Lecture II: Examples, and the planar algebra toolkit.

Today we'll introduce three important tools for the analysis of planar algebras.

① The Temperley-Lieb algebra TL_δ is initial for planar algebras with index δ^2.

$$TL_\delta \hookrightarrow P$$

② Every planar algebra with principal graph Γ embeds in the graph planar algebra for Γ.

$$P \hookrightarrow GPA(\Gamma)$$

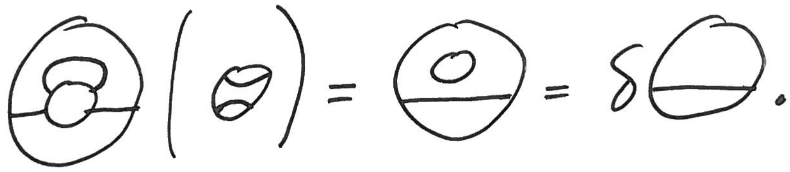
③ The annular Temperley-Lieb category acts on every planar algebra, and we can decompose the planar algebra into irreducible modules.

Temperley-Lieb

Defn $TL'_{\delta,n} = \mathbb{C}\{\text{crossingless matchings on } 2n \text{ points}\}$

e.g. $TL_{\delta,3} = \mathbb{C}\{$  $\}$

and planar tangles act by gluing, removing closed circles for a factor of δ.

Eg 

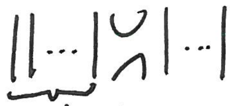
We have a map $TL'_\delta \to P$ for any planar algebra with loop value δ, because we can interpret a Temperley-Lieb diagram as a planar tangle with no inputs.

Theorem
- TL'_δ is nondegenerate except when $\delta = 2\cos\frac{\pi}{n}$.
- It is positive definite for $\delta \geq 2$.
- When $\delta = 2\cos\frac{\pi}{n}$, the radical is generated by the $(m-1)$th Jones-Wenzl idempotent, and the quotient is positive definite.

(Jones, Index for subfactors, 1983)

Let's denote $TL'_\delta/\text{radical}$ by TL_δ.

What are the Jones-Wenzl idempotents?

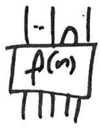
Definition 1   In TL_n, thought of as an associative algebra, we have idempotents $e_i = \frac{1}{\delta}$  for $i = 1, \ldots, n-1$.

$i-1$ strands

$$f^{(n)} = 1 - \sup(e_1, \_\_, e_n).$$

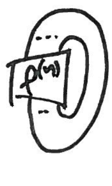
Definition 2

$f^{(n)}$ is the unique element of TL_n satisfying:

a)  $= 0$ for all caps on top or bottom

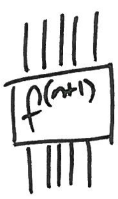
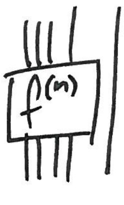
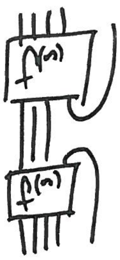
b)  $=$ 

c)  $=$ 

d)  $= [n+1]_q = \dfrac{q^{n+1} - q^{-n-1}}{q - q^{-1}}$, where $q + q^{-1} = \delta$

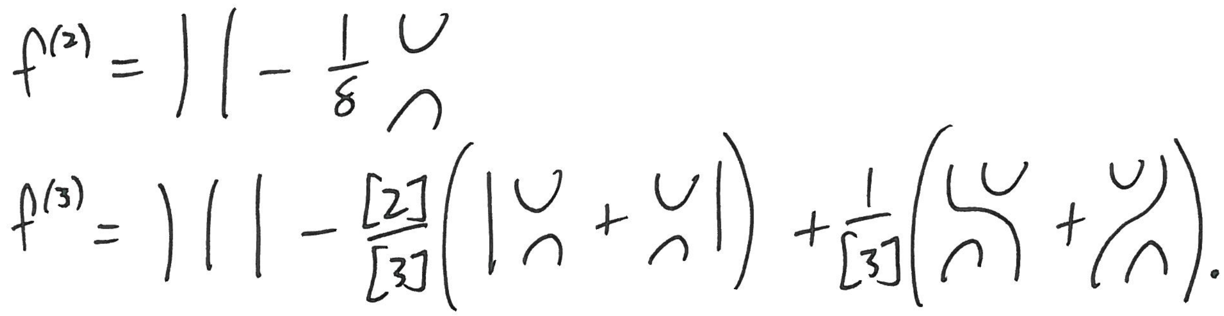
e) the coefficient of |||| in  is 1.

