Small index subfactors

The goal of these lectures is to describe recent work on the classification of small index subfactors.

We will study subfactors from the point of view of their "standard invariant", consisting of

* a pair of fusion categories $\mathcal{C}$ and $\mathcal{D}$
* a bimodule category between them ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ providing a Morita equivalence
* a 'favourite object' $X \in \mathcal{M}$.

A fusion category describes a collection of 'quantum symmetries', generalizing, e.g. finite groups and quantum groups at roots of unity.

A fusion category is an abelian $\otimes$-category, which is pivotal, and has finitely many simple objects.

An important measure of the 'size' of a subfactor is its index, $(\dim X)^2$.

The classification of subfactors with index ≤ 4 has been well-understood since the mid-90s.

At index 6 and above certain features arises making classification intractable.

<u>What happens between 4 and 6?</u>

Building on earlier work of Haagerup we've now completed the classification for the interval $(4,5]$.

This led to two surprises —

1. the classification is suprisingly sparse, with only 10 examples, coming in 5 related pairs
2. some of these examples lie in families, or arise as special cases of general constructions, but others are seemingly 'sporadic' or 'exotic' objects.

(The proofs that they even exist seem rather unenlightening so far!)

Lecture I: Subfactors, the standard invariant, and planar algebras.

For completeness, let's mention the definition of a subfactor, before immediately turning to its representation theory.

- A factor is a von Neumann algebra with trivial centre.

- A factor is type I if it has minimal projections; these are $M_n(\mathbb{C})$ or $\mathcal{B}(\mathcal{H})$.

  A factor is type II if it has no minimal projection, but there is a trace.
  It is type II_1 if that trace can be normalized so $tr(1) = 1$, and type II_∞ otherwise

  A factor is type III otherwise.
  We'll restrict our attention to II_1 factors.

- The prototypical example is the 'hyperfinite II_1 factor'
$$R = \lim_{n \to \infty} \bigotimes_{k=1}^{n} M_2(\mathbb{C}).$$

- A subfactor $A \subset B$ is 'merely' an inclusion of factors.

  The index $[B:A]$ is the dimension of $L^2(B)$ as an A-module

Given a subfactor $A \subset B$, we can analyze its bimodules, which come in 4 flavours A-A, A-B, B-A and B-B.

~~we only consider those bimodules generated by~~ B.

We can tensor bimodules together, so as well as the obvious examples ${}_A A_A$, ${}_A B_B$, ${}_B B_A$ and ${}_B B_B$, we can form ${}_A B \otimes_B B_A$ (which is just ${}_A B_A$) and ${}_B B \otimes_A B_B$, and so on.

~~we only~~ (We can work with either algebraic or L^2-bimodules; L^2-completion gives an equivalence of categories.)

We only consider those bimodules $\otimes$-generated by B.

<u>Theorem</u> (The standard invariant)

The A-A and B-B bimodules form a pair of (possibly infinite) fusion categories.

The A-B and B-A bimodules form a pair of bimodule categories between these, constituting a Morita equivalence.

We call the two categories the <u>even parts</u>. When they actually have finitely many simple objects we say the subfactor is <u>finite depth</u>.

An important invariant of a subfactor is its <u>principal graph</u>: Γ

* vertices for each simple bimodule (so four classes, A-A, A-B, B-A, B-B)
* ~~on~~ edges between P_A and Q_B for the dimension of $\mathrm{Hom}(P \otimes_A B, Q)$. (equivalently, $\mathrm{Hom}(P_A, Q \otimes_B B_A)$)

We also record the involution $*$ corresponding to duals of bimodules.

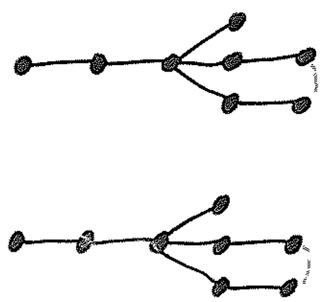
The graph has two components, for the left-A and left-B ~~Always~~ modules. Each component is bipartite for right A and B modules.

We draw the trivial bimodules at the left, with vertices arranged according to distance from $*$.

The k-th A-B bimodule at depth d is dual to the k-th B-A bimodule at depth d.

We indicate dual data on the even parts with dotted red lines.

E.g.



