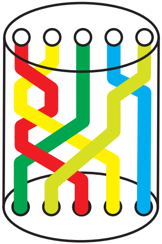
The fusion category plays the role of a symmetry group.

There is a bijective correspondence between $2+1$-dimensional topological field theories and fusion categories.

There is a proposal to build a quantum computer using these 2d topological phases!



Pan, et al., PRL 83 (1999)



We implement the unitary operator for a quantum algorithm by braiding some of the fundamental excitations around each other.

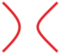
An example

The 'golden category', $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$ at a 10-th root of unity.

There are only 2 simple objects, called ι and τ, with dimensions 1 and $(1+\sqrt{5})/2$ and tensor products $\tau \otimes \tau \cong \iota \oplus \tau$.

If we write the map $\tau \otimes \tau \to \tau$ as , then we can describe all morphisms as (linear combinations) of planar trivalent graphs, modulo relations

 $= \frac{1+\sqrt{5}}{2}$  $=0$

 $-$  $= \frac{2}{1+\sqrt{5}}\Big($  $-$  $\Big)$

How do we study fusion categories?

- Imitating the study of finite groups, e.g. we have a homotopy theoretic theory for extensions of a fusion category by a finite group (but don't yet have a general theory).
- Classifying small examples (using combinatorics, number theory, representation theory and lots of algebra).
- A finite depth subfactor is essentially a Morita equivalences between fusion categories; many of our exotic examples were first discovered using subfactors.
- As an 'easy' first case towards studying n-dimensional topological field theories and n-categories.