Definition 3 (Wenzl) $f^{(0)} = \phi$, $f^{(1)} = 1$,

 $=$  $- \dfrac{[n]}{[n+1]}$ 

(You can remember the coefficient by taking traces.)

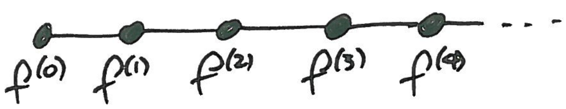
In particular we have



<u>Lemma</u> The Jones-Wenzl idempotents are minimal,
and $f^{(n)} \otimes f^{(m)} \cong f^{(|n-m|)} \oplus f^{(|n-m|+2)} \oplus \cdots \oplus f^{(n+m)}$.

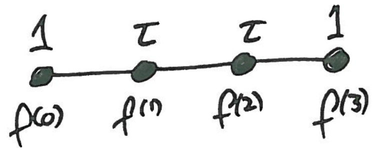
<u>Proof</u> Definition 3 gives the $m=1$ case, the rest are determined by associativity.

<u>Corollary</u> When $\delta \geq 2$, the principal graph of TL_δ is A_∞



<u>Lemma</u> When $\delta = 2\cos\frac{\pi}{m}$, $f^{(k)}$ is contained in the radical for all $k \geq m-1$.

<u>Corollary</u> The principal graph of $TL_{2\cos\frac{\pi}{m}}$ is A_{m-1}

E.g. $\Gamma\left(TL_{\frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}}\right) =$ 

Examples: the ADE classification below index 4.

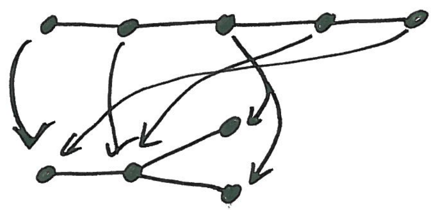
Besides the A series coming from Temperley-Lieb, there are 2 other families of planar algebras with index below 4.

- There is a unique planar algebra with principal graph D_{2n} for each n. Some constructions:

1) In A_{4n-3}, consider $X = f^{(0)} \oplus f^{(4n-4)}$

This carries the structure of an algebra object, and the fusion category of X-X bimodule objects has principal graph D_{2n}

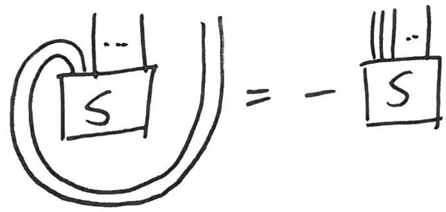
You should think of the D_{2n} principal graph as an orbifold of A_{4n-3}:

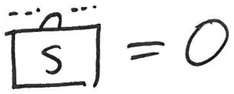


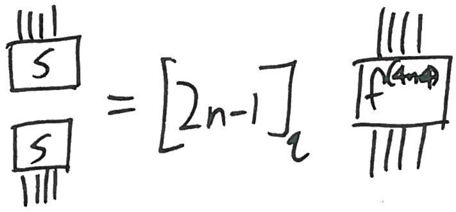
Under the map $A_{4n-3} \hookrightarrow D_{2n}$, $f^{(0)}$ and $f^{(4n-4)}$ become isomorph[ic] while $f^{(2n-2)}$ stops being simple, and splits into two pieces.

2) An explicit planar algebra; generated by $S \in P_{2n-2}$

with relations

i) 

ii) 

iii) 

(c.f. arXiv:0808.0764)

- There are a pair of complex conjugate E_6 subfactors, and a pair of complex conjugate E_8 subfactors.

Again, they can be constructed as bimodule object categories starting from A_{11} or A_{29},
or given by explicit skein theories (Bigelow math.QA/0903.0144)
or via conformal field theory (Xu MR1617550)

<u>Examples from finite groups</u>

We can realize any finite group as acting by outer automorphisms on the hyperfinite II_1 factor R (essentially uniquely!) (Jones 1980)

* Thus we have subfactors $R^G \subset R$, with index $|G|$.

The fusion category of R^G–R^G bimodules is then $\mathrm{Rep}(G)$, and the dual category of R-R bimodules generated by $R \otimes_{R^G} R$ is Vec_G.

Thus this subfactor corresponds to the Morita equivalence between $\mathrm{Rep}(G)$ and Vec_G.