The index is at least the square of the graph norm (the largest eigenvalue of the adjacency matrix)

When the graphs are finite the index <u>equals</u> the square of the graph norm.

Examples from finite groups

We can realize any finite group as acting by outer automorphism on the hyperfinite II_1 factor R (essentially uniquely!) (Jones 1980)

* Thus we have subfactors $R^G \subset R$, with index $|G|$.

The fusion category of R^G–R^G bimodules is then $\mathrm{Rep}(G)$, and the dual category of R-R bimodules generated by $R \underset{R^G}{\otimes} R$ is Vec_G.

Thus this subfactor corresponds to the Morita equivalence between $\mathrm{Rep}(G)$ and Vec_G.

* Given $H \subset G$, we can look at $R^G \subset R^H$, with index $[G:H]$

Now the R^G-R^G bimodules give $\mathrm{Rep}(G)$, the R^G-R^H bimodules give $\mathrm{Rep}(H)$,

and the functors

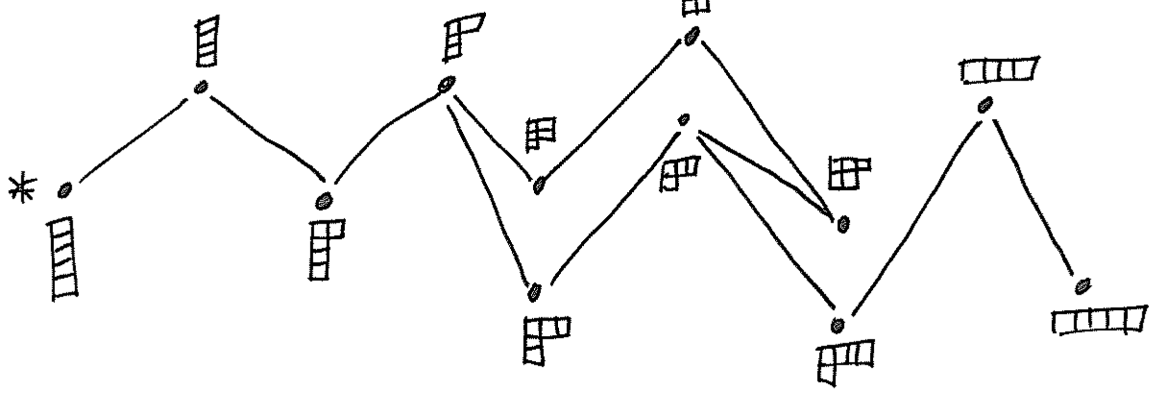
$$\mathrm{Rep}(G) \xrightarrow{-\otimes R^H} \mathrm{Rep}(H)$$

and $$\mathrm{Rep}(H) \xrightarrow{-\otimes R^H} \mathrm{Rep}(G)$$

are restriction and induction, and the principal graph is the induction restriction graph for $H \subset G$.

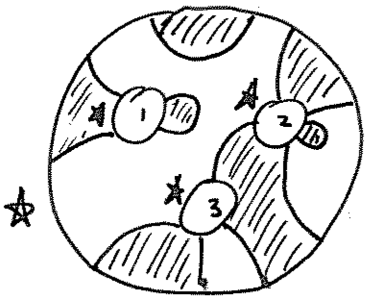
The R^H–R^H bimodules, and the dual principal graph, are harder to describe.

Example $S_4 \subset S_5$



The data of a standard invariant is equivalent to that of a *planar algebra* P:

- vector spaces $P_{n,\pm}$, $n \geq 0$
- for each planar tangle

$T =$ 

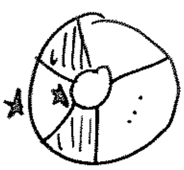
(a disc D_0 and interior discs D_i, a 1-manifold in $D_0 \setminus \cup D_i$, and shading of the regions, and a marked point on each boundary circle...)

a linear map

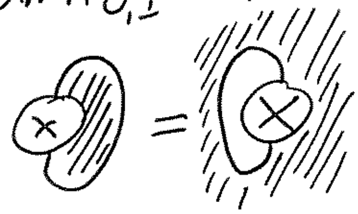
$$P(T) : P_{2,-} \otimes P_{3,+} \otimes P_{2,+} \longrightarrow P_{5,+}$$

from the vector spaces associated to the inner circles to the vector space associated to the outer circle

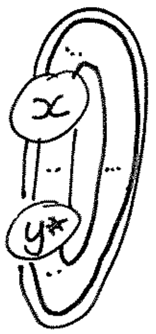
such that:

- the linear maps compose in the same way planar tangles do
- isotopic (rel ∂) tangles give the same maps
-  is the identity

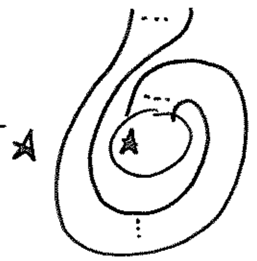
A planar algebra is

- <u>evaluable</u> if $\dim P_{0,\pm} = 1$
- <u>spherical</u> if  (as multiples of the empty diagrams in $P_{0,\pm}$)

- <u>unitary</u> if there is an antilinear $*: P_{n,\pm} \to P_{n,\pm}$, intertwining reflection of planar tangles, such that

$\langle x, y \rangle =$  is positive definite.

A <u>subfactor</u> planar algebra is one which is evaluable, bosonic spherical and unitary.

- <u>bosonic</u> if  $= 1$ on every $P_{n,\pm}$

(actually, I just made 'bosonic' up: everyone takes it as part of the definition of a planar algebra)

Reconstruction

Starting from a subfactor planar algebra, we can rebuild a subfactor.

$$A \subset B = \lim_{n \to \infty} \left(\boxed{P_{n-1,-}} \subset \boxed{P_{n,+}} \right).$$

<u>Theorem</u> (Popa) Starting with a finite depth subfactor of the hyperfinite II_1, this reconstructs the original subfactor.

Outside of these cases, the problem is more subtle.
Popa has shown that every subfactor planar algebra is realized by some subfactor of $L(F_\infty)$.

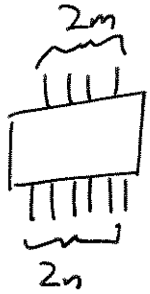
Standard invariant from a planar algebra

First define categories $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}$, $\hat{\mathcal{M}}$, $\hat{\mathcal{M}}'$.

$$\mathrm{Obj}(\hat{\mathcal{C}}) = \mathrm{Obj}(\hat{\mathcal{D}}) = 2\mathbb{N}$$

$$\mathrm{Obj}(\hat{\mathcal{M}}) = \mathrm{Obj}(\hat{\mathcal{M}}') = 2\mathbb{N}+1.$$

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(2n, 2m) = P_{n+m,+} =$$



(unshaded on both sides)

$$\mathrm{Hom}_{\hat{\mathcal{D}}}(2n, 2m) = P_{n+m,-}$$ (shaded on both sides)

$$\mathrm{Hom}_{\hat{\mathcal{M}}}(2n+1, 2m+1) = P_{n+m+1,+}$$ (unshaded - shaded)

$$\mathrm{Hom}_{\hat{\mathcal{M}}'}(2n+1, 2m+1) = P_{n+m+1,-}$$ (shading - unshaded)

Composition is vertical stacking

$\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ become $\otimes$-categories,

$\hat{\mathcal{M}}$ a $\hat{\mathcal{C}}$-$\hat{\mathcal{D}}$ bimodule category, and

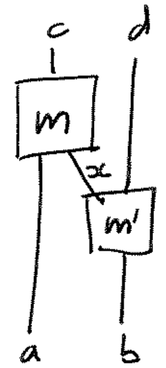
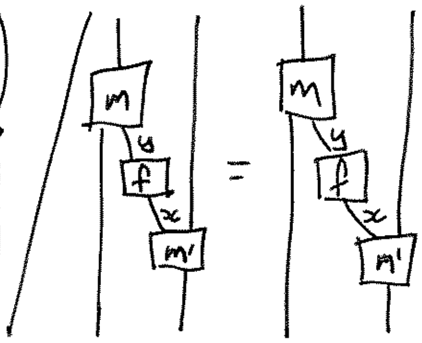
$\hat{\mathcal{M}}'$ a $\hat{\mathcal{D}}$-$\hat{\mathcal{C}}$ bimodule category

under horizontal juxtaposition.

Idempotent complete to obtain $\mathcal{C}, \mathcal{D}, \mathcal{M}, \mathcal{M}'$.