* Given $H \subset G$, we can look at $R^G \subset R^H$, with index $[G:H]$

Now the R^G-R^G bimodules give $\mathrm{Rep}(G)$,
the R^G-R^H bimodules give $\mathrm{Rep}(H)$,

and the functors

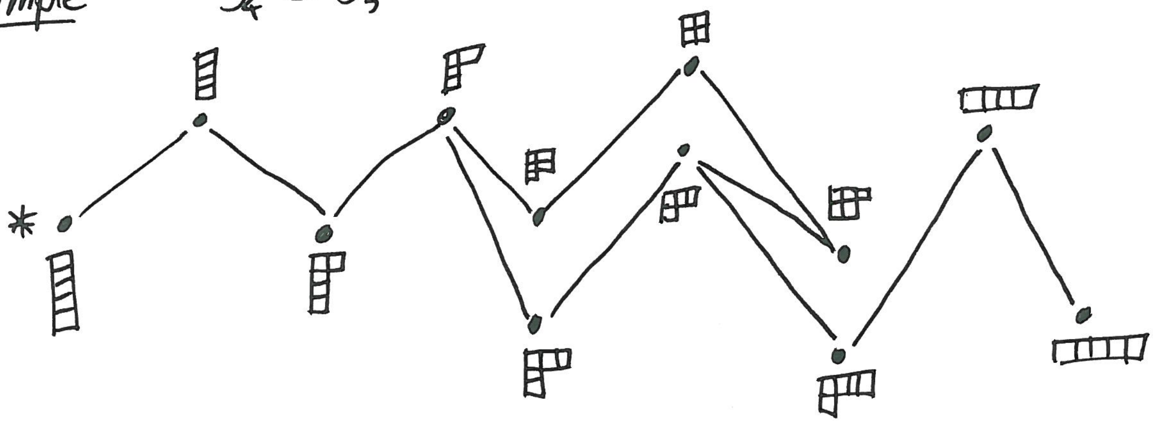
$$\mathrm{Rep}(G) \xrightarrow{- \otimes R^H} \mathrm{Rep}(H)$$

and $\mathrm{Rep}(H) \xrightarrow{- \otimes R^H} \mathrm{Rep}(G)$

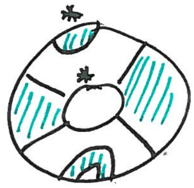
are restriction and induction, and the principal graph is the induction restriction graph for $H \subset G$.

The R^H–R^H bimodules, and the dual principal graph, are harder to describe.

Example $S_4 \subset S_5$



Annular Temperley-Lieb

The planar tangles with one input disc form a category with objects $(n \in \mathbb{N}, \pm)$ and morphisms eg.  $: (2,+) \to (5,-)$

and every planar algebra naturally becomes a representation of this category.

What are the irreducible representations?

Theorem For $\delta > 2$, the irreps of ATL are indexed by

* (n, ω) with $n > 0$, $\omega^n = 1$, or
* $(0, d)$ with $0 \leq d \leq \delta$.

These are realized by a cyclic vector in V_n satisfying relations

 $= 0$ and  $= \omega$  (for $n > 0$)

or  $= d$  (for $n = 0$)

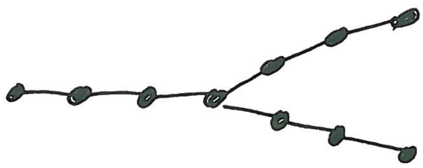
If $P = \bigoplus_n \bigoplus_z a_{n,z} V_{n,z}$, we say

P has 'magic numbers' $\left(a_n = \sum_z a_{n,z}\right)_{n \geq 0}$

If P is evaluable, $a_0 = 1$, and the corresponding irrep $V_{0,\delta}$ is the Temperley-Lieb subalgebra.

If P is k-supertransitive, TL_n exhausts P_n for $k \le n$, so $a_1, \dots, a_k = 0$.

We know the dimensions of the irreps $V_{n,z}$, and we ~~know~~ ~~the~~ can calculate $\dim P_n$ from the principal graph, so in fact the magic numbers are computable directly from the graph. $a_n = \sum_{r=0}^{n} (-1)^{r-n} \frac{2n}{n+r} \binom{n+r}{n-r} w_r$, where w_r is the # of loops of length $2r$ based at $*$

<u>Example</u> $\Gamma =$ 

$$a_n = (1,0,0,0,1,0,2,\dots)$$

When the graph is finite, the generating function for a_n is rational. Although we know $a_4 = 1$, we can't determine the rotational eigenvalue $\omega^4 = 1$ solely from the principal graph.