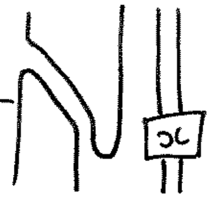
To see $\mathcal{C} \underset{\mathcal{M}'}{\overset{\mathcal{M}}{\rightleftarrows}} \mathcal{D}$ is a Morita equivalence,

"recall"

$\mathcal{M} \underset{\mathcal{D}}{\otimes} \mathcal{M}' = \Bigg\{$  $: x \in \mathrm{Obj}(\mathcal{D}),\ m \in \mathrm{Hom}_{\mathcal{M}}(a \otimes x, c),\ m' \in \mathrm{Hom}_{\mathcal{M}'}(b, x \otimes d) \Bigg\} \Big/$ 

Define $\mathcal{M} \underset{\mathcal{D}}{\otimes} \mathcal{M}' \to \mathcal{C}$ merely by interpreting the diagram above as a planar tangle.

Define $\mathcal{C} \to \mathcal{M} \underset{\mathcal{D}}{\otimes} \mathcal{M}'$ by

$x \mapsto \frac{1}{\delta}$ 

(we need some natural transformations to fit these together as an equivalence.)

Planar algebra from the standard invariant.

We can more or less reverse this recipe.

Let F denote all the (iterated) Morita equivalence functors

$$\mathcal{M}\otimes\mathcal{M}'\otimes\cdots\otimes\mathcal{M}'\longrightarrow\mathcal{C}$$
$$\mathcal{M}\otimes\mathcal{M}'\otimes\cdots\otimes\mathcal{M}\longrightarrow\mathcal{M}$$
$$\mathcal{M}'\otimes\mathcal{M}\otimes\cdots\otimes\mathcal{M}\longrightarrow\mathcal{D}$$
$$\mathcal{M}'\otimes\mathcal{M}\otimes\cdots\otimes\mathcal{M}'\longrightarrow\mathcal{M}'$$

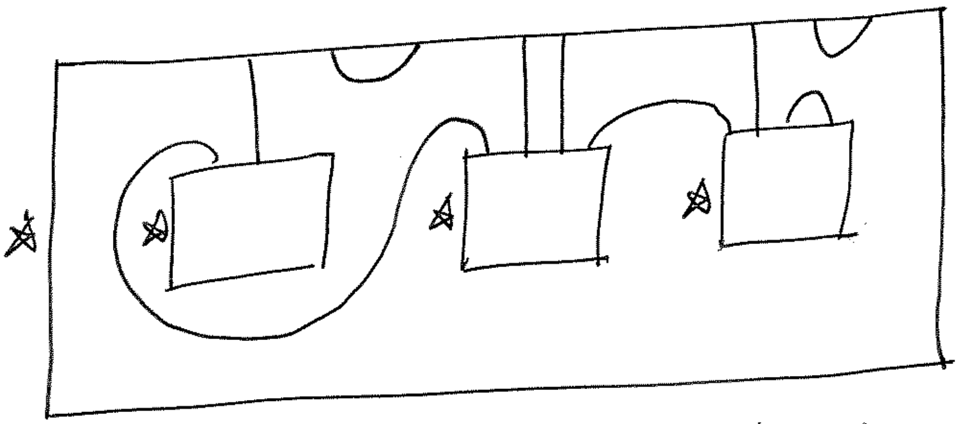
and G the functors the other way.

Define $P_{n,+} = \cancel{\mathrm{End}_{\mathcal{C}}(F(X\otimes X^*))}$

$$\mathrm{Hom}_{\mathcal{C}}\big(\mathbf{1}\longrightarrow F(\underbrace{X\otimes X^*\otimes\cdots\otimes X^*}_{2n\text{ factors}})\big)$$

and $P_{n,-} = \mathrm{Hom}_{\mathcal{D}}\big(\mathbf{1}\longrightarrow F(X^*\otimes X\otimes\cdots\otimes X)\big)$

To define the action of planar tangles, first isotope them to a standard form:



and read from bottom to top.

As we want X^* to actually be a dual of X,

$$\mathrm{Hom}_{\mathcal{C}}(0 \to F(X \otimes X^*))$$

is one dimensional. Picking a representative gives us  $\in P_{1,+}$

and similarly for ,  and .

Fix normalizations so  $=$  and  $=$ .