The graph planar algebra.

We can realize any planar algebra inside a certain combinatorially defined 'graph planar algebra'.

Given a graph Γ with dimension function d, we define

$$GPA(\Gamma, d)_n = \{\text{loops of length } 2n \text{ on } \Gamma\}^*$$

The action of planar tangles is given by

$$T(\{V_i\})(V_0)(\gamma_0) = \sum_{s \in \text{states}} \left(\prod_{c \in \text{critical points}} s(c) \right) \left(\prod_i V_i(\gamma_i) \right)$$

where

- a 'state' is a labelling of the regions in T by vertices of Γ and of strands in T by edges, extending γ_0.
- $s(c) = \sqrt{\dfrac{d(c^+)}{d(c^-)}}$ where c^+ is the label from s 'inside' the critical point and c^- is the label outside

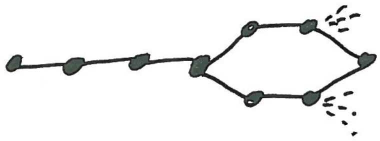
- γ_i is the path on Γ obtained by reading s around the i-th boundary circle of T

Theorem (Penneys-Jones, Morrison-Walker)
A planar algebra with principal graph Γ embeds in $GPA(\Gamma)$.

There is a beautiful explanation of this and related facts using the Turaev-Viro TQFTs.

Lemma Denote $GPA(\Gamma)_v$ the subspace of loops based at v.
$*$ $P \hookrightarrow GPA(\Gamma) \twoheadrightarrow GPA(\Gamma)_*$ is an isomorphism, $*$ $P \hookrightarrow GPA(\Gamma) \twoheadrightarrow GPA(\Gamma)_*$ is surje

Corollary A strictly k-supertransitive principal graph may not have a chain of length $> k$.

Corollary A 3-supertransitive principal graph with magic numbers 100010 is either the Haagerup subfactor or begins 

'Near-group' examples

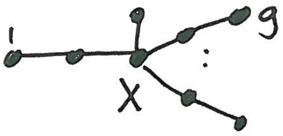
There are two interesting classes of subfactors in which the even part is 'slightly larger than a group'.

① A fusion ring of the form $G \cup \{X\}$, with G a finite group must have $gX = X$ and $X^2 = n'X + \sum_g g$.

Examples * the even part of the E_6 subfactor with $G = \mathbb{Z}/2$, $n' = 2$.

$E_6 =$ , $\frac{1}{2}E_6 =$ 

* the Tambara-Yamagami categories, $n' = 0$.

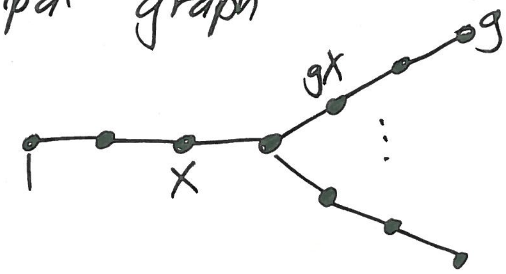
A 'near-group' subfactor with $n' = |G|$ must have principal graph 

Izumi & Xu constructed one for $G = \mathbb{Z}/3$, and Evans & Gannon many more. Existence has been reduced to a polynomial probl

When G is abelian, Izumi and independently Evans & Gannon can show $n' = |G| - 1$ or $n' \mid |G|$.

(2) Subfactors with even part fusion ring $G \cup GX$,

and principal graph



Izumi constructed examples for $G = \mathbb{Z}/3, \mathbb{Z}/5, \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$.

The $\mathbb{Z}/3$ example is the Haagerup subfactor.

Evans & Gannon constructed others, for

$G = \mathbb{Z}/7$, two for $\mathbb{Z}/9$, and none for $\mathbb{Z}/3 \times \mathbb{Z}/3$.

With Penneys & Peters we've constructed one for $\mathbb{Z}/8$.

An example for $\mathbb{Z}/4 \times \mathbb{Z}/2$ might be very helpful

for explaining the mysterious Asaeda-Haagerup subfactor,

via Grossman-Snyder's analysis of the maximal atlas for AH.

<u>Theorem</u> (Izumi) If $|G|$ is odd, G is cyclic.

New subfactors from old

The GHJ construction.

Given $\mathcal{C}$ a fusion category acting on a module category M, each simple object $X \in M$, gives rise to an algebra object in $\mathcal{C}$, the internal endomorphisms $A = \underline{\mathrm{End}}(X,X)$.

Now the 1-1, 1-A, A-1 and A-A bimodule objects in $\mathcal{C}$ give a planar algebra.

Example Take E_6 (as a fusion category) as a module category over A_{11}.

Take $X =$ 

Then $A =$ 

$f^{(0)} + f^{(8)} + f^{(10)}$

and the principal graph of the GHJ subfactor is  "3311".

with index $3+\sqrt{3}$.

~~Intermediate subfactors~~

Tensor products, free products, and intermediate subfactors

The tensor product of planar algebras P and Q is defined by

$$(P \otimes Q)_n = P_n \otimes Q_n$$

with planar tangles acting pointwise in the two factors.

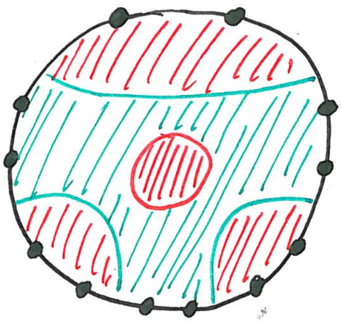
(index is multiplicative under tensor product)

The free product is harder to define!

$$(P * Q)_n = \bigoplus_{n\text{-paintings}} P_{red} \otimes Q_{blue} \Big/ \begin{matrix}\text{repainting} \\ \text{blank areas}\end{matrix}$$

Here an n-painting is a disc divided into red and blue regions, with $2n$ marked points on the boundary, such that there is a pair of red points, followed by a pair of blue points, followed by a pair of red points, and so on.

E.g.

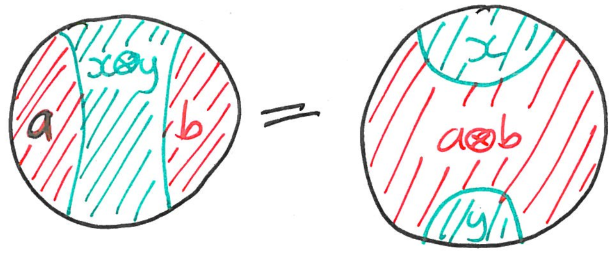


Then $P_{red} = \bigotimes_{\substack{\text{contiguous} \\ \text{red regions} \\ R}} P_{\frac{1}{2}\#R}$, where $\#R$ denotes the number of marked boundary points in the region R

and similarly for Q_{blue}.

In the example above, $P_{red} = P_1 \otimes P_1 \otimes P_1 \otimes P_0$ and $Q_{blue} = Q_3$

Finally, what is repainting?

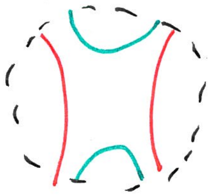


Equivalently, a 'blank' red area is considered the same as a blank blue area.

(Exercise, define amalgamated free products!)

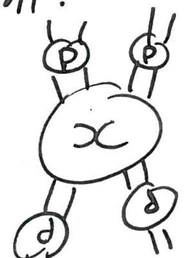
Planar tangles act by replacing strands $/ \mapsto //$.

Notice $(P * Q)_2$ contains a special element,

p = 

which is a 'biprojection':

Moreover, we can recover P and Q from the free product and this element:

$P_n \cong \Big\{$  $\Big| \ x \in (P * Q)_n \Big\}$

$Q_n \cong \Big\{$  $\Big| \ x \in (P * Q)_n \Big\}$

and with a bit of thought also the planar structure.

In fact, in any planar algebra with a biprojection p, we can define P and Q in this way, and construct a map

$$P * Q \longrightarrow F$$

of planar algebras sending the canonical biprojection to p.

On the subfactor side, this situation corresponds to an intermediate subfactor.
If F is the planar algebra for the subfactor $A \subset C$, a biprojection p gives an intermediate subfactor

$$A \subset B \subset C$$

so P and Q are the planar algebras for $A \subset B$ and $B \subset C$.

Although intermediate and composite planar algebras are extremely interesting, they don't play a significant role in the small index classification.
We'll only need:

<u>Lemma</u>: if some but not all the depth 2 bimodules have dimension 1, the sum of these and $\frac{1}{\delta}\genfrac{}{}{0pt}{}{\cup}{\cap}$ is a biprojection, hence there is an intermediate subfactor, so the index is composite.